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ORIGINAL RESEARCH

Steiner Wiener index of Line graphs

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Abstract Let S be a set of vertices of a connected graph G. The Steiner distance of S is the minimum size among all connected subgraphs whose vertex sets contain S. The sum of all Steiner distances on sets of size k is called the Steiner k-Wiener index. We study inequalities on Steiner Wiener index of line graphs. Also we study relations among Steiner Wiener index of line graphs, Steiner edge Wiener index and Steiner Gutman index.

Keywords Distance in graphs · Steiner distance · Wiener index · k-Steiner Wiener index · Line graphs · Steiner Gutman index

Mathematics Subject Classifications 05C09 · 05C12 · 05C76

1 Introduction

All graphs in this paper are simple, finite and undirected. If G is a connected graph and $u, v \in V(G)$, then the distance $d(u, v)$ between u and v is the number of edges on a shortest path connecting u and v. The Wiener index $W(G)$ of a connected graph G is defined by

$$W(G) = \sum_{\{u,v\} \in V(G)} d(u, v).$$

The first investigation of this distance-based graph invariant was done by Wiener in 1947, who realized in [15] that there exist correlations between the boiling points of paraffins and their molecular structure. The Steiner distance of a graph was introduced in [2] by Chartrand et.al., as a natural generalization of the usual graph distance. For a connected graph G and $S = \{x_1, \ldots, x_k\} \subseteq V(G)$, the Steiner distance of S, $d(S) = d(x_1, \ldots, x_k)$ is the minimum size among all connected subgraphs whose vertex sets contain S. Note that any such subgraph H is a tree, called a Steiner tree connecting the vertices of S. The vertices of S are called the terminal vertices of tree H, while the other vertices are called the inner vertices. If $S = \{u, v\}$, then $d(S) = d(u, v)$ coincides with the usual distance between u and v. In [4] Dankelmann et.al. studied the average k-Steiner distance $\mu_k(G)$. In [11],

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Li et.al. introduced a generalization of the Wiener index by using the Steiner distance. The Steiner k-Wiener index $SW_k(G)$ of a connected graph G is defined as

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d(S).$$

For $k = 2$, the Steiner k-Wiener index coincides with the Wiener index. The average k-Steiner distance $\mu_k(G)$ is related to the Steiner k-Wiener index, via the equality $\mu_k(G) = SW_k(G)/\binom{n}{k}$.

The edge Wiener index of a graph is defined [8] as follows, $W_e(G) = \sum_{\{f,g\} \in E(G)} D(f, g)$ where $D(f, g) = \min\{d(u, x), d(u, y), d(v, x), d(v, y)\}$ for $f = uv$, $g = xy$.

The line graph $L(G)$ of G, has its vertex set $V(L(G)) = E(G)$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. The iterated line graphs are defined as follows:

$$L^i(G) = \begin{cases} G, \text{ for } i = 0 \\ L(L^{i-1}(G)), \text{ for } i > 0 \end{cases}$$

For any nontrivial tree T on n vertices we have $W(L(T)) = W(T) - \binom{n}{2}$ [1].

In this paper, we first study the Steiner Wiener index of line graphs of unicyclic graphs. We study relations among Steiner Wiener index of line graphs, Steiner edge Wiener index and Steiner Gutman index. We conclude the paper with some bounds on 3-Steiner Wiener index of line graphs.

2 Inequalities for Steiner Wiener index of line graphs

In [7], Gutman et.al. proved the following result.

Theorem 21 *If G is a connected unicyclic graph of order n, then $W(L(G)) \leq W(G)$ with equality only if G is the cycle-graph C_n.*

Now, we generalise the theorem as follows.

Theorem 22 *For a unicyclic graph G of order n,*

$$SW_k(L(G)) \leq SW_k(G)$$

and the equality holds if and only if G is a cycle C_n.

Proof If $G = C_n$, then $L(G)$ is isomorphic to G. Therefore, assume that G differs from C_n.

A connected unicyclic graph of order n has equal number of vertices and edges. Therefore we can construct bijective mappings between the vertex and edge sets of G. We choose the following mapping f of the vertices onto the edges:

Let the vertices belonging to the (unique)cycle C of G be $v_1, \ldots, v_p$, such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, p$ and $v_{p+1} = v_1$. Let $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, p$. Define $f(v_i) = e_i$ for $i = 1, 2, \ldots, p$. To define f for vertices not belonging to C, we start from a pendant vertex u. Since G is a unicyclic graph and differs from C_n, there exist at least one pendant vertex. If uv is the pendant edge incident to u, then $f(u) = uv$. Now consider $V(G) \setminus \{u\}$. Map a pendant vertex u_1 in $V(G) \setminus \{u\}$ to its pendant edge. Then, consider $V(G) \setminus \{u, u_1\}$. Repeat the process until all the vertices in the graph, except the vertices in the cycle C are mapped. Alternatively, this map can be defined as follows. For a vertex v' not in the unique cycle C, let $f(v') = e'$ if e' is incident to v' and e' is in the unique path to the cycle C. This map is well defined since G is connected and contains exactly one cycle.

The bijection f has the following property: If the vertices x and y are adjacent, then the edges $f(x)$ and $f(y)$ are incident. In other words, if the vertices x and y of G are adjacent, then the vertices $f(x)$ and $f(y)$ of $L(G)$ are adjacent. This implies that G is isomorphic to a spanning subgraph of $L(G)$. Theorem now follows from the fact that the Steiner Wiener index of a (connected) spanning subgraph of a graph does not exceed the Steiner Wiener index of the graph itself. If G differs from C_n then $L(G)$ has more edges than G and the inequality in the theorem must be strict. □

3 Steiner edge Wiener index and Steiner Gutman index

Definition 31 For a connected graph G, the Steiner edge Wiener index

$$SW_{ke}(G) = \sum_{S \subseteq E(G), |S|=k} D_G(S)$$

where $D_G(S)$ is the minimum size among all connected subgraphs whose edge sets contain S.

Khalifeh et.al. [8] gave the relation between Wiener index of a line graph and edge Wiener index. Using the definition 31, the relation can be rephrased as

Lemma 32 *Let G be a connected graph. Then, for $n \geq 2$,*

$$W_e(G) - W(L(G)) = \binom{|E(G)|}{2} = \binom{|V(L(G))|}{2} \tag{1}$$

and

$$W_e(L^{n-1}G) - W(L^n(G)) = \binom{|V(L^n(G))|}{2}$$

We generalise the Lemma 32 as follows.

Lemma 33 *For a connected graph G,*

$$SW_{ke}(G) - SW_k(L(G)) = \binom{|E(G)|}{k} = \binom{|V(L(G))|}{k} \tag{2}$$

and

$$SW_{ke}(L^{n-1}(G)) - SW_k(L^n(G)) = \binom{|V(L^n(G))|}{k}$$

Proof Suppose $S \subseteq V(L(G))$, then $S \subseteq E(G)$, and $d_{L(G)(S)} = D_G(S) - 1$

$$\begin{aligned} SW_k(L(G)) &= \sum_{S \subseteq V(L(G))} d_{L(G)}(S) \\ &= \sum_{S \subseteq E(G)} D_G(S) - 1 \\ &= SW_{ke}(G) - \binom{|E(G)|}{k} \end{aligned}$$

In general

$$SW_{ke}(L^{n-1}(G)) - SW_k(L^n(G)) = \binom{|V(L^n(G))|}{k}$$

□

Now we prove the following proposition.

Proposition 1 *Let G be a connected graph of order n. Then, $SW_{ke}(G) \geq k\binom{n-1}{k}$ with equality if and only if G is a star.*

Proof A connected graph G of order n has at least $n-1$ edges and the Steiner distance between any k edges is at least k. Hence,

$$SW_{ke}(G) = \sum_{\{e_1,\ldots,e_k\} \subseteq E(G)} d(e_1, \ldots, e_k) \geq k\binom{|E(G)|}{k} \geq k\binom{n-1}{k}.$$

We have the equality above for a graph G if and only if G has exactly $n-1$ edges and the distance between any k edges is k. Then, G is a tree and $L(G)$ is a complete graph. Thus G is a star and the result follows. □

This proposition generalises the following in [3]. Let G be a connected graph of order n. Then, $W_e(G) \geq \binom{n-1}{2}$ with equality if and only if G is a star.
The Gutman index is a well studied graph invariant introduced by Gutman [6]. Mao et.al. [13] generalised the concept using Steiner distance as follows.

Definition 34 [13] The Steiner Gutman index of a graph G,

$$Gut_k(G) = \sum_{\{x_1,\ldots,x_k\}\subseteq V(G)} d_{x_1}\ldots d_{x_k} d(x_1,\ldots,x_k).$$

Definition 35 [3] Let $G = (V, E)$ be a connected graph and c be a real valued weight function on the vertices of G. Then the Steiner Wiener index of G with respect to c is

$$SW_k(G,c) = \sum_{\{x_1,\ldots,x_k\}\subseteq V(G)} c(x_1)\ldots c(x_k) d_G(x_1,\ldots,x_k)$$

We note that for c=1 this yields the usual Wiener index, while for $c(x) = d_x$ we obtain the Gutman index.

Lemma 36 *Let G be a graph and H be the graph obtained from G by subdividing each edge once. For an edge $e \in E(G)$, v_e represents the vertex of degree 2 in $V(H) - V(G)$. Then, for any k edges, $e_1,\ldots,e_k \in E(G)$,*

$$d_H(v_{e_1},\ldots,v_{e_k}) = 2d_G(e_1,\ldots,e_k) - p,$$

where p denotes the number of pendant vertices.

Proof We prove the statement by induction on number of pendant vertices. Suppose, the graph has 1 pendant vertex, then the pendant edge $\in \{e_1,\ldots,e_k\}$. Let it be $e_1 = uv$ such that u is pendant vertex. Finding $d_H(v_{e_1},\ldots,v_{e_k})$, we can omit the edge uv_{e_1} in considering the Steiner tree for $v_{e_1},\ldots,v_{e_k}$. All the other edges remain the same. Then, $d_H(v_{e_1},\ldots,v_{e_k}) = 2d_G(e_1,\ldots,e_k) - 1$. Now, assume the case for m pendant vertices. We prove for $m+1$. Let G be a graph with $m+1$ pendant vertices. Let e_1 be a pendant edge. We delete e_1 and its incident edges until we get a graph G' with m pendant vertices. Applying the induction assumption, we get, $d_{H'}(v_{e_2},\ldots,v_{e_k}) = 2d_{G'}(e_2,\ldots,e_k) - m$. If $e_1 = uv$ such that u is the pendant edge, then adding all the deleted edges of G except uv_{e_1}, we get a Steiner tree for $v_{e_1},\ldots,v_{e_k}$. Then, $d_H(v_{e_1},\ldots,v_{e_k}) = 2d_G(e_1,\ldots,e_k) - (m+1)$. Thus, the result follows. □

Dankelmann et.al. [3] studied the relation between Wiener index of line graphs and Gutman index.

Theorem 37 *[3] For a graph G of order n,*

$$|W(L(G)) - \frac{Gut_2(G)}{4}| \leq \frac{n^4}{8}$$

Now, we can generalise the Theorem 37 as follows.

Theorem 38 *Let G be a connected graph of order n. Then,*

$$|SW_k(L(G)) - \frac{Gut_k(G)}{2^k}| \leq \frac{n^{2k}}{(2^{k+1})(k-1)!}$$

Proof Consider the graph H obtained from G by subdividing each edge once. Consider the following functions a and b on $V(H)$ defined as follows.

$$a(v) = \begin{cases} d_v & \text{if } v \in V(G), \\ 0 & \text{if } v \in V(H) - V(G). \end{cases} \quad b(v) = \begin{cases} 0 \text{ if } v \in V(G), \\ 2 \text{ if } v \in V(H) - V(G). \end{cases}$$

Since for any k vertices $x_1,\ldots,x_k \in V(G)$, we have $d_H(x_1,\ldots,x_k) = 2d_G(x_1,\ldots,x_k)$, it follows that

$$\begin{aligned} SW_k(H,a) &= \sum_{\{x_1,\ldots,x_k\}\subseteq V(H)} a_{x_1}\ldots a_{x_k} d_H(x_1,\ldots,x_k) \\ &= \sum_{\{x_1,\ldots,x_k\}\subseteq V(G)} 2d_{x_1}\ldots d_{x_k} d_G(x_1,\ldots,x_k) \\ &= 2Gut_k(G) \end{aligned}$$

Denote by v_e the vertex of degree 2 in $V(H) - V(G)$ that subdivides the edge $e \in E(G)$. Then $b(x) \neq 0$ only if $x = v_e$ for some edge e of G. By the Lemma 36, we have $d_H(v_{e_1}, \ldots, v_{e_k}) = 2d_G(e_1, \ldots, e_k) - p$, for any k edges $e_1, \ldots, e_k \in E(G)$. Here, p denotes the number of pendant vertices of the Steiner tree in G containing the edges $e_1, \ldots, e_k$. Let m be the number of edges in the graph G. Also, $p \geq 2$. Then,

$$\begin{aligned} SW_k(H, b) &= \sum_{\{x_1, \ldots, x_k\} \subseteq V(H) - V(G)} b_{x_1}, \ldots, b_{x_k} d_H(x_1, \ldots, x_k) \\ &= \sum_{\{x_1, \ldots, x_k\} \subseteq V(H) - V(G)} 2^k (2d_G(e_1, \ldots, e_k) - p) \\ &= 2^{k+1} SW_{ke}(G) - 2^k \sum_{x_1, \ldots, x_k \subseteq V(H) - V(G)} p \\ &\leq 2^{k+1} SW_{ke}(G) - 2^{k+1} \binom{m}{k} \end{aligned}$$

We now compare $SW_k(H, a)$ and $SW_k(H, b)$. Clearly, the weight function of a is obtained from the weight function of b by moving one weight unit of a vertex v_{uw} to vertex u and the other weight unit to a vertex w for all $uw \in E(G)$. Hence no weight has been moved over a distance of more than one, so no distance between two weights has moved a distance more than 2. Since we have $2|E(G)|$ weight units in total, the sum of the distances between the weight units has changed by at most $k\binom{2m}{k}$. Now,

$$|SW_k(H, b) - SW_k(H, a)| \leq |2^{k+1} SW_{ke}(G) - 2^{k+1} \binom{m}{k} - 2Gut_k(G)| \leq k \binom{2m}{k}.$$

Dividing both sides by 2^{k+1}, we get,

$$|SW_{ke}(G) - \frac{Gut_k(G)}{2^k} - \binom{m}{k}| \leq \frac{k}{2^{k+1}} \binom{2m}{k}$$

Then from Lemma 33, we get,

$$|SW_k(L(G)) - \frac{Gut_k(G)}{2^k}| \leq \frac{k}{2^{k+1}} \binom{2m}{k} \leq \frac{n^{2k}}{(2^{k+1})(k-1)!}$$

□

4 Bounds on 3-Steiner Wiener index of line graphs

For any three edges $e_1 = (a_1, a_2)$, $e_2 = (b_1, b_2)$ and $e_3 = (c_1, c_2)$ of a connected graph G, let $s(e_1, e_2, e_3) = \sum_{i,j,k \in \{1,2,3\}} d(a_i, b_j, c_k)$.

Lemma 41 *Let G be a connected graph. Then for any three edges $e_1 = (a_1, a_2)$, $e_2 = (b_1, b_2)$ and $e_3 = (c_1, c_2)$,*

$$D(e_1, e_2, e_3) \geq \frac{1}{2^3} s(e_1, e_2, e_3) + \frac{1}{2}.$$

Proof Without loss of generality, we assume that $d(a_1, b_1, c_1) = min\{d(a_i, b_j, c_k)\}$ for $i, j, k \in \{1, 2, 3\}$. The term $s(e_1, e_2, e_3)$ contains eight summands and clearly, $s(e_1, e_2, e_3) \leq 8d(a_1, b_1, c_1) + 12$. $D(e_1, e_2, e_3)$ is obtained by adding the number of pendant edges (two or three) to $d(a_1, b_1, c_1)$. In any case,

$$\begin{aligned} s(e_1, e_2, e_3) &\leq 8(D(e_1, e_2, e_3) - 2) + 12 \\ &= 8D(e_1, e_2, e_3) - 4 \end{aligned}$$

which proves the result. □

Lemma 42 *[13] For a connected graph G with minimum degree $\delta(G)$,*

$$\frac{1}{\delta(G)^k} Gut_k(G) \geq SW_k(G).$$

Lemma 43 *For a graph G,*

$$SW_{3e}(G) \geq \frac{1}{2^3}\{Gut_3(G) - \sum_{ab\in E(G);\{bc,ac\}\notin E(G)} (d_c)d(a,b,c) - \sum_{\{ab,bc\}\in E(G);ac\notin E(G)} 2(d_c+d_a) - \sum_{\{ab,bc,ac\}\in E(G)} 2(d_a+d_b+d_c)\} + \frac{1}{2}\binom{m}{3}$$

Proof We have,

$$SW_{3e}(G) = \sum_{\{e_1,e_2,e_3\}\subseteq E(G)} D(e_1,e_2,e_3).$$

By definition,

$$s(e_1,e_2,e_3) = \sum_{i,j,k\in\{1,2,3\}} d(a_i,b_j,c_k)$$

To find $\sum_{\{e_1,e_2,e_3\}\in E(G)} s(e_1,e_2,e_3)$, we count the contribution of each triplet of vertices to the sum.

$$\begin{aligned}
\sum_{\{e_1,e_2,e_3\}\in E(G)} s(e_1,e_2,e_3) &= \sum_{\{ab,bc,ac\}\notin E(G)} d_a d_b d_c d(a,b,c) \\
&\quad + \sum_{ab\in E(G);\{bc,ac\}\notin E(G)} (d_a d_b d_c - d_c)d(a,b,c) \\
&\quad + \sum_{\{ab,bc\}\in E(G);ac\notin E(G)} (d_a d_b d_c - d_c - d_a)d(a,b,c) \\
&\quad + \sum_{\{ab,bc,ac\}\in E(G)} (d_a d_b d_c - d_a - d_b - d_c)d(a,b,c) \\
&= Gut_3(G) - \sum_{ab\in E(G);\{bc,ac\}\notin E(G)} (d_c)d(a,b,c) \\
&\quad - 2\sum_{\{ab,bc\}\in E(G);ac\notin E(G)} (d_c+d_a) - 2\sum_{\{ab,bc,ac\}\in E(G)} (d_a+d_b+d_c)
\end{aligned}$$

Using lemma 41, we have,

$$\begin{aligned}
SW_{3e}(G) &\geq \sum_{\{e_1,e_2,e_3\}\in E(G)} \{\frac{1}{2^3} s(e_1,e_2,e_3) + \frac{1}{2}\} \\
&= \frac{1}{2^3}\{Gut_3(G) - \sum_{ab\in E(G);\{bc,ac\}\notin E(G)} (d_c)d(a,b,c) \\
&\quad - \sum_{\{ab,bc\}\in E(G);ac\notin E(G)} 2(d_c+d_a) - \sum_{\{ab,bc,ac\}\in E(G)} 2(d_a+d_b+d_c)\} + \frac{1}{2}\binom{m}{3}
\end{aligned}$$

□

Theorem 44 *For a graph G with $\delta(G) \geq 2$,*

$$SW_3(L(G)) \geq SW_3(G) - \frac{1}{2^3}\{ \sum_{ab \in E(G); \{bc, ac\} \notin E(G)} (d_c) d(a, b, c) + \sum_{\{ab, bc\} \in E(G); ac \notin E(G)} 2(d_c + d_a) + \sum_{\{ab, bc, ac\} \in E(G)} 2(d_a + d_b + d_c)\} - \frac{1}{2}\binom{m}{3}$$

Proof Using Lemma 42 for $\delta(G) \geq 2$ and Lemma 43, the result follows. □

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