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ORIGINAL RESEARCH

Hyperbolic *k*-Jacobsthal and *k*-Jacobsthal-Lucas Quaternions

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Abstract In this paper, we introduce the hyperbolic $k-$Jacobsthal and $k-$Jacobsthal-Lucas quaternions. We present generating functions, Binet formula, Catalan's identity, Vajda's identity etc. for the hyperbolic k-Jacobsthal and $k-$Jacobsthal-Lucas quaternions.

Keywords Hyperbolic $k-$Jacobsthal quaternions · Hyperbolic $k-$Jacobsthal-Lucas quaternions · Binet formula · Cassini identity · Catalan identity

1 Introduction

The Fibonacci sequence is a renowned integer sequence. This sequence has excited mathematicians for a long time. The Fibonacci sequence has been applied in diverse fields such as Architecture, Engineering, Nature and Art, Physics, Computer science and so on. The generalized Fibonacci sequence is a generalization of the Fibonacci sequence which is obtained from the Fibonacci sequence by changing recurrence relation, initial condition or both of them. Many authors have studied generalized Fibonacci sequences of second order. The well-known examples of such sequences are the Lucas sequence, k-Fibonacci sequence, k-Lucas sequence, Jacobsthal sequence, Pell sequence, Pell-Lucas sequence, etc. Many authors also studied Fibonacci sequences of a higher order such as Tribonacci sequence, Tetranacci sequence, Pentanacci sequence, etc.

The Jacobsthal numbers J_n are defined by the recurrence relation

$$J_{n+2} = J_{n+1} + 2J_n, \text{ for } n \geq 0$$

with the initial conditions $J_0 = 0$ and $J_1 = 1$.

Binet Formula for Jacobsthal numbers J_n is given by

$$J_n = \frac{a^n - b^n}{a - b}$$

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where $a = 2$ and $b = 1$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

Similarly, the Jacobsthal-Lucas numbers j_n are given by the recurrence relation

$$j_{n+2} = j_{n+1} + 2j_n, \text{ for } n \geq 0$$

with the initial conditions $j_0 = 2$ and $j_1 = 1$.

Binet formula of Jacobsthal-Lucas numbers j_n

$$j_n = a + b$$

with $a = 2$ and $b = 1$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

Uygun and Eldoğan [2] defined the $k-$ Jacobsthal numbers by the recurrence relation

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \text{ for } n \geq 2$$

with the initial conditions $J_{k,0} = 0$ and $J_{k,1} = 1$.

Binet Formula of $k-$Jacobsthal numbers J_n is gıven by

$$J_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where

$$\alpha = \frac{k + \sqrt{k^2 + 8}}{2}$$

and

$$\beta = \frac{k - \sqrt{k^2 + 8}}{2}$$

are the roots of the characteristic equation $x^2 - kx - 2 = 0$.

Uygun and Eldoğan [2] defined $k-$ Jacobsthal-Lucas numbers by the recurrence relation

$$j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}, \text{ for } n \geq 2$$

with the initial conditions $j_{k,0} = 2$ and $j_{k,1} = k$.

Binet Formula of $k-$ Jacobsthal-Lucas numbers $j_{k,n}$ is given by

$$j_{k,n} = \alpha^n + \beta^n$$

where

$$\alpha = \frac{k + \sqrt{k^2 + 8}}{2}$$

and

$$\beta = \frac{k - \sqrt{k^2 + 8}}{2}$$

are the roots of the characteristic equation $x^2 - kx - 2 = 0$.

The negative $k-$ Jacobsthal-Lucas numbers are defined by the relation in [5,6]

$$J_{k,-n} = (-1)^{n-1} 2^{-n} J_{k,n}.$$

In the recent years, these types of sequences have become an interesting topic of research in Mathematics, see for example [13,14], and the references cited therein.

The quaternion is an algebraic structure, first originated in the year 1843 by Irish mathematician William Rowan Hamilton [3]. He proved that the set of quaternions form a 4-dimensional real vector space with a multiplicative operation.

In the year 1963, Horodam [12] defined the n^{th} Fibonacci and n^{th} Lucas quaternions and studied various properties of these quaternions.

Table 1 Multiplication rule for hyperbolic quaternion units

.	i_1	i_2	i_3	i_4
i_1	1	i_2	i_3	i_4
i_2	i_2	1	i_4	$-i_3$
i_3	i_3	$-i_4$	1	i_2
i_4	i_4	i_3	$-i_2$	1

Table 2 Some initial terms of the hyperbolic k−Jacobsthal quaternions

n	$\bar{\mathcal{H}}J_{k,n}$
0	$i_2 + ki_3 + (k^2+2)i_4$
1	$i_1 + ki_2 + (k^2+2)i_3 + (k^3+4k)i_4$
2	$ki_1 + (k^2+2)i_2 + (k^3+4k)i_3 + (k^4+6k^2+4)i_4$
3	$(k^2+2)i_1 + (k^3+4k)i_2 + \left(k^4+6k^2+4\right) i_3 + (k^5+8k^3+12)i_4$
4	$(k^3+4k)i_1 + \left(k^4+6k^2+4\right) i_2 + \left(k^5+8k^3+12\right) i_3 + (k^6+10k^4+24k^2+8)i_4$

In the year 2016, Szynal-Liana and Włoch [4] defined the n^{th} Jacobsthal and n^{th} Jacobsthal-Lucas quaternions as

$$JQ_n = J_n + J_{n+1}i_1 + J_{n+2}i_2 + J_{n+3}i_3$$
$$JLQ_n = j_n + j_{n+1}i_1 + j_{n+2}i_2 + j_{n+3}i_3$$

In the year 2017, Taşçı[7] defined n^{th} $k-$ Jacobsthal and n^{th} $k-$ Jacobsthal-Lucas quaternions by

$$QJ_{k,n} = J_{k,n} + J_{k,n+1}i_1 + J_{k,n+2}i_2 + J_{k,n+3}i_3$$
$$Qj_{k,n} = j_{k,n} + j_{k,n+1}i_1 + j_{k,n+2}i_2 + j_{k,n+3}i_3$$

where $\mathrm{J_{k,n}}$ is the n^{th} $k-$Jacobsthal sequence and $\mathrm{j_{k,n}}$ is the n^{th} $k-$Jacobsthal-Lucas sequence.

In the year 1900, Macfarlane [8] discovered the hyperbolic quaternions and investigated their properties.

A hyperbolic quaternion h is an expression of the form

$$h = h_1i_1 + h_2i_2 + h_3i_3 + h_4i_4 = (h_1, h_2, h_3, h_4)$$

with real components h_1, h_2, h_3, h_4 and i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the non-commutative multiplication rules as in Table 1.

Godase [9-11] established the hyperbolic $k-$Fibonacci and hyperbolic $k-$Lucas quaternions and studied various properties of these quaternions.

We are influenced from the idea of Fibonacci quaternion and Lucas quaternion and motivated to establish the hyperbolic $k-$Jacobsthal and $k-$ Jacobsthal-Lucas Quaternions.

2 Hyperbolic $k-$ Jacobsthal and $k-$ Jacobsthal-Lucas Quaternions

Definition 1.1 The hyperbolic $k-$ Jacobsthal and $k-$ Jacobsthal -Lucas quaternions $\bar{\mathcal{H}}J_{k,n}$ *and* $\bar{\mathcal{H}}j_{k,n}$ for $n \geq 0$ are defined by

$$\bar{\mathcal{H}}J_{k,n} = J_{k,n}i_1 + J_{k,n+1}i_2 + J_{k,n+2}i_3 + J_{k,n+3}i_4 = (J_{k,n}, J_{k,n+1}, J_{k,n+2}, J_{k,n+3}) \tag{2.1}$$

and

$$\bar{\mathcal{H}}j_{k,n} = j_{k,n}i_1 + j_{k,n+1}i_2 + j_{k,n+2}i_3 + j_{k,n+3}i_4 = (j_{k,n}, j_{k,n+1}, j_{k,n+2}, j_{k,n+3}) \tag{2.2}$$

respectively, where $J_{k,n}$ is n^{th} $k-$Jacobsthal sequence and $j_{k,n}$ is n^{th} $k-$Jacobsthal-Lucas sequence. Here, i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the multiplication rule as in Table 1.

Table 3 Some initial terms of the hyperbolic k −Jacobsthal-Lucas quaternions

n	$\bar{\mathcal{H}}j_{k,n}$
0	$2i_1 + ki_2 + (k^2+4)i_3 + (k^3+6k)i_4$
1	$ki_1 + (k^2+4)i_2 + (k^3+6k)i_3 + (k^4+8k^2+8)i_4$
2	$(k^2+4)i_1 + (k^3+6k)i_2 + (k^4+8k^2+8)i_3 + (k^5+10k^3+20k)i_4$
3	$(k^3+6k)i_1 + (k^4+8k^2+8)i_2 + \left(k^5+10k^3+20k\right)i_3 + (k^6+12k^4+36k^2+16)i_4$
4	$(k^4+8k^2+8)i_1 + \left(k^5+10k^3+20k\right)i_2 + \left(k^6+12k^4+36k^2+16\right)i_3 + (k^7+14k^5+56k^3+56k)i_4$

3 Properties of Hyperbolic k −Jacobsthal and k−Jacobsthal- Lucas Quaternions

In this section, we investigate some identities of the hyperbolic k −Jacobsthal and k −Jacobsthal-Lucas quaternions.

Theorem 3.1 *For $n \geq 0$, the k − Jacobsthal and k − Jacobsthal-Lucas quaternions satisfy the following recurrence relations*

(i) $\bar{\mathcal{H}}J_{k,n+2} = k\bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n}$,
(ii) $\bar{\mathcal{H}}j_{k,n+2} = k\bar{\mathcal{H}}j_{k,n+1} + 2\bar{\mathcal{H}}j_{k,n}$,
(iii) $\bar{\mathcal{H}}j_{k,n} = \bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n-1}$.

Proof i) From equation (2.1) and recurrence relation of k−Jacobsthal numbers, we can write

$$\begin{aligned} k\bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n} &= k\left(J_{k,n+1}i_1 + J_{k,n+2}i_2 + J_{k,n+3}i_3 + J_{k,n+4}i_4\right) \\ &\quad +2(J_{k,n}i_1 + J_{k,n+1}i_2 + J_{k,n+2}i_3 + J_{k,n+3}i_4) \\ &= \left(kJ_{k,n+1} + 2J_{k,n}\right)i_1 + \left(kJ_{k,n+2} + 2J_{k,n+1}\right)i_2 + \left(kJ_{k,n+3} + 2J_{k,n+2}\right)i_3 \\ &\quad + \left(kJ_{k,n+4} + 22J_{k,n+2}\right)i_4 \\ &= J_{k,n+2}i_1 + J_{k,n+3}i_2 + J_{k,n+4}i_3 + J_{k,n+5}i_4 \\ &= \bar{\mathcal{H}}J_{k,n+2} \end{aligned}$$

The proofs of (ii) and (iii) are similar to (i), using equations (2.1) and (2.2). □

Theorem 3.2 (Binet Formula) *For $n \geq 0$, prove that*

$$(i)\ \bar{\mathcal{H}}J_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}, \tag{3.1}$$

$$(ii)\ \bar{\mathcal{H}}j_{k,n} = \bar{\alpha}\alpha^n + \bar{\beta}\beta^n. \tag{3.2}$$

where

$$\begin{aligned} \bar{\alpha} &= i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = (1, \alpha, \alpha^2, \alpha^3), \\ \bar{\beta} &= i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = (1, \beta, \beta^2, \beta^3) \end{aligned}$$

and i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the multiplication rule as in Table 1.

Proof (i) We prove Binet formula of the hyperbolic k −Jacobsthal quaternions.

For this, the characteristic equation of recurrence relation $\bar{\mathcal{H}}J_{k,n+2} = k\bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n}$ is,

$$r^2 - kr - 2 = 0.$$

The roots of the characteristic equation are

$$\alpha = \frac{k + \sqrt{k^2+8}}{2} \ \text{and}\ \beta = \frac{k - \sqrt{k^2+8}}{2}$$

where $\alpha + \beta = k,\ \alpha\beta = -2$ and $\alpha - \beta = \sqrt{k^2+8}$.

The Binet formula of the hyperbolic k− Jacobsthal quaternions is

$$\bar{\mathcal{H}}J_{k,n} = A\alpha^n + B\beta^n.$$

By using the recurrence relations and initial conditions, we have

$$\bar{\mathcal{H}}J_{k,0} = \left(i_2 + ki_3 + \left(k^2+2\right)i_4\right) = (0, 1, k, \left(k^2+2\right))$$

and

$$\bar{\mathcal{H}}J_{k,1} = \left(i_1 + ki_2 + \left(k^2+2\right)i_3 + (k^3+4k)i_4\right) = (1, k, \left(k^2+2\right), (k^3+4k)).$$

Next, to find the coefficients A and B, we can write for $n=0$, $\bar{\mathcal{H}}J_{k,0} = A+B$ and for $n=1$, $\bar{\mathcal{H}}J_{k,1} = A\alpha + B\beta$.

Thus, we obtain

$$A = \frac{\bar{\mathcal{H}}J_{k,1} - \beta\bar{\mathcal{H}}J_{k,0}}{\alpha-\beta} = \frac{i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4}{\alpha-\beta},$$
$$B = \frac{\bar{\mathcal{H}}J_{k,1} - \alpha\bar{\mathcal{H}}J_{k,0}}{\beta-\alpha} = \frac{i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4}{\beta-\alpha}.$$

where $\left(i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4\right) = \left(1, \alpha, \alpha^2, \alpha^3\right) = \bar{\alpha}$ and $(i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4) = \left(1, \beta, \beta^2, \beta^3\right) = \bar{\beta}$.

Hence, the Binet formula of the hyperbolic $k-$ Jacobsthal quaternions is

$$\bar{\mathcal{H}}J_{k,n} = \frac{\bar{\alpha}\alpha}{\alpha-\beta} - \frac{\bar{\beta}\beta}{\alpha-\beta}.$$

(ii) Now, we prove Binet formula of the hyperbolic $k-$Jacobsthal-Lucas quaternions.

The characteristic equation of recurrence relation $\bar{\mathcal{H}}j_{k,n+2} = k\bar{\mathcal{H}}j_{k,n+1} + 2\bar{\mathcal{H}}j_{k,n}$ is

$$t^2 - kt - 2 = 0.$$

The roots of the characteristic equation are

$$\alpha = \frac{k+\sqrt{k^2+8}}{2} \ and \ \beta = \frac{k-\sqrt{k^2+8}}{2}.$$

where $\alpha+\beta = k$, $\alpha\beta = -2$ and $\alpha-\beta = \sqrt{k^2+8} = \sqrt{\rho}$.

The Binet formula of the hyperbolic $k-$ Jacobsthal quaternions is

$$\bar{\mathcal{H}}j_{k,n} = C\alpha^n + D\beta^n.$$

Now, using the recurrence relation and initial conditions, we get

$$\bar{\mathcal{H}}j_{k,0} = \left(2i_1 + ki_2 + \left(k^2+4\right)i_3 + (k^3+6k)i_4\right) = (2, k, \left(k^2+4\right), (k^3+6k))$$

and

$$\bar{\mathcal{H}}j_{k,1} = \left(ki_1 + \left(k^2+4\right)i_2 + \left(k^3+6k\right)i_3 + \left(k^4+8k^2+8\right)i_4\right) = (k, \left(k^2+4\right), \left(k^3+6k\right), (k^4+8k^2+8)).$$

Next, we find the coefficients C and D, for $n=0$,

$$\bar{\mathcal{H}}j_{k,0} = A + B = \left(2i_1 + ki_2 + \left(k^2+4\right)i_3 + (k^3+6k)i_4\right)$$

and

for $n=1$,

$$\bar{\mathcal{H}}J_{k,1} = A\alpha + B\beta = \left(ki_1 + \left(k^2+4\right)i_2 + \left(k^3+6k\right)i_3 + (k^4+8k^2+8)i_4\right).$$

Thus, we obtain

$$A = \frac{\bar{\mathcal{H}}J_{k,1} - \beta\bar{\mathcal{H}}J_{k,0}}{\alpha - \beta} = (i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4),$$
$$B = \frac{\bar{\mathcal{H}}J_{k,1} - \alpha\bar{\mathcal{H}}J_{k,0}}{\beta - \alpha} = (i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4).$$

where

$$\left(i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4\right) = \left(1, \alpha, \alpha^2, \alpha^3\right) = \bar{\alpha}$$

and

$$(i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4) = \left(1, \beta, \beta^2, \beta^3\right) = \bar{\beta}.$$

Therefore, the Binet formula of the hyperbolic k −Jacobsthal quaternions is

$$\bar{\mathcal{H}}j_{k,n} = \bar{\alpha}\alpha^n + \bar{\beta}\beta^n.$$

□

Lemma 3.3 *For* $\bar{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = (1, \alpha, \alpha^2, \alpha^3)$ *and* $\bar{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = (1, \beta, \beta^2, \beta^3)$, *we have*

(i) $\bar{\alpha} - \bar{\beta} = \sqrt{\rho}\bar{\mathcal{H}}J_{k,0}$,
(ii) $\bar{\alpha} + \bar{\beta} = \bar{\mathcal{H}}j_{k,0}$,
(iii)
$$\begin{aligned}\bar{\alpha}\bar{\beta} &= -5 + (5\beta - 3\alpha)\, i_2 + \left(3\beta^2 - \alpha^2\right) i_3 + \left(\alpha^3 + \beta^3 - 2\beta + 2\alpha\right) i_4 \\ &= (-5, (5\beta - 3\alpha)\,, \left(3\beta^2 - \alpha^2\right), \left(\alpha^3 + \beta^3 - 2\beta + 2\alpha\right)),\end{aligned}$$
(iv)
$$\begin{aligned}\bar{\beta}\bar{\alpha} &= -5 + (5\alpha - 3\beta)\, i_2 + \left(3\alpha^2 - \beta^2\right) i_3 + \left(\alpha^3 + \beta^3 - 2\alpha + 2\beta\right) i_4 \\ &= (-5, (5\alpha - 3\beta)\,, \left(3\alpha^2 - \beta^2\right), \left(\alpha^3 + \beta^3 - 2\alpha + 2\beta\right)),\end{aligned}$$
(v) $\bar{\alpha}^2 = 2\bar{\alpha} - 1 + \alpha^2 + \alpha^4 + \alpha^6$,
(vi) $\bar{\beta}^2 = 2\bar{\beta} - 1 + \beta^2 + \beta^4 + \beta^6$,
(vii) $\bar{\alpha}\bar{\beta} + \bar{\beta}\bar{\alpha} = 2(-7 + \bar{\mathcal{H}}j_{k,0})$,
(viii) $\bar{\alpha}\bar{\beta} - \bar{\beta}\bar{\alpha} = -4\sqrt{\rho}\,(2i_2 + ki_3 + i_4) = -4\sqrt{\rho}(0, 2, k, 1)$,
(viiii) $\bar{\alpha}^2 - \bar{\beta}^2 = \sqrt{\rho}(2\bar{\mathcal{H}}J_{k,0} + J_{k,2} + J_{k,4} + J_{k,6})$,
(ix) $\bar{\alpha}^2 - \bar{\beta}^2 = \sqrt{\rho}(2\bar{\mathcal{H}}J_{k,0} + J_{k,2} + J_{k,4} + J_{k,6})$,
(x) $\bar{\alpha}^2 + \bar{\beta}^2 = 2\bar{\mathcal{H}}j_{k,0} - j_{k,0} + j_{k,2} + j_{k,4} + j_{k,6}$.

Proof (iii) Consider

$$\begin{aligned}\bar{\alpha}\bar{\beta} &= \left(i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4\right)(i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4) \\ &= 1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 + \alpha i_2 i_1 + \alpha\beta + \alpha\beta^2 i_2 i_3 \\ &\quad + \alpha\beta^3 i_2 i_4 + \alpha^2 i_3 + \alpha^2\beta i_3 i_2 + \alpha^2\beta^2 + \alpha^2\beta^3 i_3 i_4 \\ &\quad + \alpha^3 i_4 + \alpha^3\beta i_4 i_2 + \alpha^3\beta^2 i_4 i_3 + \alpha^3\beta^3 \\ &= -5 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 + \alpha i_2 + \alpha\beta^2 i_4 - \alpha\beta^3 i_3 + \alpha^2 i_3 - \alpha^2\beta i_4 \\ &\quad + \alpha^2\beta^3 i_2 + \alpha^3 i_4 + \alpha^3\beta i_3 \\ &\quad - \alpha^3\beta^2 i_2 \\ &= -5 + \left(\beta + \alpha + \alpha^2\beta^3 - \alpha^3\beta^2\right) i_2 + \left(\beta^2 - \alpha\beta^3 + \alpha^3\beta + \alpha^2\right) i_3 \\ &\quad + (\beta^3 + \alpha\beta^2 - \alpha^2\beta + \alpha^3) i_4.\end{aligned}$$

Thus, we get

$$\begin{aligned}\bar{\alpha}\bar{\beta} &= -5 + (5\beta - 3\alpha)\, i_2 + \left(3\beta^2 - \alpha^2\right) i_3 + \left(\alpha^3 + \beta^3 - 2\beta + 2\alpha\right) i_4 \\ &= (-5, (5\beta - 3\alpha)\,, \left(3\beta^2 - \alpha^2\right), \left(\alpha^3 + \beta^3 - 2\beta + 2\alpha\right)).\end{aligned}$$

The proof of (i) , (ii) , (iv) , (x) are similar to (iii). So, we omit them. □

Theorem 3.4 *For all $t \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, the generating functions for the hyperbolic $k-$ Jacobsthal quaternions and the hyperbolic* k− *Jacobsthal -Lucas quaternions are given by*

$$\begin{aligned}(i)\ \ GJ\,(t) &= \sum_{n=0}^{\infty} \bar{\mathcal{H}}J_{k,n}t^n = \frac{\bar{\mathcal{H}}J_{k,0}\,(1-kt) + \bar{\mathcal{H}}J_{k,1}t}{1-kt-2t^2} = \frac{\bar{\mathcal{H}}J_{k,0} + t\,(i_1 + 2i_3 + 2ki_4)}{1-kt-2t^2}, \\ (ii)\ \ Gj\,(t) &= \sum_{n=0}^{\infty} \bar{\mathcal{H}}j_{k,n}t^n = \frac{\bar{\mathcal{H}}j_{k,0}\,(1-kt) + \bar{\mathcal{H}}j_{k,1}t}{1-kt-2t^2} \\ &= \frac{\bar{\mathcal{H}}j_{k,0} + t\,(-ki_1 + 4i_2 + 2ki_3 + 2k^2i_4)}{1-kt-2t^2}.\end{aligned}$$

Proof (i) The power series representation of the generating function of $\bar{\mathcal{H}}J_{k,n}$ is

$$GJ\,(t) = \bar{\mathcal{H}}J_{k,0} + \bar{\mathcal{H}}J_{k,1}t + \bar{\mathcal{H}}J_{k,2}t^2 + \cdots + \bar{\mathcal{H}}J_{k,n}t^n + \cdots \tag{3.3}$$

Now, multiplying the equation (3.3) by *1, kt* and *2t*^{2} respectively, we get

$$\begin{aligned}GJ\,(t) &= \bar{\mathcal{H}}J_{k,0} + \bar{\mathcal{H}}J_{k,1}t + \bar{\mathcal{H}}J_{k,2}t^2 + \cdots + \bar{\mathcal{H}}J_{k,n}t^n + \cdots \\ ktGJ\,(t) &= \bar{\mathcal{H}}J_{k,0}kt + \bar{\mathcal{H}}J_{k,1}kt^2 + \bar{\mathcal{H}}J_{k,2}kt^3 + \cdots \\ 2t^2GJ\,(t) &= \bar{\mathcal{H}}J_{k,0}2t^2 + \bar{\mathcal{H}}J_{k,1}2t^3 + \bar{\mathcal{H}}J_{k,2}2t^4 + \cdots\end{aligned}$$

After necessary calculations and using the recurrence relation, we obtain

$$\begin{aligned}GJ\,(t)\left(1-kt-2t^2\right) &= \bar{\mathcal{H}}J_{k,0} + \bar{\mathcal{H}}J_{k,1}t - \bar{\mathcal{H}}J_{k,0}kt = \bar{\mathcal{H}}J_{k,0}\,(1-kt) + \bar{\mathcal{H}}J_{k,1}t, \\ GJ\,(t) &= \frac{\bar{\mathcal{H}}J_{k,0} + t\,(i_1 + 2i_3 + 2ki_4)}{1-kt-2t^2}.\end{aligned}$$

The proof of (ii) is similar to (i). So, we omit it. □

Theorem 3.5 *Let $S\bar{\mathcal{H}}J_{k,n}\,(x)$ and $S\bar{\mathcal{H}}j_{k,n}\,(x)$ be sum of the hyperbolic $k-$Jacobsthal quaternions and the hyperbolic $k-$Jacobsthal-Lucas, respectively. Then we have*

(i) $S\bar{\mathcal{H}}J_{k,n} = \sum_{s=0}^{n} \bar{\mathcal{H}}J_{k,s} = \frac{1}{k+1}\left[\bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n} - \bar{\mathcal{H}}J_{k,0} - i_1 - 2i_3 - 2i_4\right]$,

(ii) $S\bar{\mathcal{H}}j_{k,n} = \sum\limits_{s=0}^{n} \bar{\mathcal{H}}j_{k,s} = \frac{1}{k+1}\left[\bar{\mathcal{H}}j_{k,n+2} - \bar{\mathcal{H}}j_{k,0} + ki_1 - 2i_3 - 2k^2i_4\right]$.

Proof (i) From the definition of hyperbolic $k-$Jacobsthal quaternions, we have

$$S\bar{\mathcal{H}}J_{k,n} = \sum_{s=0}^{n} \bar{\mathcal{H}}J_{k,s} = i_1 \sum_{s=0}^{n} J_{k,s} + i_2 \sum_{s=0}^{n} J_{k,s+1} + i_3 \sum_{s=0}^{n} J_{k,s+2} + i_4 \sum_{s=0}^{n} J_{k,s+3}.$$

Since $\sum_{i=1}^{n} J_{k,i} = \frac{1}{k+1}(J_{k,n+1} + 2J_{k,n} - 1)$, we get

$$\begin{aligned}
S\bar{\mathcal{H}}J_{k,n} &= \frac{1}{k+1}\left[\left(J_{k,n+1} + 2J_{k,n} - 1\right) + i_2\left(J_{k,n+2} + 2J_{k,n+1} - 1\right) i_1\right.\\
&\quad \left. + \left(i_3\left(J_{k,n+3} + 2J_{k,n+2} - 1\right) - 1\right) + i_4\left(\left(J_{k,n+4} + 2J_{k,n+3} - 1\right) - (k+1)\right)\right]\\
&= \frac{1}{k+1}\left[J_{k,n+1}i_1 + J_{k,n+2}i_2 + J_{k,n+3}i_3 + J_{k,n+4}i_4\right.\\
&\quad \left. +2\left(J_{k,n}i_1 + J_{k,n+1}i_2 + J_{k,n+2}i_3 + J_{k,n+3}i_4\right) - i_1 - i_2 - i_3 - i_4 - (k+1)i_4\right]\\
&= \frac{1}{k+1}\left[\bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n} - i_1 - i_2 - (k+2)\, i_3 - (k^2+2k+2)i_4\right]\\
&= \frac{1}{k+1}\left[\bar{\mathcal{H}}J_{k,n+1} + 2\bar{\mathcal{H}}J_{k,n} - \bar{\mathcal{H}}J_{k,0} - i_1 - 2i_3 - 2i_4\right]
\end{aligned}$$

This completes the proof of (i).

The proof of (ii) is analogous to (i). So, we omit the proof.

Theorem 3.6 (Cassini Identity) *For all $n \geq 1$, show that*

(i) $\bar{\mathcal{H}}J_{k,n-1}\bar{\mathcal{H}}J_{k,n+1} - \bar{\mathcal{H}}J_{k,n}^2 = (-2)^{n-1}\left(5J_{k,1},\ -5J_{k,2},\ 2-3J_{k,3},\ -J_{k,4}\right)$,

(ii) $\bar{\mathcal{H}}j_{k,n-1}\bar{\mathcal{H}}j_{k,n+1} - \bar{\mathcal{H}}j_{k,n}^2 = (-2)^{n-1}\,\rho(-\frac{5}{2}j_{k,0},\ 5j_{k,1},\ 3j_{k,2}-8,\ j_{k,3}-2j_{k,1})$.

Proof (i) Using the Binet formula, we can write

$$\begin{aligned}
\bar{\mathcal{H}}J_{k,n-1}\bar{\mathcal{H}}J_{k,n+1} - \bar{\mathcal{H}}J_{k,n}^2 &= \left[\frac{\bar{\alpha}\alpha^{n-1} - \bar{\beta}\beta^{n-1}}{\alpha-\beta}\right]\left[\frac{\bar{\alpha}\alpha^{n+1} - \bar{\beta}\beta^{n+1}}{\alpha-\beta}\right] - \left[\frac{\bar{\alpha}\alpha^{n} - \bar{\beta}\beta^{n}}{\alpha-\beta}\right]^2\\
&= \left[\frac{-\bar{\alpha}\bar{\beta}\alpha^{n-1}\beta^{n+1} - \bar{\beta}\bar{\alpha}\beta^{n-1}\alpha^{n+1} + \bar{\alpha}\bar{\beta}\alpha^{n}\beta^{n} + \bar{\beta}\bar{\alpha}\beta^{n}\alpha^{n}}{(\alpha-\beta)^{\mathbf{2}}}\right]\\
&= \frac{\bar{\alpha}\bar{\beta}\alpha^n\beta^n\left[1-\frac{\beta}{\alpha}\right] + \bar{\beta}\bar{\alpha}\alpha^n\beta^n\left[1-\frac{\alpha}{\beta}\right]}{(\alpha-\beta)^{\mathbf{2}}}\\
&= \frac{\bar{\alpha}\bar{\beta}(-2)^n\left(\frac{\alpha-\beta}{\alpha}\right) + \bar{\beta}\bar{\alpha}(-2)^n\left(\frac{\beta-\alpha}{\beta}\right)}{(\alpha-\beta)^{\mathbf{2}}}\\
&= \frac{(-2)^n(\alpha-\beta)}{(\alpha-\beta)^{\mathbf{2}}}\left(\frac{\bar{\alpha}\bar{\beta}}{\beta} - \frac{\bar{\beta}\bar{\alpha}}{\alpha}\right) = \frac{(-2)^{n-1}}{(\alpha-\beta)}\left(\beta\bar{\alpha}\bar{\beta} - \alpha\bar{\beta}\bar{\alpha}\right).
\end{aligned}$$

Since, we have

$$\begin{aligned}
\left(\bar{\alpha}\bar{\beta} - \alpha\bar{\beta}\bar{\alpha}\right) &= 5(\alpha-\beta) + \left(5\beta^2 - 3\alpha\beta - 5\alpha^2 + 3\alpha\beta\right) i_2 + (-3\beta^3 - \alpha^2\beta - 3\alpha^3 + \beta^2\alpha)i_3\\
&\quad + \left(\alpha^3\beta + \beta^4 - 2\beta^2 + 2\alpha\beta - \alpha^4 - \beta^3\alpha + 2\alpha^2 - 2\alpha\beta\right) i_4\\
&= (\alpha-\beta)(5 - 5(\alpha+\beta)\, i_2 + \left(2 - 3(\alpha^2+\alpha\beta+\beta^2\right))i_3 + (-(\alpha+\beta)(\alpha^2+\beta^2))i_4\\
&= (\alpha-\beta)(5 - 5ki_2 + \left(2 - 3(k^2+2)\right) i_3 + (-k^3-4k)i_4.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\bar{\mathcal{H}}J_{k,n+1}\bar{\mathcal{H}}J_{k,n-1} - \bar{\mathcal{H}}J_{k,n}^2\\
&= \frac{(-2)^{n-1}}{(\alpha-\beta)}(\alpha-\beta)\left(5 - 5ki_2 + \left(2 - 3(k^2+2)\right) i_3 + (-k^3-4k)i_4\right)\\
&= (-2)^{n-1}\left(5 + 3ki_2 + \left(k^2-4\right) i_3 + \left(-k^3-8k\right) i_4\right)\\
&= (-2)^{n-1}\left(5J_{k,1}, -5J_{k,2}, 2-3J_{k,3}, -J_{k,4}\right).
\end{aligned}$$

This concludes the proof of (i). □

The proof of (ii) is similar to (i). So, we omit the proof.

Theorem 3.7 (Catalan Identity) *For all $n \geq 1$, prove that*

(i)

$$\begin{aligned}
&\bar{\mathcal{H}}J_{k,n-r}\bar{\mathcal{H}}J_{k,n+r}-\bar{\mathcal{H}}J_{k,n}^{2}\\
&=(-2)^{n-r}J_{k,r}\left(5J_{k,r}+\left(-5J_{k,r+1}-6J_{k,r-1}\right)i_2+\left(-3J_{k,r+2}+4J_{k,r-2}\right)i_3\right.\\
&\quad\left.+\left(-J_{k,r+3}+2J_{k,r+1}+4J_{k,r-1}+8J_{k,r-3}\right)i_4\right)\\
&=(-2)^{n-r}J_{k,r}(5J_{k,r},-5J_{k,r+1}-6J_{k,r-1},-3J_{k,r+2}+J_{k,r-2},\\
&\quad -J_{k,r+3}+2J_{k,r+1}+4J_{k,r-1}+8J_{k,r-3})
\end{aligned}$$

(ii)

$$\begin{aligned}
&\bar{\mathcal{H}}j_{k,n-r}\bar{\mathcal{H}}j_{k,n+r}-\bar{\mathcal{H}}j_{k,n}^{2}\\
&=(-2)^{n-r}J_{k,r}\rho(-5j_{k,1}i_1+\left(5J_{k,r+1}+6J_{k,r-1}\right)i_2\\
&\quad+\left(J_{k,r+2}-4J_{k,r-2}\right)i_3+(J_{k,r+3}-2J_{k,r+1}-8J_{k,r-3}+4J_{k,r-1})i_4
\end{aligned}$$

Proof (i) Using the Binet formula, we have

$$\begin{aligned}
\bar{\mathcal{H}}J_{k,n-r}\bar{\mathcal{H}}J_{k,n+r}-\bar{\mathcal{H}}J_{k,n}^{2}&=\left[\frac{\bar{\alpha}\alpha^{n-r}-\bar{\beta}\beta^{n-r}}{\alpha-\beta}\right]\left[\frac{\bar{\alpha}\alpha^{n+r}-\bar{\beta}\beta^{n+r}}{\alpha-\beta}\right]-\left[\frac{\bar{\alpha}\alpha^{n}-\bar{\beta}\beta^{n}}{\alpha-\beta}\right]^2\\
&=\left[\frac{-\bar{\alpha}\bar{\beta}\alpha^{n-r}\beta^{n+r}-\bar{\beta}\bar{\alpha}\beta^{n-r}\alpha^{n+r}+\bar{\alpha}\bar{\beta}\alpha^{n}\beta^{n}+\bar{\beta}\bar{\alpha}\beta^{n}\alpha^{n}}{(\alpha-\beta)^{2}}\right]\\
&=\frac{\bar{\alpha}\bar{\beta}\alpha^{n}\beta^{n}\left[1-\left(\frac{\beta}{\alpha}\right)^{r}\right]+\bar{\beta}\bar{\alpha}\alpha^{n}\beta^{n}\left[1-\left(\frac{\alpha}{\beta}\right)^{r}\right]}{(\alpha-\beta)^{2}}\\
&=\frac{\bar{\alpha}\bar{\beta}(-2)^{n}\left(\frac{\alpha^{r}-\beta^{r}}{\alpha^{r}}\right)+\bar{\beta}\bar{\alpha}(-2)^{n}\left(\frac{\beta^{r}-\alpha^{r}}{\beta^{r}}\right)}{(\alpha-\beta)^{2}}\\
&=\frac{(-2)^{n}\left(\alpha^{r}-\beta^{r}\right)}{(\alpha-\beta)^{2}}\left(\frac{\bar{\alpha}\bar{\beta}}{\beta^{r}}-\frac{\bar{\beta}\bar{\alpha}}{\alpha^{r}}\right)\\
&=\frac{(-2)^{n-r}(\alpha^{r}-\beta^{r})}{(\alpha-\beta)^{2}}\left(\beta^{r}\bar{\alpha}\bar{\beta}-\alpha^{r}\bar{\beta}\bar{\alpha}\right).
\end{aligned}$$

Now, we can write

$$\begin{aligned}
\left(\beta^{r}\bar{\alpha}\bar{\beta}-\alpha^{r}\bar{\beta}\bar{\alpha}\right)&=5\left(\alpha^{r}-\beta^{r}\right)+\left(5\beta^{r+1}-3\alpha\beta^{r}-5\alpha^{r+1}+3\beta\alpha^{r}\right)i_2\\
&\quad+(3\beta^{r+2}-\alpha^{2}\beta^{r}-3\alpha^{r+2}+\beta^{2}\alpha^{r})i_3+\left(\alpha^{3}\beta^{r}+\beta^{r+3}-2\beta^{r+1}+2\alpha\beta^{r}-\alpha^{r+3}-\beta^{3}\alpha^{r}\right.\\
&\quad\left.+2\alpha^{r+1}-2\beta\alpha^{r}\right)i_4\\
&=5\left(\alpha^{r}-\beta^{r}\right)+\left(-5\left(\alpha^{r+1}-\beta^{r+1}\right)\right)i_2+(-3\left(\alpha^{r+2}-\beta^{r+2}\right)+\alpha^{2}\beta^{2}\left(\alpha^{r-2}-\beta^{r+2}\right)i_3\\
&\quad+(-\alpha^{3}\beta^{3}\left(\alpha^{r-3}-\beta^{r-3}\right)-\left(\alpha^{r+3}-\beta^{r+3}\right)+2\left(\alpha^{r+1}-\beta^{r+1}\right)-2\alpha\beta(\alpha^{r-1}-\beta^{r-1}))i_4.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\bar{\mathcal{H}}J_{k,n-r}\bar{\mathcal{H}}J_{k,n+r}-\bar{\mathcal{H}}J_{k,n}^{2}&=(-2)^{n-r}J_{k,r}\left(5J_{k,r}+\left(-5J_{k,r+1}-6J_{k,r-1}\right)i_2+\left(-3J_{k,r+2}+4J_{k,r-2}\right)i_3\right.\\
&\quad\left.+\left(8J_{k,r-3}-J_{k,r+3}+2J_{k,r+1}+4J_{k,r-1}\right)i_4\right)\\
&=(-2)^{n-r}J_{k,r}(5J_{k,r},-5J_{k,r+1}-6J_{k,r-1},-3J_{k,r+2}+J_{k,r-2},-J_{k,r+3}+2J_{k,r+1}\\
&\quad+4J_{k,r-1}+8J_{k,r-3}).
\end{aligned}$$

This completes the proof of (i). □

The proof of (ii) is analogous to (i). So, we omit it.

In particular, if we put r = 1 in Theorem 3.7 then it reduces to Cassini's identity

$$\begin{aligned}\bar{\mathcal{H}}J_{k,n-1}\bar{\mathcal{H}}J_{k,n+1}-\bar{\mathcal{H}}J_{k,n}^2 &= (-2)^{n-1}\,J_{k,1}\left(5J_{k,1}+\left(-5J_{k,2}-6J_{k,0}\right)i_2+\left(-3J_{k,3}+4J_{k,-1}\right)i_3\right.\\ &\quad \left.+\left(-J_{k,4}+2J_{k,2}+4J_{k,0}+8J_{k,-2}\right)i_4\right)\\ &= (-2)^{n-1}\left(5J_{k,1},-5J_{k,2},2-3J_{k,3},-J_{k,4}\right).\end{aligned}$$

where $J_{k,-n}=(-1)^{n-1}2^{-n}J_{k,n}$, for all $n\geq 0$.

Theorem 3.8 (Vajda's Identity) *If i, j are any two natural numbers then*

(i) $$\begin{aligned}\bar{\mathcal{H}}J_{k,n+i}\bar{\mathcal{H}}J_{k,n+j}-\bar{\mathcal{H}}J_{k,n}\bar{\mathcal{H}}J_{k,n+i+j} &= J_{k,i}(-2)^n\left(-5J_{k,j}+\left(5J_{k,j+1}+6J_{k,j-1}\right)i_2+\left(3J_{k,j+2}-4J_{k,j-2}\right)i_3\right.\\ &\quad\left.+\left(-8J_{k,j-3}+J_{k,j+3}-2J_{k,j+1}-4J_{k,j-1}\right)i_4\right),\end{aligned}$$

(ii) $$\begin{aligned}\bar{\mathcal{H}}j_{k,n+i}\bar{\mathcal{H}}j_{k,n+j}-\bar{\mathcal{H}}j_{k,n}\bar{\mathcal{H}}j_{k,n+i+j} &= (-2)^n\,J_{k,i}\rho(5J_{k,j}-5J_{k,j+1}i_2\\ &\quad+\left(-3J_{k,j+2}-4J_{k,j-2}\right)i_3+\left(-8J_{k,j-3}-J_{k,j+3}+2J_{k,j+1}+4J_{k,j-1}\right)i_4).\end{aligned}$$

Proof (i) Using the Binet formula, we obtain

$$\begin{aligned}&\bar{\mathcal{H}}J_{k,n+i}\bar{\mathcal{H}}J_{k,n+j}-\bar{\mathcal{H}}J_{k,n}\bar{\mathcal{H}}J_{k,n+i+j}\\ &=\left[\frac{\bar{\alpha}\alpha^{n+i}-\bar{\beta}\beta^{n+i}}{\alpha-\beta}\right]\left[\frac{\bar{\alpha}\alpha^{n+j}-\bar{\beta}\beta^{n+j}}{\alpha-\beta}\right]-\left[\frac{\bar{\alpha}\alpha^{n}-\bar{\beta}\beta^{n}}{\alpha-\beta}\right]\left[\frac{\bar{\alpha}\alpha^{n+i+j}-\bar{\beta}\beta^{n++i+j}}{\alpha-\beta}\right]\\ &=\left[\frac{-\bar{\alpha}\bar{\beta}\alpha^{n+i}\beta^{n+j}-\bar{\beta}\bar{\alpha}\beta^{n+i}\alpha^{n+j}+\bar{\alpha}\bar{\beta}\alpha^{n}\beta^{n+i+j}+\bar{\beta}\bar{\alpha}\beta^{n}\alpha^{n+i+j}}{(\alpha-\beta)^{\mathbf{2}}}\right]\\ &=\frac{-\bar{\alpha}\bar{\beta}\alpha^{n}\beta^{n+j}\left(\alpha^i-\beta^i\right)+\bar{\beta}\bar{\alpha}\alpha^{n+j}\beta^{n}\left(\alpha^i-\beta^i\right)}{(\alpha-\beta)^{\mathbf{2}}}\\ &=\frac{-\bar{\alpha}\bar{\beta}\alpha^{n}\beta^{n+j}}{(\alpha-\beta)}J_{k,i}+\frac{\bar{\beta}\bar{\alpha}\alpha^{n+j}\beta^{n}}{(\alpha-\beta)}J_{k,i}\\ &=\left[\frac{\alpha^n\beta^n J_{k,i}}{\alpha-\beta}\right]\left(\alpha^j\bar{\beta}\bar{\alpha}-\beta^j\bar{\alpha}\bar{\beta}\right).\end{aligned}$$

Thus, it gives that

$$\begin{aligned}\bar{\mathcal{H}}J_{k,n+i}\bar{\mathcal{H}}J_{k,n+j}-\bar{\mathcal{H}}J_{k,n}\bar{\mathcal{H}}J_{k,n+i+j} &= J_{k,i}(-2)^n\left(-5J_{k,j}+\left(5J_{k,j+1}+6J_{k,j-1}\right)i_2\right.\\ &\quad+\left(3J_{k,j+2}-4J_{k,j-2}\right)i_3\\ &\quad\left.+\left(-8J_{k,j-3}+J_{k,j+3}-2J_{k,j+1}-4J_{k,j-1}\right)i_4\right).\end{aligned}$$

The proof of (ii) is similar to (i). So, we omit the proof.

Theorem 3.9 (D'ocagne identity) *Let n be any non-negative integer and t be any natural number. If $t\geq n+1$, then we have*

(i) $$\begin{aligned}&\bar{\mathcal{H}}J_{k,t}\bar{\mathcal{H}}J_{k,n+1}-\bar{\mathcal{H}}J_{k,t+1}\bar{\mathcal{H}}J_{k,n}\\ &=(5\,(-2)^t\,J_{k,n-1-t},\left(5\,(-2)^{n+1}\,J_{k,t-n-1}+3\,(-2)^{t+1}\,J_{k,n-t-1}\right),\\ &\left(3\,(-2)^{n+2}\,J_{k,t-n-2}+(-2)^{t+2}\,J_{k,n-t-2}\right),((-2)^{t+3}\,J_{k,n-t-3}+(-2)^{n+3}\,J_{k,t-n-3}\\ &+(-2)^{n+2}\,J_{k,t-n-1}+(-2)^{t+2}J_{k,t-n-1}),\end{aligned}$$

(ii) $$\begin{aligned}&\bar{\mathcal{H}}j_{k,t}\bar{\mathcal{H}}j_{k,n+1}-\bar{\mathcal{H}}j_{k,t+1}\bar{\mathcal{H}}J_{k,n}\\ &=-\rho(5\,(-2)^t\,J_{k,n-1-t},\left(5\,(-2)^{n+1}\,J_{k,t-n-1}+3\,(-2)^{t+1}\,J_{k,n-t-1}\right),\\ &\left(3\,(-2)^{n+2}\,J_{k,t-n-2}+(-2)^{t+2}\,J_{k,n-t-2}\right),\\ &\left((-2)^{t+3}\,J_{k,n-t-3}+(-2)^{n+3}\,J_{k,t-n-3}+(-2)^{n+2}\,J_{k,t-n-1}+(-2)^{t+2}J_{k,t-n-1}\right).\end{aligned}$$

Proof The proof is analogous to Theorem 3.9. □

Theorem 3.10 *For any integer $r, s \geq t$, prove that*

$$\bar{\mathcal{H}}J_{k,r+s}\bar{\mathcal{H}}j_{k,r+t} - \bar{\mathcal{H}}J_{k,r+t}\bar{\mathcal{H}}j_{k,r+s} = (-2)^{r+t}\,2^{r+t+1}J_{k,s-t}(-7+\bar{\mathcal{H}}j_{k,0}).$$

Proof Using the Binet formula, we can write

$$\begin{aligned}
&\bar{\mathcal{H}}J_{k,r+s}\bar{\mathcal{H}}j_{k,r+t} - \bar{\mathcal{H}}J_{k,r+t}\bar{\mathcal{H}}j_{k,r+s}\\
&= \left(\frac{\bar{\alpha}\alpha^{r+s} - \bar{\beta}\beta^{r+s}}{\alpha-\beta}\right)\left(\bar{\alpha}\alpha^{r+t} + \bar{\beta}\beta^{r+t}\right) - \left(\frac{\bar{\alpha}\alpha^{r+t} - \bar{\beta}\beta^{r+t}}{\alpha-\beta}\right)\left(\bar{\alpha}\alpha^{r+s} + \bar{\beta}\beta^{r+s}\right)\\
&= \frac{\bar{\alpha}\bar{\beta}\beta^{r+t}\alpha^{r+s} - \bar{\beta}\bar{\alpha}\alpha^{r+t}\beta^{r+s} - \bar{\alpha}\bar{\beta}\beta^{r+s}\alpha^{r+t} + \bar{\beta}\bar{\alpha}\alpha^{r+s}\beta^{r+t}}{\alpha-\beta}\\
&= \frac{\alpha^{r+s}\beta^{r+t}\left(\bar{\beta}\bar{\alpha} + \bar{\alpha}\bar{\beta}\right) - \alpha^{r+t}\beta^{r+s}(\bar{\alpha}\bar{\beta} + \bar{\beta}\bar{\alpha})}{\alpha-\beta}\\
&= \frac{\left(\bar{\alpha}\bar{\beta} + \bar{\beta}\bar{\alpha}\right)\alpha^{r+t}\beta^{r+t}\left(\alpha^{s-t} - \beta^{s-t}\right)}{\alpha-\beta}\\
&= (-2)^{r+t}\,2^{r+t+1}J_{k,s-t}(-7+\bar{\mathcal{H}}j_{k,0}).
\end{aligned}$$

This completes the proof. □

Theorem 3.11 *If $r, t \leq s$ then*

(*i*) $\bar{\mathcal{H}}J_{k,s+t} + (-2)^t\bar{\mathcal{H}}J_{k,r+t} = \bar{\mathcal{H}}J_{k,s}j_{k,t}$,
(*ii*) $\bar{\mathcal{H}}J_{k,s+t} - (-2)^t\bar{\mathcal{H}}J_{k,s-t} = \bar{\mathcal{H}}j_{k,s}J_{k,t}$,
(*iii*) $\bar{H}j_{k,s+t} + (-2)^t\bar{\mathcal{H}}j_{k,s-t} = \bar{\mathcal{H}}j_{k,s}j_{k,t}$.

Proof (i) By making the use of the Binet formula, we can write

$$\begin{aligned}
\bar{\mathcal{H}}J_{k,s}j_{k,t} &= \left(\frac{\bar{\alpha}\alpha^{s} - \bar{\beta}\beta^{s}}{\alpha-\beta}\right)\left(\alpha^{t} + \beta^{t}\right)\\
&= \frac{\bar{\alpha}\alpha^{s+t} + \bar{\alpha}\alpha^{s}\beta^{t} - \bar{\beta}\beta^{s}\alpha^{t} - \bar{\beta}\beta^{s+t}}{\alpha-\beta}\\
&= \frac{\bar{\alpha}\alpha^{s+t} - \bar{\beta}\beta^{s+t}}{\alpha-\beta} + \frac{\alpha^{t}\beta^{t}(\bar{\alpha}\alpha^{s-t} - \bar{\beta}\beta^{s-t})}{\alpha-\beta}\\
&= \bar{\mathcal{H}}J_{k,s+t} + (-2)^t\bar{\mathcal{H}}J_{k,r+t}.
\end{aligned}$$

Thus, the proof is completed. □

The proof of (ii) and (iii) are similar to (i). So, we omit the proofs.

Theorem 3.12 *For $s \leq t$, prove that*

(*i*) $\bar{\mathcal{H}}J_{k,s}\bar{\mathcal{H}}J_{k,t} - \bar{\mathcal{H}}J_{k,t}\bar{\mathcal{H}}J_{k,s} = (-2)^s\,J_{k,t-s} - 4(0,2,k,1)$,
(*ii*) $\bar{\mathcal{H}}j_{k,s}\bar{\mathcal{H}}J_{k,t} - \bar{\mathcal{H}}j_{k,t}\bar{\mathcal{H}}j_{k,s} = (-2)^s\,J_{k,t-s}4\rho(0,2,k,1)$,
(*iii*) $\bar{\mathcal{H}}J_{k,t}\bar{\mathcal{H}}j_{k,s} - \bar{\mathcal{H}}J_{k,s}\bar{\mathcal{H}}j_{k,t} = J_{k,t-s}\,(-1)^s\,2^{s+1}(-7+\bar{\mathcal{H}}j_{k,0})$.

Proof The proofs are similar to Theorem 3.11. Hence, we omit the proofs.

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