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ORIGINAL RESEARCH

Generalized Hybrid Fibonacci and Lucas p-numbers

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Abstract The hybrid numbers are a generalization of complex, hyperbolic and dual numbers. Until this time, many researchers have studied related to hybrid numbers. In this paper, using the generalized Fibonacci and Lucas p-numbers, we introduce the generalized hybrid Fibonacci and Lucas p-numbers. Also, we give some special cases and algebraic properties of the generalized hybrid Fibonacci and Lucas p-numbers.

Keywords Fibonacci p-numbers · Lucas p-numbers · Hybrid numbers

Mathematics Subject Classification 11B37 · 11B39 · 11E88 · 15A66

1 Introduction

Fibonacci and Lucas p-numbers are introduced by Stakhov and Rozin in [12]. The Pell and Pell-Lucas p-numbers are introduced by Kocer and Tuglu in [7]. Afterwards, the m-extension of Fibonacci and Lucas p-numbers and bivariate Fibonacci like p-polynomials are defined by Tuglu, Kocer and Stakhov in [8,18]. Now, we shall give the definitions and some properties of the m-extension of Fibonacci and Lucas p-numbers.

In [8], for given integer $p \geq 0,\ n > p$ and positive real number $m > 0$, the m-extension of Fibonacci p-numbers are defined by the following recurrence relation

$$F_{p,m}(n) = mF_{p,m}(n-1) + F_{p,m}(n-p-1) \tag{1}$$

with initial conditions $F_{p,m}(0) = 0,\ F_{p,m}(n) = m^{n-1}$ for $n = 1, 2, \ldots, p+1$.

Similarly, the m-extension of Lucas p-numbers are defined by the following recurrence relation

$$L_{p,m}(n) = mL_{p,m}(n-1) + L_{p,m}(n-p-1) \tag{2}$$

with initial conditions $L_{p,m}(0) = p+1,\ L_{p,m}(n) = m^n$ for $n = 1, 2, \ldots, p$.

For different values of m and p, the recursive relations generates different numerical sequences. Some special cases of the m-extension of Fibonacci and Lucas p-numbers are as follows:

1. For $m = p = 1$, $F_{p,m}(n)$ and $L_{p,m}(n)$ reduce to the classical Fibonacci and Lucas numbers

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2. For $m = 2$ and $p = 1$, $F_{p,m}(n)$ and $L_{p,m}(n)$ reduce to the Pell and Pell-Lucas numbers
3. For $m = k$ and $p = 1$, $F_{p,m}(n)$ and $L_{p,m}(n)$ reduce to the k-Fibonacci and k-Lucas numbers
4. For $m = 1$, $F_{p,m}(n)$ and $L_{p,m}(n)$ reduce to the classical Fibonacci and Lucas p-numbers
5. For $m = 2$, $F_{p,m}(n)$ and $L_{p,m}(n)$ reduce to the classical Pell and Pell-Lucas p-numbers

For more information relation with these number sequences, please refer to [3,7,9,12] and closely related references therein. In [18], the authors have defined the bivariate Fibonacci and Lucas p-polynomials and gave some properties of these polynomials. Using some properties of bivariate Fibonacci and Lucas p-polynomials, we can obtain some identities of $F_{p,m}(n)$ and $L_{p,m}(n)$. Firstly, the generalized Honsberger formula for $F_{p,m}(n)$ is

$$F_{p,m}(k+n) = F_{p,m}(k)F_{p,m}(n+1) + \sum_{t=1}^{p} F_{p,m}(k-t)F_{p,m}(n-p+t) \tag{3}$$

where $k,\ n$ are positive integers.

Also, we have

$$L_{p,m}(n) = F_{p,m}(n+1) + pF_{p,m}(n-p) \tag{4}$$

for $F_{p,m}(n)$ and $L_{p,m}(n)$ generalized Fibonacci and Lucas p-numbers from [18].

In [11], Ozdemir defined the set of hybrid numbers denoted by $\mathbb{K}$ which contains complex, dual and hyperbolic numbers. The author gave some theorems, properties and matrix form related to hybrid numbers. The set of hybrid numbers is defined as

$$\mathbb{K} = \left\{a + b\mathbf{i} + c\epsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \epsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \epsilon + \mathbf{i}\right\}.$$

We can perfom some properties and operations with the hybrid numbers. Namely, taking two hybrid numbers $Z_1 = a_1 + b_1\mathbf{i} + c_1\epsilon + d_1\mathbf{h}$ and $Z_2 = a_2 + b_2\mathbf{i} + c_2\epsilon + d_2\mathbf{h}$, we get

$Z_1 = Z_2$, if and only if, $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$ (Equality),
$Z_1 + Z_2 = (a_1 + a_2) + (b_1 + b_2)\,\mathbf{i} + (c_1 + c_2)\,\epsilon + (d_1 + d_2)\,\mathbf{h}$ (addition),
$Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)\,\mathbf{i} + (c_1 - c_2)\,\epsilon + (d_1 - d_2)\,\mathbf{h}$ (subtraction),
$sZ_1 = sa_1 + sb_1\mathbf{i} + sc_1\epsilon + sd_1\mathbf{h}$ (multiplication by scalar $s \in \mathbb{R}$).

The hybrid product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and then writing the values of the following substituting each product of units by the equalities $\mathbf{i}^2 = -1$, $\epsilon^2 = 0$, $\mathbf{h}^2 = 1$, $\mathbf{ih} = -\mathbf{hi} = \epsilon + \mathbf{i}$. From these equalities, we can obtain the product of any two hybrid units. The multiplication table of the basis of hybrid numbers are as follows:

.	$\mathbf{1}$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	ϵ	$\mathbf{h}+\mathbf{1}$	0	$-\epsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	$\mathbf{1}$

The multiplication operation of the hybrid numbers is not commutative, but it has the property of associativity. The set of the hybrid numbers form a non-commutative ring with respect to the addition and multiplication.

Recently, many researchers have studied related to hybrid numbers. For example, in [15], Szynal-Liana and Wloch considered the Fibonacci hybrid numbers and obtained some properties of this numbers. In [13,16], the authors also defined and examined the Pell and Pell–Lucas hybrid numbers and the Jacosthal and Jacosthal–Lucas hybrid numbers, respectively. In [14], Szynal-Liana defined the Horadam hybrid numbers and gave the some properties. In [5], Kızılates defined the another generalization of hybrid numbers which called the q-Fibonacci hybrid numbers and q-Lucas hybrid numbers. Moreover, the author gave some important algebraic properties of these numbers. In [6], the author introduced a new hybrinomials with Horadam polynomials and gave some properties of Horadam hybrinomials. For more information, please refer to [1,2,4–6,10,11,17] and closely related references therein.

Motivated by some of the above-mentioned recent papers, we introduce generalized hybrid Fibonacci and Lucas p-numbers. This definition brings about more general hybrid number sequence by taking components from Fibonacci and Lucas p-numbers. With the help of this generalization, we obtain the hybrid Fibonacci and Lucas numbers, the hybrid Pell and Pell-Lucas numbers, the hybrid k-Fibonacci and k-Lucas numbers, the hybrid Fibonacci and Lucas p-numbers and the hybrid Pell and Pell-Lucas p-numbers. We also obtain various results for the generalized hybrid Fibonacci and Lucas p-numbers.

2 Generalized Hybrid Fibonacci and Lucas p-Numbers

In this section, we define the generalized hybrid Fibonacci and Lucas p-numbers. Afterwards, we give some special cases of generalized hybrid Fibonacci and Lucas p-numbers such as the hybrid Fibonacci and Lucas numbers, the hybrid Pell and Pell-Lucas numbers, the hybrid k-Fibonacci and k-Lucas numbers, the hybrid Fibonacci and Lucas p-numbers, the hybrid Pell and Pell-Lucas p-numbers. Finally, we obtain generating functions, summation formulas and some identities.

Definition 1 For $p > 0$ and $n \geq p$, the n^{th} generalized hybrid Fibonacci and Lucas p-numbers are defined by

$$\mathbb{H}F_{p,m}(n) = F_{p,m}(n) + \mathbf{i}F_{p,m}(n+1) + \epsilon F_{p,m}(n+2) + \mathbf{h}F_{p,m}(n+3) \tag{5}$$

and

$$\mathbb{H}L_{p,m}(n) = L_{p,m}(n) + \mathbf{i}L_{p,m}(n+1) + \epsilon L_{p,m}(n+2) + \mathbf{h}L_{p,m}(n+3). \tag{6}$$

Some special cases of generalized hybrid Fibonacci and Lucas p-numbers are as follows:

1. For $m = 1$, the generalized hybrid Fibonacci and Lucas p-numbers $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ become the $\mathbb{H}F_p(n)$ hybrid Fibonacci p-numbers and $\mathbb{H}L_p(n)$ hybrid Lucas p-numbers,
2. For $m = 2$, the generalized hybrid Fibonacci and Lucas p-numbers $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ become the $\mathbb{H}P_p(n)$ hybrid Pell p-numbers and $\mathbb{H}Q_p(n)$ hybrid Pell-Lucas p-numbers,
3. For $m = p = 1$, the generalized hybrid Fibonacci and Lucas p-numbers $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ become the $\mathbb{H}F_n$ hybrid Fibonacci and $\mathbb{H}L_n$ hybrid Lucas numbers,
4. For $m = 2$ and $p = 1$, the generalized hybrid Fibonacci and Lucas p-numbers $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ become the $\mathbb{H}P_n$ hybrid Pell and $\mathbb{H}Q_n$ hybrid Pell-Lucas numbers,
5. For $p = 1$ and k instead of m, the generalized hybrid Fibonacci and Lucas p-numbers $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ become the hybrid k-Fibonacci and k-Lucas numbers,

From the recurrence relations (1) and (5), we obtain that for $n > p$,

$$\begin{aligned}
\mathbb{H}F_{p,m}(n) &= F_{p,m}(n) + \mathbf{i}F_{p,m}(n+1) + \epsilon F_{p,m}(n+2) + \mathbf{h}F_{p,m}(n+3) \\
&= \big(mF_{p,m}(n-1) + F_{p,m}(n-p-1)\big) \\
&\quad + \big(mF_{p,m}(n) + F_{p,m}(n-p)\big)\,\mathbf{i} \\
&\quad + \big(mF_{p,m}(n+1) + F_{p,m}(n-p+1)\big)\,\epsilon \\
&\quad + \big(mF_{p,m}(n+2) + F_{p,m}(n-p+2)\big)\,\mathbf{h} \\
&= m\mathbb{H}F_{p,m}(n-1) + \mathbb{H}F_{p,m}(n-p-1).
\end{aligned}$$

So, for $n > p$, the recurrence relation of generalized hybrid Fibonacci p-numbers

$$\mathbb{H}F_{p,m}(n) = m\mathbb{H}F_{p,m}(n-1) + \mathbb{H}F_{p,m}(n-p-1). \tag{7}$$

Similarly, we can obtain the recurrence relation of generalized hybrid Lucas p-numbers as

$$\mathbb{H}L_{p,m}(n) = m\mathbb{H}L_{p,m}(n-1) + \mathbb{H}L_{p,m}(n-p-1). \tag{8}$$

Taking $m = 1$ in (7) and (8), we have the recurrence relations of hybrid Fibonacci and Lucas p-numbers are follows

$$\mathbb{H}F_p(n) = \mathbb{H}F_p(n-1) + \mathbb{H}F_p(n-p-1)$$

and

$$\mathbb{H}L_p(n) = \mathbb{H}L_p(n-1) + \mathbb{H}L_p(n-p-1)$$

for $n > p$.

Now, we give the generating functions for the generalized hybrid Fibonacci and Lucas p-numbers.

Theorem 1 *For $p \geq 1$, the generating function for the generalized hybrid Fibonacci p-numbers is*

$$g(z) = \frac{\mathbb{H}F_{p,m}(0) + z + z^{p-1}\mathbf{h} + z^{p}(\epsilon + m\mathbf{h})}{1 - mz - z^{p+1}}. \tag{9}$$

For $p > 1$, the generating function for the generalized hybrid Lucas p-numbers is

$$h(z) = \frac{\left[\begin{array}{c} HL_{p,m}(0) - mpz + ((p+1)h)z^{p-2} + \\ ((p+1)\epsilon + mh)z^{p-1} + \left((p+1)i + m\epsilon + m^2 h\right)z^{p} \end{array}\right]}{1 - mz - z^{p+1}}. \tag{10}$$

Proof We begin with the formal power series representation of the generating function for $\left\{\mathbb{H}F_{p,m}(n)\right\}_{n=0}^{\infty}$,

$$g(z) = HF_{p,m}(0) + HF_{p,m}(1)z + \cdots + HF_{p,m}(p)z^{p} + \cdots. \tag{11}$$

Hence,

$$mzg(z) = mHF_{p,m}(0)z + mHF_{p,m}(1)z^{2} + \cdots + mHF_{p,m}(p)z^{p+1} + \cdots \tag{12}$$

$$z^{p+1}g(z) = HF_{p,m}(0)z^{p+1} + HF_{p,m}(1)z^{p+2} + \cdots + HF_{p,m}(p)z^{2p+1} + \cdots \tag{13}$$

From (11), (12) and (13), we find that

$$\begin{aligned}(1 - mz - z^{p+1})g(z) = \mathbb{H}F_{p,m}(0) &+ \left(\mathbb{H}F_{p,m}(1) - m\mathbb{H}F_{p,m}(0)\right)z \\ &+ \left(\mathbb{H}F_{p,m}(2) - m\mathbb{H}F_{p,m}(1)\right)z^{2} \\ &+ \left(\mathbb{H}F_{p,m}(3) - m\mathbb{H}F_{p,m}(2)\right)z^{3} \\ &+ \cdots + \left(\mathbb{H}F_{p,m}(p) - m\mathbb{H}F_{p,m}(p-1)\right)z^{p} \\ &+ \left(\mathbb{H}F_{p,m}(p+1) - m\mathbb{H}F_{p,m}(p) - \mathbb{H}F_{p,m}(0)\right)z^{p+1} + \cdots\end{aligned}$$

Using (7), we have

$$g(z) = \sum_{n=0}^{\infty}\mathbb{H}F_{p,m}(n)z^{n} = \frac{\mathbb{H}F_{p,m}(0) + z + z^{p-1}\mathbf{h} + z^{p}(\epsilon + m\mathbf{h})}{1 - mz - z^{p+1}}.$$

Similarly, for $p > 1$, we can obtain the generating function for the generalized hybrid Lucas p-numbers as

$$h(z) = \frac{\left[\begin{array}{c} HL_{p,m}(0) - mpz + ((p+1)h)z^{p-2} + \\ ((p+1)\epsilon + mh)z^{p-1} + \left((p+1)i + m\epsilon + m^2 h\right)z^{p} \end{array}\right]}{1 - mz - z^{p+1}}.$$

□

Taking $m = p = 1$ in (9), we give the generating function of hybrid Fibonacci numbers as following corollary.

Corollary 1 *The generating function of hybrid Fibonacci numbers is*

$$g(z) = \frac{(\mathbf{i} + \epsilon + 2\mathbf{h}) + (1 + \epsilon + \mathbf{h})z}{1 - z - z^{2}}.$$

Theorem 2 *Let $F_{p,m}(n)$ and $\mathbb{H}F_{p,m}(n)$ be n^{th} generalized Fibonacci p-numbers and generalized hybrid Fibonacci p-numbers, respectively. Then*

$$\mathbb{H}F_{p,m}(k+n) = F_{p,m}(k)\,\mathbb{H}F_{p,m}(n+1) + \sum_{t=1}^{p}F_{p,m}(k-t)\,\mathbb{H}F_{p,m}(n-p+t) \tag{14}$$

where k and n are positive integers.

Proof From (5), we give

$$\mathbb{H}F_{p,m}(k+n) = F_{p,m}(k+n) + \mathbf{i}F_{p,m}(k+n+1) + \epsilon F_{p,m}(k+n+2) + \mathbf{h}F_{p,m}(k+n+3).$$

Using (3), we have

$$\begin{aligned}\mathbb{H}F_{p,m}(k+n) = & \left(F_{p,m}(k)F_{p,m}(n+1) + \sum_{t=1}^{p} F_{p,m}(k-t)F_{p,m}(n-p+t)\right) \\ & +\mathbf{i}\left(F_{p,m}(k)F_{p,m}(n+2) + \sum_{t=1}^{p} F_{p,m}(k-t)F_{p,m}(n-p+t+1)\right) \\ & +\epsilon\left(F_{p,m}(k)F_{p,m}(n+3) + \sum_{t=1}^{p} F_{p,m}(k-t)F_{p,m}(n-p+t+2)\right) \\ & +\mathbf{h}\left(F_{p,m}(k)F_{p,m}(n+4) + \sum_{t=1}^{p} F_{p,m}(k-t)F_{p,m}(n-p+t+3)\right)\end{aligned}$$

Then, we get

$$\begin{aligned}\mathbb{H}F_{p,m}(k+n) &= F_{p,m}(k)\left(F_{p,m}(n+1)+\mathbf{i}F_{p,m}(n+2)+\epsilon F_{p,m}(n+3)+\mathbf{h}F_{p,m}(n+4)\right) \\ &\quad + \sum_{t=1}^{p} F_{p,m}(k-t)\begin{pmatrix} F_{p,m}(n-p+t)+\mathbf{i}F_{p,m}(n-p+t+1)+ \\ \epsilon F_{p,m}(n-p+t+2)+\mathbf{h}F_{p,m}(n-p+t+3)\end{pmatrix} \\ &= F_{p,m}(k)\,\mathbb{H}F_{p,m}(n+1) + \sum_{t=1}^{p} F_{p,m}(k-t)\,\mathbb{H}F_{p,m}(n-p+t).\end{aligned}$$

□

Theorem 3 *Sum of the generalized hybrid Fibonacci and Lucas* p*-numbers are*

$$\sum_{n=0}^{k} \mathbb{H}F_{p,m}(n) = \frac{1}{m}\begin{pmatrix} \mathbb{H}F_{p,m}(k+p+1) - \mathbb{H}F_{p,m}(p)+ \\ (1-m)\sum_{n=0}^{p-1}\left[\mathbb{H}F_{p,m}(n+k+1) - \mathbb{H}F_{p,m}(n)\right]\end{pmatrix} \tag{15}$$

and

$$\sum_{n=0}^{k} \mathbb{H}L_{p,m}(n) = \frac{1}{m}\begin{pmatrix} \mathbb{H}L_{p,m}(k+p+1) - \mathbb{H}L_{p,m}(p)+ \\ (1-m)\sum_{n=0}^{p-1}\left[\mathbb{H}L_{p,m}(n+k+1) - \mathbb{H}L_{p,m}(n)\right]\end{pmatrix} \tag{16}$$

for $m \geq 1$.

Proof Using the recurrence relation of generalized hybrid Fibonacci p-numbers, we have

$$m\mathbb{H}F_{p,m}(n+p) = \mathbb{H}F_{p,m}(n+p+1) - \mathbb{H}F_{p,m}(n).$$

Then, we obtain

$$m\sum_{n=0}^{k}\mathbb{H}F_{p,m}(n+p) = \sum_{n=0}^{k}\mathbb{H}F_{p,m}(n+p+1) - \sum_{n=0}^{k}\mathbb{H}F_{p,m}(n). \tag{17}$$

Now, we consider LHS of (17)

$$\begin{aligned}m\sum_{n=0}^{k}\mathbb{H}F_{p,m}(n+p) &= m\sum_{n=0}^{k}\mathbb{H}F_{p,m}(n) + m\sum_{n=k+1}^{k+p}\mathbb{H}F_{p,m}(n) - m\sum_{n=0}^{p-1}\mathbb{H}F_{p,m}(n) \\ &= m\sum_{n=0}^{k}\mathbb{H}F_{p,m}(n) + m\sum_{n=0}^{p-1}\left(\mathbb{H}F_{p,m}(n+k+1) - \mathbb{H}F_{p,m}(n)\right).\end{aligned}$$

Also, RHS of (17) is

$$\sum_{n=0}^{k} \mathbb{H}F_{p,m}(n+p+1) - \sum_{n=0}^{k} \mathbb{H}F_{p,m}(n) = \mathbb{H}F_{p,m}(k+p+1) - \mathbb{H}F_{p,m}(p) + \sum_{n=0}^{p-1} \left(\mathbb{H}F_{p,m}(n+k+1) - \mathbb{H}F_{p,m}(n)\right).$$

From equality of LHS and RHS of (17), we have

$$\sum_{n=0}^{k} \mathbb{H}F_{p,m}(n) = \frac{1}{m}\left(\begin{array}{c} \mathbb{H}F_{p,m}(k+p+1) - \mathbb{H}F_{p,m}(p) + \\ (1-m)\sum_{n=0}^{p-1}\left[\mathbb{H}F_{p,m}(n+k+1) - \mathbb{H}F_{p,m}(n)\right] \end{array}\right).$$

Similarly, we can have summation formula for the generalized hybrid Lucas $p-$ numbers. □

We can give some special cases of this theorem as following corollaries.

Corollary 2 *Let $\mathbb{H}F_p(n)$ and $\mathbb{H}L_p(n)$ be n^{th} hybrid Fibonacci and Lucas p-numbers, respectively. Then*

$$\sum_{n=0}^{k} \mathbb{H}F_p(n) = \mathbb{H}F_p(k+p+1) - \mathbb{H}F_p(p)$$

and

$$\sum_{n=0}^{k} \mathbb{H}L_p(n) = \mathbb{H}L_p(k+p+1) - \mathbb{H}L_p(p).$$

Proof Taking $m = 1$ in (15) and (16), the result is clear. □

Corollary 3 *Let $\mathbb{H}P_n$ and $\mathbb{H}Q_n$ be n^{th} hybrid Pell and Pell-Lucas numbers, respectively. Then*

$$\sum_{n=0}^{k} \mathbb{H}P_n = \frac{1}{2}\left(\mathbb{H}P_{k+2} - \mathbb{H}P_{k+1} - \mathbb{H}P_1 + \mathbb{H}P_0\right)$$

and

$$\sum_{n=0}^{k} \mathbb{H}Q_n = \frac{1}{2}\left(\mathbb{H}Q_{k+2} - \mathbb{H}Q_{k+1} - \mathbb{H}Q_1 + \mathbb{H}Q_0\right).$$

Proof Taking $m = 2$ and $p = 1$ in (15) and (16), the result is clear. □

Similarly, we can obtain summation formulas for the hybrid Fibonacci and Lucas numbers ($m = p = 1$), hybrid k-Fibonacci and k-Lucas numbers ($m = k$ and $p = 1$) and hybrid Pell and Pell-Lucas p-numbers ($m = 2$).

Theorem 4 *Let $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ be n^{th} generalized hybrid Fibonacci and Lucas p-numbers, respectively. Then*

$$\mathbb{H}L_{p,m}(n) = \mathbb{H}F_{p,m}(n+1) + p\mathbb{H}F_{p,m}(n-p).$$

Proof From (6), we have

$$\mathbb{H}L_{p,m}(n) = L_{p,m}(n) + \mathbf{i}L_{p,m}(n+1) + \epsilon L_{p,m}(n+2) + \mathbf{h}L_{p,m}(n+3).$$

Using (4), we obtain

$$\mathbb{H}L_{p,m}(n) = \mathbb{H}F_{p,m}(n+1) + p\mathbb{H}F_{p,m}(n-p).$$

□

Corollary 4 *Let $\mathbb{H}F_{p,m}(n)$ and $\mathbb{H}L_{p,m}(n)$ be n^{th} generalized hybrid Fibonacci and Lucas p-numbers, respectively. Then*

$$\mathbb{H}L_{p,m}(n) + mp\mathbb{H}F_{p,m}(n) = (p+1)\,\mathbb{H}F_{p,m}(n+1).$$

The matrix representation of the bivariate Fibonacci p-polynomials is introduced in [18]. For $p = 0, 1, 2, 3, \ldots$, the matrix representation of the generalized Fibonacci p-numbers is

$$Q_p = \begin{pmatrix} m & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(p+1)\times(p+1)}.$$

For $p = 1$ and $m = k$, we have the matrix representation of the k-Fibonacci numbers (see [3]). Similarly, for $m = 1$, Q_p is the matrix representation of the Fibonacci p-numbers (see [12]). Using the Q_p matrix, we derive the matrix representation of the generalized hybrid Fibonacci p-numbers as

$$\mathbf{F} = \begin{pmatrix} \mathbb{H}F_{p,m}(p+1) & \mathbb{H}F_{p,m}(p) & \cdots & \mathbb{H}F_{p,m}(1) \\ \mathbb{H}F_{p,m}(p) & \mathbb{H}F_{p,m}(p-1) & \cdots & \mathbb{H}F_{p,m}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{H}F_{p,m}(2) & \mathbb{H}F_{p,m}(1) & \cdots & \mathbb{H}F_{p,m}(2-p) \\ \mathbb{H}F_{p,m}(1) & \mathbb{H}F_{p,m}(0) & \cdots & \mathbb{H}F_{p,m}(1-p) \end{pmatrix} Q_p^n \tag{18}$$

where the matrix $\mathbf{F}$ is

$$\mathbf{F} = \begin{pmatrix} \mathbb{H}F_{p,m}(n+p+1) & \mathbb{H}F_{p,m}(n+p) & \cdots & \mathbb{H}F_{p,m}(n+1) \\ \mathbb{H}F_{p,m}(n+p) & \mathbb{H}F_{p,m}(n+p-1) & \cdots & \mathbb{H}F_{p,m}(n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{H}F_{p,m}(n+2) & \mathbb{H}F_{p,m}(n+1) & \cdots & \mathbb{H}F_{p,m}(n-p+2) \\ \mathbb{H}F_{p,m}(n+1) & \mathbb{H}F_{p,m}(n) & \cdots & \mathbb{H}F_{p,m}(n-p+1) \end{pmatrix}.$$

Using the induction method on n, we can see easily (18).

Theorem 5 *(Generalized Cassini Formula) Let $\mathbb{H}F_{p,m}(n)$ be n^{th} generalized hybrid Fibonacci p-numbers, then*

$$|\mathbf{F}| = (-1)^{pn} \begin{vmatrix} \mathbb{H}F_{p,m}(p+1) & \mathbb{H}F_{p,m}(p) & \cdots & \mathbb{H}F_{p,m}(1) \\ \mathbb{H}F_{p,m}(p) & \mathbb{H}F_{p,m}(p-1) & \cdots & \mathbb{H}F_{p,m}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{H}F_{p,m}(2) & \mathbb{H}F_{p,m}(1) & \cdots & \mathbb{H}F_{p,m}(2-p) \\ \mathbb{H}F_{p,m}(1) & \mathbb{H}F_{p,m}(0) & \cdots & \mathbb{H}F_{p,m}(1-p) \end{vmatrix}.$$

It's essential to note that, because of the multiplication operation of the hybrid numbers is noncommutative, if one expands the determinant according to different row or column, the results may be different. Therefore, the determinant is expanded using the first row when finding the results in this article.

Considering the generalized Cassini formula, we have Cassini formulas for hybrid Fibonacci p-numbers, hybrid Pell p-numbers, hybrid Fibonacci, hybrid k-Fibonacci and hybrid Pell numbers. For example; taking $m = p = 1$, we have Cassini formula for the hybrid Fibonacci numbers as follows.

Corollary 5 *Let $\mathbb{H}F_n$ be n^{th} hybrid Fibonacci numbers. Cassini identity for the hybrid Fibonacci numbers is*

$$\mathbb{H}F_{n+2}\mathbb{H}F_n - \mathbb{H}F_{n+1}^2 = (-1)^n \left(\mathbb{H}F_2\mathbb{H}F_0 - \mathbb{H}F_1^2\right).$$

For $m = 1$ and $p = 2$, we can give Cassini formula for the hybrid Fibonacci 2-numbers as following corollary.

Corollary 6 *Let $\mathbb{H}F_{2,1}(n)$ be n^{th} hybrid Fibonacci 2-numbers. Generalized Cassini formula for the hybrid Fibonacci 2-numbers is*

$$\begin{vmatrix} \mathbb{H}F_{2,1}(n+3) & \mathbb{H}F_{2,1}(n+2) & \mathbb{H}F_{2,1}(n+1) \\ \mathbb{H}F_{2,1}(n+2) & \mathbb{H}F_{2,1}(n+1) & \mathbb{H}F_{2,1}(n) \\ \mathbb{H}F_{2,1}(n+1) & \mathbb{H}F_{2,1}(n) & \mathbb{H}F_{2,1}(n-1) \end{vmatrix} = \begin{vmatrix} \mathbb{H}F_{2,1}(3) & \mathbb{H}F_{2,1}(2) & \mathbb{H}F_{2,1}(1) \\ \mathbb{H}F_{2,1}(2) & \mathbb{H}F_{2,1}(1) & \mathbb{H}F_{2,1}(0) \\ \mathbb{H}F_{2,1}(1) & \mathbb{H}F_{2,1}(0) & \mathbb{H}F_{2,1}(-1) \end{vmatrix}.$$

3 Conclusion

In present article, we have introduced generalized hybrid Fibonacci and Lucas p-numbers which are defined via m-extension of Fibonacci and Lucas p-numbers. We have derived several interesting properties of generalized hybrid Fibonacci and Lucas p-numbers. According to the special cases of m and p, all the results which are given in our research are reducible to all hybrid numbers mentioned in this paper.

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