Indian J Pure Appl Math (2022) 53:907–914
https://doi.org/10.1007/s13226-021-00196-4

ORIGINAL RESEARCH

Uniqueness of a meromorphic function sharing two values CM with its difference operator

Mahesh Barki · Renukadevi S. Dyavanal · Subhas S. Bhoosnurmath

Received: 16 March 2018 / Accepted: 11 October 2021 / Published online: 7 January 2022

Abstract This research investigates the uniqueness of meromorphic function $f(z)$ and its n^{th} order difference operator with two shared values counting multiplicities. It generalizes the results of Yeyang Jiang and Zongxuan Chen [5] and for the case $n \geq 2$, it admits a very special generalization. Moreover, we give an example to illustrate the validity of our main result.

Keywords Meromorphic functions · Difference operators · Sharing values · Small functions

Mathematics Subject Classification 30D35 · 39A10

1 Introduction, definitions and results

For a meromorphic function *f* in the complex plane $\mathbb{C}$, we shall use the standard notations, definitions and basic results of Nevanlinna theory of meromorphic functions (see [10,11]). A meromorphic function $a(z)$ is said to be small function of $f(z)$, if $T(r, a) = S(r, f)$, as $r \to \infty$, without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of *r* of finite linear measure.

Two meromorphic functions f and g share a CM (counting multiplicity) or IM (ignoring multiplicity), if $f - a$ and $g - a$ have the same set of zeros counting multiplicities or ignoring multiplicities, respectively.

In 1926, Nevanlinna [14] investigated the following two theorems for shared values.

Theorem A *If two meromorphic functions f and g share five distinct values IM, then $f \equiv g$.*

Theorem B *If two meromorphic functions f and g share four distinct values CM, then $f \equiv g$ or $f \equiv T \cdot g$, where T is a mobious transformation.*

In 1977, Rubel and Yang [15], investigated uniqueness of entire function f sharing values with its derivatives f'. Mues and Steinmetz [9,12] and Gunder Sen [2] improved their results.

Communicated by Kaushal Verma.

M. Barki · R. S. Dyavanal · S. S. Bhoosnurmath (✉)
Department of Mathematics, Karnatak University, Dharwad, Karnataka 580 003, India
E-mail: ssbmath@gmail.com

R. S. Dyavanal
E-mail: renukadyavanal@gmail.com

Present address
M. Barki
Department of Mathematics, Davangere University, Davangere, Karnataka 577 007, India
E-mail: mgbarkidud@gmail.com

Recently Nevanlinna theory has been established for difference operators [3,4,13,16]. Many authors [7,8,13] have considered shared values of meromorphic function with their difference operators or shifts.

The following two theorems, due to Heittokangas et al. [8,13] give the idea of shared values of meromorphic function with their shifts.

Theorem C *Let f be a meromorphic function of finite order and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $a_1, a_2, a_3 \in \acute{S}(f) = S(f) \cup \infty$ with period c CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem D *Let f be a meromorphic function of finite order and let $c \in \mathbb{C}$ and let $a_1, a_2, a_3 \in \acute{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

In 2013, Jiang and Chen [5], proved the following theorem for two distinct shared CM with some conditions.

Theorem E *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let η be a constant such that $f(z+\eta) - f(z) \not\equiv 0$ and let a, b be two non zero distinct finite complex constants. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share a, b CM then $f(z+\eta) = 2f(z)$.*

For a meromophic function $f(z)$ and a nonzero complex constant c, for $n \in \mathbb{N}$ we define its Shift by $f(z+c)$ and its n^{th} order difference operator by

$$\Delta_c^n f(z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f(z+jc),$$

where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

In this article we generalize Theorem E to the case of higher order difference operators defined above.

Theorem 1.1 *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let $c \in \mathbb{C}$ be a constant such that $\Delta_c^n f(z) \not\equiv 0$ and let a, b be two non zero distinct finite complex constants. If $\Delta_c^n f(z)$ and $f(z)$ share a, b CM then $\Delta_c^n f(z) \equiv f(z)$.*

Corollary 1.1 *Let f be a non constant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let $c \in \mathbb{C}$ be a constant such that $\Delta_c^n f(z) \not\equiv 0$ and let a, b be two non zero distinct finite complex constants. If $\Delta_c^n f(z)$ and $f(z)$ share a, b CM then $f(z+nc) = 2^n f(z)$.*

Remark 1.1 The corollary 1.1 generalizes Theorem E.

Example 1.1 Let $Q(z)$ be a periodic function of period 1 such that $\sigma(Q) = 2$ and $f(z) = \frac{Q(z)e^{z\log 2}}{\sin 2\pi z}$. Then $\Delta_c^n f(z) = \sum_{j=0}^{n} {}^nC_j\, (-1)^{n-j} f(z+jc)$ and $f(z)$ share 1, 2 CM and $N(r, f) = S(r, f)$, using corollary 1.1, we get $\Delta_c^n f(z) = f(z)$. Also $f(z+nc) = 2^n f(z)$.

2 Some lemmas

We need the following lemmas to prove our main result.

Lemma 2.1 ([11]). *Let $f(z)$ be a non constant meromorphic function, $a_j (j = 0, 1, \ldots, q)$ be q distinct complex numbers. Then we have*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O(1) \tag{2.1}$$

In 2013, Chen and Feng [3] and in 2006, Halburd and Korhonen [5] studied Value distribution theory of meromorphic function for difference operators. Analogous result of the Lemma on logarithemic derivatives is as follows

Lemma 2.2 ([7]). *Let $f(z)$ be a meromorphic function of finite order and let c be a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f). \tag{2.2}$$

These lemmas help us in establishing the following

Lemma 2.3 *Let $f(z)$ be a non constant meromorphic function in $\mathbb{C}$. Let $a_j(j = 0, 1, \ldots, q)$ be q distinct complex numbers. Then we have*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \tag{2.3}$$

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + S(r, f) \tag{2.4}$$

Proof By Lemmas 2.1, 2.2, we have

$$\begin{aligned}
\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) &= m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) + O(1) \\
&= m\left(r, \sum_{j=1}^{q} \frac{f^{(k)}}{f^{(k)}(f - a_j)}\right) + O(1) \\
&\leq m\left(r, \frac{1}{f^{(k)}}\right) + m\left(r, \sum_{j=1}^{q} \frac{f^{(k)}}{f - a_j}\right) + O(1) \\
&= m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).
\end{aligned}$$

Hence

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Now by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) &= m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) + O(1) \\
&= m\left(r, \sum_{j=1}^{q} \frac{\Delta_c^n f}{\Delta_c^n f(f - a_j)}\right) + O(1) \\
&\leq m\left(r, \frac{1}{\Delta_c^n f}\right) + m\left(r, \sum_{j=1}^{q} \frac{\Delta_c^n f}{f - a_j}\right) + O(1) \\
&\leq m\left(r, \frac{1}{\Delta_c^n f}\right) + \sum_{j=1}^{q} m\left(r, \frac{\Delta_c^n f}{f - a_j}\right) + O(1) \\
&\leq m\left(r, \frac{1}{\Delta_c^n f}\right) + S(r, f).
\end{aligned}$$

Hence

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) \leq m\left(r, \frac{1}{\Delta_c^n f}\right) + S(r, f).$$

□

Lemma 2.4 ([3]) *Let $f(z)$ be a meromorphic function with order $\sigma(f) = \sigma < \infty$, and c be a fixed non-zero complex number, then for each $\epsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O(r^{\sigma-1+\epsilon}) + S(r, f).$$

Lemma 2.5 ([7]) *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function a(z) with period c, with respect to f (z),*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = o(T(r, f)) = S(r, f),$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

3 Proof of the Theorem 1.1

Proof The assumption that $\Delta_c^n f(z)$ and $f(z)$ share a, b CM and by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r, f) &\leq N(r, f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &= N\left(r, \frac{1}{\Delta_c^n f - a}\right) + N\left(r, \frac{1}{\Delta_c^n f - b}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{\Delta_c^n f - f}\right) + S(r, f), \end{aligned}$$

that is

$$T(r, f) \leq N\left(r, \frac{1}{\Delta_c^n f - f}\right) + S(r, f). \tag{3.1}$$

Since $N(r, f) = S(r, f)$ and using first fundamental theorem, we have

$$\begin{aligned} N\left(r, \frac{1}{\Delta_c^n f - f}\right) + S(r, f) &= N\left(r, \frac{f}{\Delta_c^n f - f}\right) + S(r, f) \\ &= N\left(r, \frac{1}{\frac{\Delta_c^n f}{f} - 1}\right) + S(r, f) \\ &\leq T\left(r, \frac{1}{\frac{\Delta_c^n f}{f} - 1}\right) + S(r, f) \\ &= T\left(r, \frac{\Delta_c^n f}{f}\right) + S(r, f) \\ &= m\left(r, \frac{\Delta_c^n f}{f}\right) + N\left(r, \frac{\Delta_c^n f}{f}\right) + S(r, f) \\ &= N\left(r, \frac{\Delta_c^n f}{f}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f}\right) + N\left(r, \Delta_c^n f\right) + S(r, f). \end{aligned}$$

Hence by using the Lemma 2.4, we have

$$\begin{aligned} N\left(r, \frac{1}{\Delta_c^n f - f}\right) + S(r,f) &\le N\left(r, \frac{1}{f}\right) + 2^n N(r,f) + S(r,f) \\ &= N\left(r, \frac{1}{f}\right) + S(r,f) \\ &\le T\left(r, \frac{1}{f}\right) + S(r,f) \end{aligned}$$

that is

$$N\left(r, \frac{1}{\Delta_c^n f - f}\right) + S(r,f) \le T(r,f) + S(r,f) \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$T(r,f) = N\left(r, \frac{1}{f}\right) + S(r,f) = N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r,f). \tag{3.3}$$

Let $\Phi = \frac{(\Delta_c^n f)'}{\Delta_c^n f - a} - \frac{f'}{f-a}$. Using the logarithmic derivative theorem, we have

$$m(r,\Phi) \le m\left(r, \frac{(\Delta_c^n f)'}{\Delta_c^n f - a}\right) + m\left(r, \frac{f'}{f-a}\right) + O(1).$$

$$m(r,\Phi) \le S(r, \Delta_c^n f) + S(r,f) \tag{3.4}$$

Since

$$\begin{aligned} T(r, \Delta_c^n f) &= m(r, \Delta_c^n f) + N(r, \Delta_c^n f) \\ &\le m\left(r, f\frac{\Delta_c^n f}{f}\right) + \sum_{j=0}^{n}\binom{n}{j} N(r,f) + S(r,f) \\ &\le m(r,f) + m\left(r, \frac{\Delta_c^n f}{f}\right) + 2^n N(r,f) + S(r,f) \\ &= m(r,f) + S(r,f) \\ &\le T(r,f) + S(r,f) \end{aligned}$$

$$T(r, \Delta_c^n f) \le T(r,f) + S(r,f). \tag{3.5}$$

Since $S(r, \Delta_c^n f) = S(r,f)$, from (3.4), we have $m(r,\Phi) = S(r,f)$. Since Φ is the logarithmic derivative of $\dfrac{\Delta_c^n f - a}{f-a}$, the poles of Φ derive from the zeros and poles of $\dfrac{\Delta_c^n f - a}{f-a}$. Since f, $\Delta_c^n f$ share the value a CM, then $\frac{\Delta_c^n f - a}{f-a}$ has no zeros and has atmost $N(r,f)$ poles. Thus $N(r,\Phi) = N(r,f) = S(r,f)$. Combining this and (3.4), we have $T(r,\Phi) = S(r,f)$.

Suppose that $\Phi \not\equiv 0$, then from

$$\frac{\Phi}{f-b} = \frac{(\Delta_c^n f)'}{\Delta_c^n f - a}.\frac{\Delta_c^n f}{f-b} - \frac{f'}{(f-a)(f-b)},$$

we obtain

$$\begin{aligned} m\left(r, \frac{1}{f-b}\right) &\le m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{\Phi}{f-b}\right) \\ &\le m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{(\Delta_c^n f)'}{\Delta_c^n f - a}\right) + m\left(r, \frac{\Delta_c^n f}{f-b}\right) \end{aligned}$$

$$
\begin{aligned}
&+ m\left(r, \frac{f'}{(f-a)(f-b)}\right) + O(1)\\
&\leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{(\Delta_c^n f)'}{a}\left(\frac{1}{\Delta_c^n f - a} - \frac{1}{\Delta_c^n f}\right)\right)\\
&+ m\left(r, \frac{\Delta_c^n f}{f-b}\right) + m\left(r, \frac{f'}{(a-b)}\left(\frac{1}{(f-a)} - \frac{1}{f-b}\right)\right) + O(1)\\
&\leq T(r, \Phi) + S(r, \Delta_c^n f) + S(r, f)\\
&= S(r, f).
\end{aligned}
$$

Hence

$$m\left(r, \frac{1}{f-b}\right) = S(r, f).$$

By the first fundamental theorem we get

$$T(r, f) = N\left(r, \frac{1}{f-b}\right) + S(r, f). \tag{3.6}$$

Thus, by (3.3) and (3.6), we have

$$N\left(r, \frac{1}{f-a}\right) = S(r, f).$$

By the assumption that $\Delta_c^n f(z)$ and $f(z)$ share a CM, we have that

$$N\left(r, \frac{1}{\Delta_c^n f - a}\right) = N\left(r, \frac{1}{f-a}\right) = S(r, f).$$

Hence

$$N\left(r, \frac{1}{\Delta_c^n f - a}\right) = S(r, f).$$

Since $\Delta_c^n f(z)$ and $f(z)$ share b CM, by equation (3.5)and (3.6) and first fundamental theorem, we have

$$
\begin{aligned}
m\left(r, \frac{1}{\Delta_c^n f - b}\right) + N\left(r, \frac{1}{\Delta_c^n f - b}\right) &= T(r, \Delta_c^n f) + S(r, f)\\
&\leq T(r, f) + S(r, f)\\
&= N\left(r, \frac{1}{f-b}\right) + S(r, f)\\
&= N\left(r, \frac{1}{\Delta_c^n f - b}\right) + S(r, f).
\end{aligned}
$$

Thus,

$$m\left(r, \frac{1}{\Delta_c^n f - b}\right) = S(r, \Delta_c^n f) = S(r, f).$$

By lemma (2.3), we have that

$$m\left(r, \frac{1}{\Delta_c^n f}\right) + m\left(r, \frac{1}{\Delta_c^n f - a}\right) + m\left(r, \frac{1}{\Delta_c^n f - b}\right) \leq m\left(r, \frac{1}{(\Delta_c^n f)'}\right) + S(r, f) \tag{3.7}$$

$$m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) \leq m\left(r, \frac{1}{\Delta_c^n f}\right) + S(r, f) \tag{3.8}$$

By using (3.3) and the previous inequalities for counting functions, we have

$$N\left(r, \frac{1}{\Delta_c^n f - a}\right) + N\left(r, \frac{1}{\Delta_c^n f - b}\right) \leq N\left(r, \frac{1}{\Delta_c^n f - b}\right) + S(r, f) \tag{3.9}$$

$$N\left(r, \frac{1}{f - a}\right) + N\left(r, \frac{1}{f - b}\right) \leq T(r, f) + S(r, f). \tag{3.10}$$

Using equations (3.7) - (3.10), we get

$$\begin{aligned} T\left(r, \frac{1}{\Delta_c^n f - a}\right) + T\left(r, \frac{1}{\Delta_c^n f - b}\right) + T\left(r, \frac{1}{f - a}\right) + T\left(r, \frac{1}{f - b}\right) \\ \leq m\left(r, \frac{1}{(\Delta_c^n f)'}\right) + N\left(r, \frac{1}{\Delta_c^n f - b}\right) + T(r, f) + S(r, f) \\ \leq T\left(r, \frac{1}{(\Delta_c^n f)'}\right) + T\left(r, \frac{1}{\Delta_c^n f - b}\right) + T(r, f) + S(r, f). \end{aligned} \tag{3.11}$$

Since $N(r, f) = S(r, f)$, we get

$$\begin{aligned} T\left(r, \frac{1}{(\Delta_c^n f)'}\right) &= m\left(r, (\Delta_c^n f)'\right) + N\left(r, (\Delta_c^n f)'\right) + O(1) \\ &= m\left(r, \Delta_c^n f \frac{(\Delta_c^n f)'}{\Delta_c^n f}\right) + N\left(r, \Delta_c^n f\right) + \overline{N}\left(r, \Delta_c^n f\right) + O(1) \\ &\leq m\left(r, \Delta_c^n f\right) + m\left(r, \frac{(\Delta_c^n f)'}{\Delta_c^n f}\right) + S(r, f) \\ &\leq T\left(r, \Delta_c^n f\right) + S(r, f). \end{aligned}$$

Hence

$$T\left(r, \frac{1}{(\Delta_c^n f)'}\right) \leq T\left(r, \Delta_c^n f\right) + S(r, f). \tag{3.12}$$

Considering (3.5),(3.11),(3.12) and first fundamental theorem, we get

$$T(r, f) = S(r, f),$$

which is a contradiction.
Therefore $\Phi \equiv 0$, that is

$$\frac{(\Delta_c^n f)'}{\Delta_c^n f - a} = \frac{f'}{f - a}. \tag{3.13}$$

By integrating (3.13), we obtain

$$\frac{\Delta_c^n f - a}{f - a} \equiv C_1, \tag{3.14}$$

where C_1 is some non-zero constant.
Using the same procedure as above, by the assumption, $\Delta_c^n f$ and $f(z)$ share b CM, we get

$$\frac{\Delta_c^n f - a}{f - a} \equiv C_2, \tag{3.15}$$

where C_2 is some non-zero constant.
If $C_1 \equiv 1$ or ($C_2 \equiv 1$), then by (3.14) (or(3.15)), we obtain

$$\Delta_c^n f(z) \equiv f(z).$$

Hence conclusion holds.

If $C_1 \neq 1$ and $C_2 \neq 1$, then by (3.14) and (3.15), we get

$$(\Delta_c^n f - a) - (\Delta_c^n f - b) = (C_1 f - C_1 a) - (C_2 f - C_2 b)$$
$$b - a = (C_1 - C_2) f + C_2 b - C_1 a$$

$$(C_1 - C_2) f = b - a + C_2 b - C_1 a \tag{3.16}$$

If $C_1 \neq C_2$, then f is a constant.
which is contradiction.
Hence $C_1 = C_2$, thus by (3.16), we have

$$a - b = C_1 (a - b)$$

Therefore $C_1 = 1 = C_2$. Which is contradiction.
This completes the proof.

□

Proof of the corollary 1.1. Using Theorem 1.1 , we have

$$\Delta_c^n f(z) \equiv f(z).$$

This implies that,

$$\begin{aligned} f(z + nc) &= \left[\sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n-j} 2^j + 1 \right] f(z) \\ &= 2^n f(z) \end{aligned}$$

Hence the proof. □

Acknowledgements The authors wish to thank the referee for his/her valuable remarks and suggestions. The first and third authors are supported by the SERB(DST) Project,No:SB/S4/MS/842/2013, dated:10/12/2015 and the Second author is thankful to the Karnatak University, Dharwad - India for financial support under University Research Seed Grant Policy for research project No.: KU/PMEB/2021-22/76

References

1. Chiang, Yik-Man; Feng, Shao-Ji - *On the Nevanlinna Characteristic of f(z) and difference equations in the complex plane*, Ramanujan J., 16(2008), 105-129.
2. Gundersen G. G. - *Meromorphic function that share two finite values with their derivatives*, Pacific J. Math., 105(1983), 299-309.
3. Chen, Zong-Xuan; Yi, Hong-Xun. - *On charing values of meromorphic functions and their derivatives*, Results math., 63(2013), 557-565.
4. Chen, B; Chen, Z; Li, S. - *Uniqueness of difference operators of meromorphic functions*, J.Inequal. Appl., 2012, 2012:48,9pp.
5. Jiang, Yeyang; Chen, Zongxuan - *Meromoprhic functions share two values with its difference operator*,Analele Stintifice Ale Universitatii "Al.I. Cuza" Din Iasi (S.N.).
6. Dyavanal, R. S; Mathai, M. M. - *Uniqueness of difference-differential polynomials of meromorphic functions and its applications*, Indian J. Math. Math. Sci., 12, No. 1, 11-30 (2016).
7. Halburd, R. G; Korhonen, R. J. - *Difference analogue of the lemma on the logarithmic derivative with application to difference equations*, J. Math. Anal. Appl., 314 (2006), 477-487.
8. Heittokangas, J; Korhonen, R; Laine, I; Rieppo J. - *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ., 56 (2011), 81-92.
9. Mues, E; Steinmetz, N. - *Meromorphe Funktiotnen die mit ither Ableitung Werte teilen*, Manucripta Math., 29(1979), 195-206.
10. W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).
11. C.C.Yang and H.X. Yi,*Uniqueness theory of meromorphic functions*, Kluwer, Dordrect, 2004.
12. Mues. E.; Steinmetz, N. - *Meromorphe die mit ithrer Ableitung zwei Werte teit*, (German),[Meromorphic functions that share two values with their derivatives], Results Math., 6 (1983), 48-55.
13. Heittokangas, J.; Korhonen, R; Laine, I.; Rieppo, J.; Zhang, J. - *Value sharing results for shift of meromorphic functions and sufficient condition for periodicity*, J. Math. Anal. Appl., 355 (2009), 352-391.
14. Nevanlinna, R. *Einige Eindeutigkeitsstze in der theorie der Meromorphen Funkionen*, Acta Math., 48 (1926), 367-391.
15. Rubel, L.A; Yang, C. C. *Values shared by an entire function and its derivative.* In: Buckholtz J.D., Suffridge T.J. (eds) Complex Analysis. Lecture Notes in Mathematics, vol 599(1977). Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0096830
16. Qi, Xiaoguang; Liu, Kai. - *Uniqueness and value distribution of differences of entire functions*,J. Math. Anal. Appl., 379 (2011), 180-187.