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ORIGINAL RESEARCH

Curvature invariants of slant submanifolds in S-space forms

P. Alegre · J. Barrera · A. Carriazo · L. M. Fernández

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Abstract In this paper, we establish some relationships between the main intrinsic invariants, scalar and Ricci curvatures, and the main extrinsic invariant, the mean curvature vector, for slant submanifolds of S-space-forms. In addition to that, we study those slant submanifolds satisfying the equality case between the above invariants, due to the great importance that these submanifolds have had in both complex and contact geometry. Moreover, we present some examples on these conditions.

Keywords slant submanifolds · S-space-forms · Ricci curvature · scalar curvature · mean curvature

Mathematics Subject Classification 53C40 · 53C25 · 53C15

1 Introduction

The main intrinsic and extrinsic invariants of submanifolds have been studied in both complex and contact geometry in the last years. In complex geometry, A. Ros and F. Urbano established the following inequality for a Lagrangian submanifold of $\mathbb{C}^m$ in [16].

$$\|H\|^2 \geq \frac{2(m+2)}{m^2(m-1)}\tau.$$

Moreover, it was proved that the submanifold satisfies the equality case at every point, if and only if its second fundamental form σ is given by

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P. Alegre (✉) · A. Carriazo · L. M. Fernández
Department of Geometry and Topology, Faculty of Mathematics, University of Sevilla, C./Tarfia, s.n., 41012 Sevilla, Spain
E-mail: palegre@us.es

A. Carriazo
E-mail: carriazo@us.es

L. M. Fernández
E-mail: lmfer@us.es

J. Barrera
Departamento de Coordinación de Gestión de Recursos Humanos, Consejería de Educación y Deporte, Junta de Andalucía, c/ Juan Antonio de Vizarrón, S/N. 41092 Sevilla, Spain
E-mail: barreralopezjoaquin@gmail.com

$$\sigma(X,Y) \;=\; \frac{m}{m+2}\{g(X,Y)H + g(JX,H)JY + g(JY,H)JX\},$$

for any tangent vector fields X, Y.

On the same line, in contact geometry, D. E. Blair and A. Carriazo established an analogue inequality, for anti-invariant submanifolds in $\mathbf{R}^{2m+1}$ with its standard Sasakian structure in [3]:

$$\|H\|^2 \geq \frac{2(m+2)}{(m+1)^2(m-1)}\tau,$$

and they gave the following characterization for the equality case, using the second fundamental form:

$$\begin{aligned}\sigma(X,Y) = \frac{m+1}{m+2}\{&(g(X,Y)-\eta(X)\eta(Y))H + (g(\phi X,H)\\ &-\frac{m+2}{m+1}\eta(X))\phi Y + (g(\phi Y,H) - \frac{m+2}{m+1}\eta(Y))\phi X\},\end{aligned}$$

for all X, Y tangent to the submanifold.

On the other hand, slant submanifolds were defined by B.-Y. Chen in [7] as a generalization of both complex and totally real submanifolds of almost Hermitian manifolds, while A. Lotta defined them on the almost contact metric setting in [15]. Later, M. B. Hans Uber in [12] introduced them on the S-manifolds.

For slant submanifolds, A. Song and X. Liu obtained in [17] two inequalities concerning the Ricci curvature, the scalar curvature and the squared mean curvature of slant submanifolds in generalized complex space forms.

Recently, in contact geometry, P. Alegre, J. Barrera and A. Carriazo have established the following inequality for slant submanifolds of generalized Sasakian space forms:

$$\|H\|^2 \geq \frac{2(m+2)}{(m+1)^2(m-1)}\left[\tau - \frac{1}{2}m(m+1)f_1 - \frac{3}{2}m\cos^2\theta f_2 + mf_3 + \alpha^2 m\sin^2\theta\right], \tag{1}$$

and they obtained the following characterization for the equality case:

$$\begin{aligned}\sigma(X,Y) \;=\; \frac{m+1}{m+2}\Bigg\{&(g(X,Y)-\eta(X)\eta(Y))\,H\\ &+\left(\frac{1}{\sin^2\theta}g(\phi X,H) - \alpha\frac{m+2}{m+1}\eta(X)\right)NY\\ &+\left(\frac{1}{\sin^2\theta}g(\phi Y,H) - \alpha\frac{m+2}{m+1}\eta(Y)\right)NX\Bigg\}.\end{aligned}$$

Generalizing complex and contact manifolds, K. Yano in [18] introduced the notion of f-structure of an $(2m+s)$-dimensional manifold as a tensor field f of type (1, 1) and rank $2m$ satisfying $f^3 + f = 0$. In this context, D. E. Blair in [2] defined K-manifolds (and particular cases of S-manifolds and C-manifolds).

About the main intrinsic and extrinsic invariants of the submanifolds, it is worth noting that L. M. Fernández and A. Prieto-Martín established in [11] the following inequality for anti-invariant submanifolds of $\mathbb{R}^{2m+s}$:

$$\|H\|^2 \geq \frac{2(m+2)}{(m+s)^2(m-1)}\tau,$$

and they characterized the equality case by the following expression of the second fundamental form:

$$\begin{aligned}\sigma(X,Y) \;=\; \frac{m+s}{m+2}\Bigg\{&\left(g(X,Y) - \sum_{\alpha=1}^{t}\eta_\alpha(X)\eta_\alpha(Y)\right)H\\ &+\left(g(fX,H) - \frac{m+2}{m+s}\sum_{\alpha=1}^{t}\eta_\alpha(X)\right)fY\\ &+\left(g(fY,H) - \frac{m+2}{m+s}\sum_{\alpha=1}^{t}\eta_\alpha(Y)\right)fX\Bigg\}.\end{aligned}$$

Nevertheless, little has been studied about invariants of slant submanifolds in S-manifolds. In this sense, L. M. Fernández and A. M. Fuentes in [8] established some inequalities of Chen's type between intrinsic and extrinsic invariants of submanifolds of a generalized S-space-forms. On the other hand, L. M. Fernández and M. B. Hans Uber in [9] presented new relationships between the Ricci curvature and the squared mean curvature for slant submanifolds in S-space-forms. Moreover, they studied the shape operator associated with the mean curvature vector.

Continuing this line, we establish some relationships between intrinsic and extrinsic invariants, characterizing the equality case through a specific expression of the second fundamental form of the submanifold. Specifically, in the third section we establish an inequality between the scalar curvature and the mean curvature vector, characterizing the equality case and giving some examples. In the last section, for slant submanifolds that satisfy the above equality case, we establish the relationship between the Ricci curvature and the mean curvature vector, and give lower and upper bounds for the Ricci curvature.

2 Preliminaries.

A Riemannian manifold $(\widetilde{M}, g)$ of dimension $2m+s$, endowed with a f-structure f, that is, a tensor f of type $(1, 1)$ and rank $2m$ satisfying the equality $f^3 + f = 0$ (see [18]), it is said to be a *f-metric manifold* if the fields $\xi_1, \dots, \xi_s$ of $\widetilde{M}$ exist, and in that case they are called *structure fields*, such that if $\eta_1, \dots, \eta_s$ are the dual 1-forms of $\xi_1, \dots, \xi_s$, then

$$\eta_\alpha(\xi_\alpha) = 1;\ \ f\xi_\alpha = 0;\ \ \eta_\alpha \circ f = 0;\ \ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha; \tag{2}$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{3}$$

for all X, Y of $\widetilde{M}$. By definition, the metric g satisfies that

$$g(fX, Y) = -g(X, fY), \tag{4}$$

for all X, Y of $\widetilde{M}$. Let F be the 2-form of $\widetilde{M}$ defined as $F(X, Y) = g(X, fY)$. We know that f has rank $2m$, then $\eta_1 \wedge \dots \wedge \eta_s \wedge F^m \neq 0$ and, in particular, $\widetilde{M}$ is orientable. The f-structure f is *normal* if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ denotes the Nijenhuis tensor of f.

A f-metric manifold it is said to be a *f-contact metric manilfold* if $F = d\eta_\alpha$, for all $\alpha = 1, \dots, s$. On the other hand, a f-contact metric manifold it is said to be a *f-K-contact metric manifold* if all the structure fields are Killing fields. A f-metric manifold it is said to be a *K-manifold* (see [2]) if it is normal and $dF = 0$. Moreover, a K-manifold it is said to be a *S-manifold* if $F = \mathrm{d}\eta_\alpha$, for all $\alpha = 1, \dots, s$. When $s \geq 2$, we can find non-trivial examples in [2] and [13]. Moreover, the Riemannian connection $\widetilde{\nabla}$ of a S-variedad satisfies for all X, Y tangent fields and any $\alpha = 1, \dots, s$:

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \tag{5}$$

$$(\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} (g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X). \tag{6}$$

A flat section Π of a f-metric manifold $\widetilde{M}$ it is said to be a *f-section* if it is generated by a unitary vector X, orthogonal to the structure fields, and by fX. The sectional curvature of Π is called *f-sectional curvature*. A

S-manifold is a S-space-form if its f-sectional curvature is constant c and, in that case, it is denoted by $\widetilde{M}(c)$. The curvature tensor $\widetilde{R}$ de $\widetilde{M}(c)$ satisfies (see [14]):

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = \sum_{\alpha,\beta=1}^{s} &(g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W)\\ &+g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z))\\ &+\frac{c+3s}{4}(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW))\\ &+\frac{c-s}{4}(F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)),\end{aligned} \tag{7}$$

for all X, Y, Z, W in $\widetilde{M}$.

Then, denoting by R and $\widetilde{R}$ the curvature tensors of M and $\widetilde{M}$, respectively, we have the Gauss equation:

$$\widetilde{R}(X,Y;Z,W) \;=\; R(X,Y;Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)). \tag{8}$$

On the other hand, for any tangent field X of M, we denote

$$fX = TX + NX, \tag{9}$$

where TX and NX are the tangent and normal components of fX, respectively. M is *invariant* if N is identically null, and M is *anti–invariant* if T is identically null.

For any normal field V of M, we denote

$$fV \;=\; tV + nV, \tag{10}$$

where tV (resp. nV) denotes the tangent component (resp. normal) of fV.

From (4), (9) and (10), we have

$$g(TX,Y) = -g(X,TY),$$
$$g(NX,V) = -g(X,tV),$$

for all X, Y tangent and all V normal to M. Moreover, if $\widetilde{M}$ is a S-manifold and the structure fields are tangent to M, from (5), (9) and taking into account the Gauss formula, $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ it is easy to obtain

$$\nabla_X \xi_\alpha = -TX,$$

$$\sigma(X,\xi_\alpha) = -NX, \tag{11}$$

for all X tangent to M and any $\alpha = 1,\ldots,s$ and, in particular, as we know that $f\xi_\alpha = 0$, for any α:

$$\sigma(\xi_\alpha,\xi_\beta) = 0,\ \alpha,\beta = 1,\ldots,s. \tag{12}$$

As the structure fields are tangent to M, we can denote as $\mathcal{M}$ to the distribution of M generated by the structure fields and as $\mathcal{L}$ its complementary orthogonal distribution. Thus, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for all $\alpha = 1,\ldots,s$ and if $X \in \mathcal{M}$, then $fX = 0$. Now, we define the main intrinsic and extrinsic invariants of M. The *scalar curvature* of M is defined as

$$\tau = \sum_{1\le i<j\le m+s} K(e_i,e_j), \tag{13}$$

where $K(e_i,e_j)$ denotes the sectional curvature of M associated with the flat section generated by e_i y e_j, for all e_i, e_j of a local orthonormal reference of M.

The *Ricci curvature* of M is defined as

$$Ric(X) = \sum_{1\le i\le m+s} K(X,e_i). \tag{14}$$

Moreover, we know that the Ricci curvature and the scalar curvature are related by the following equality:

$$\tau = \frac{1}{2} \sum_{1 \leq j \leq m+s} Ric(e_j).$$

On the other hand, the *mean curvature vector* of M is defined as

$$H = \frac{1}{m+s} \text{traza } \sigma = \frac{1}{m+s} \sum_{i=1}^{m+s} \sigma(e_i, e_i), \tag{15}$$

where $m+s$ is the dimension of M.

Finally, we remember the notion of *slant submanifolds* in S-manifolds. These submanifolds were introduced by M. B. Hans Uber in [12] as those that satisfy that, for all $x \in M$ and $X \in T_xM$ linearly independent of $\xi_1, \ldots, \xi_s$, the angle between fX and T_xM is a constant $\theta \in [0, \pi/2]$, called *slant angle* of M in $\widetilde{M}$. Moreover, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion that is neither invariant nor anti-invariant is said to be a *proper slant* immersion, and the submanidold is said to be a *proper slant* submanifold. It should be noted that, if the submanifold is proper slant, all the structure fields $\xi_1, \ldots, \xi_s$ should be tangent fields, because from [12], if it would exist a normal structure field ξ_α, for $\alpha = 1, \ldots, s$, then the submanifold M would be anti-invariant. Thus, the dimension of the submanifold M should be $m+s$. Moreover, it is established that

$$T^2X = \cos^2\theta(-X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha), \tag{16}$$

$$tNX = \sin^2\theta(-X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha), \tag{17}$$

$$NTX + nNX = 0, \tag{18}$$

for any X tangent vector field.

On the other hand, if M is a non-anti-invariant submanifold (that is, if $\theta \in [0, \pi/2)$), it was proved in [10] that

$$(\overline{f}, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

is a f-metric structure in M, where $\overline{f} = (\sec\theta)T$, then if $\dim(M) = m+s$, m is even. Moreover, in a θ-slant submanifold of a f-metric manifold, we have:

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)), \tag{19}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)), \tag{20}$$

for all X, Y tangent to the submanifold (see [5]). Moreover, from (19) y (20) we can deduce that

$$\|T\|^2 = m\cos^2\theta, \quad \|N\|^2 = m \,\mathrm{sen}^2\theta. \tag{21}$$

It is said that a proper slant submanifold of a S-manifold is a *S-slant* submanifold if

$$(\nabla_X T)Y = \cos^2\theta \sum_{\alpha=1}^{s} (g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X),$$

for all X, Y tangent to M, where θ is the slant angle. It is easy to see that, if $X, Y \in \mathcal{L}$, then $(\nabla_X T)Y = (\nabla_Y T)X$. Moreover, it has been proved in [5] that any $(2+s)-dimensional$ proper slant submanifold of a S-manifold is a S-slant submanifold. We can observe that $2+s$ is the smallest possible dimension for a proper slant submanifold of a S-manifold, tangent to the structure fields.

3 The Scalar Curvature

In this section, we establish an inequality between one of the main intrinsic invariants, the scalar curvature, τ, and the main extrinsic invariant, the mean curvature vector H, of a slant submanifold in a S-space-form. This inequality generalizes the one obtained by L. M. Fernández and A. Prieto-Martín in [11] for anti-invariant submanifolds of $\mathbb{R}^{2m+s}$. Moreover, we characterize the equality case based on the second fundamental form of the inequality.

At the beginning of the main theorem, we choose an adapted slant frame

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1^*}, \ldots, e_{m^*}\},$$

using the same procedure detailed in [4], that is, for any unitary tangent vector field e_1, we select $e_2 = \frac{1}{\cos\theta} T e_1$, and then we choose e_3 a unitary vector field orthogonal to e_1 and e_2 and repeat the procedure. Finally $e_{m+1} = \xi_1, \ldots, e_{m+s} = \xi_s$ and $e_{i^*} = \frac{1}{\sin\theta} N e_i$, for $i = 1, \ldots, m$. Moreover, we prove in the following lemma that we can make the base choosing e_{1^*} in the same direction of H.

Lemma 1 *Let M^{m+s} be a slant submanifold of a f-metric manifold $\widetilde{M}^{2m+s}$. It can be chosen*

$$e_{1^*} = \frac{1}{\|H\|} H, \quad e_1 = -\csc\theta t e_{1^*},$$

in such a case, it holds $e_{1^*} = \csc\theta N e_1$.

Proof For a $m+s$ dimensional slant submanifold of a $2m+s$ dimensional f-metric manifold, it is deduced from (16-18) that $NtV = -\sin^2\theta V$ for any V normal vector field. Therefore, taking such e_{1^*}, one has $e_{1^*} = \csc\theta N e_1$. □

Now, we introduce another lemma about the symmetries of the second fundamental form, which can be easily proved by denoting $\sigma^i_{jk} = g(\sigma(e_j, e_k), e_{i^*})$, $i, j, k = 1, \ldots, m$ for any three vector fields of an adapted slant frame and taking into account the f-structure of $\widetilde{M}$.

Lemma 2 *Let M^{m+s} be a slant submanifold of a f-metric manifold $\widetilde{M}^{2m+s}$. Then, with respect to an adapted slant frame,*

$$\begin{aligned} &\sigma^i_{jk} = \sigma^j_{ik} = \sigma^k_{ij}, && \\ &\sigma^i_{\xi_\alpha \xi_\beta} = 0, && \alpha, \beta = 1, \ldots, s, \\ &\sigma^i_{\xi_\alpha j} = \sigma^j_{\xi_\alpha i} = -\sin\theta \delta_{ij}, && \alpha = 1, \ldots, s, \end{aligned}$$

with $i, j, k = 1, \ldots, m$.

Theorem 1 *Let M^{m+s} be a slant submanifold ($m \geq 2$) tangent to the structure fields $\xi_1, \ldots, \xi_s$ of a S-space-form $\widetilde{M}^{2m+s}(c)$. Then, its squared mean curvature $\|H\|^2$ and its scalar curvature τ satisfy at each point the following inequality:*

$$\|H\|^2 \geq \frac{2(m+2)}{(m+s)^2(m-1)} \left[\tau - \frac{1}{2} m(m-1) \frac{c+3s}{4} - \frac{3}{2} m \cos^2\theta \frac{c-s}{4} - sm\cos^2\theta \right]. \tag{22}$$

Proof Let M^{m+s} be a slant submanifold of a S-space-form $\widetilde{M}^{2m+s}(c)$. Let be $\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1^*}, \ldots, e_{m^*}\}$ an adapted slant frame. From (15), we know that

$$\begin{aligned} \|H\|^2 =& g(H, H) = \frac{1}{(m+s)^2} g(\sum_{i=1}^{m+s} \sigma(e_i, e_i), \sum_{j=1}^{m+s} \sigma(e_j, e_j)) = \\ =& \frac{1}{(m+s)^2} \left(\sum_{i=1}^{m+s} g(\sigma(e_i, e_i), \sigma(e_i, e_i)) + \sum_{i \neq j}^{m+s} g(\sigma(e_i, e_i), \sigma(e_j, e_j)) \right), \end{aligned} \tag{23}$$

where, from (12), $\sigma(\xi_\alpha, \xi_\beta) = 0$, for all $\alpha, \beta = 1, \ldots, s$.

On the other hand, from (3), (4), (7), (8) and (13), we can write the scalar curvature in the following way:

$$\begin{aligned}
2\tau &= \sum_{i\neq j}^{m+s} K(e_i, e_j) = \sum_{i\neq j}^{m+s} R(e_i, e_j, e_j, e_i) \\
&= \sum_{i\neq j}^{m+s} \left(\widetilde{R}(e_i, e_j; e_j, e_i) - g(\sigma(e_i, e_j), \sigma(e_i, e_j)) + g(\sigma(e_i, e_i), \sigma(e_j, e_j))\right) \\
&= \sum_{i\neq j}^{m+s} \Big(\frac{c+3s}{4}\left\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\right\} \\
&\quad + \frac{c-s}{4}\left\{g(e_i, \phi e_j)g(\phi e_j, e_i) - g(e_j, \phi e_j)g(\phi e_i, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\right\} \\
&\quad + \sum_{\alpha,\beta=1}^{s} \left\{g(fe_i, fe_i)\eta_\alpha(e_j)\eta_\beta(e_j) - g(fe_i, fe_j)\eta_\alpha(e_j)\eta_\beta(e_i)\right. \\
&\quad \left. + g(fe_j, fe_j)\eta_\alpha(e_i)\eta_\beta(e_i) - g(fe_j, fe_i)\eta_\alpha(e_i)\eta_\beta(e_j)\right\} \\
&\quad - g(\sigma(e_i, e_j), \sigma(e_i, e_j)) + g(\sigma(e_i, e_i), \sigma(e_j, e_j))\Big),
\end{aligned}$$

from where, by a direct computation and taking into account (21), we easily obtain

$$\begin{aligned}
2\tau &= \frac{c+3s}{4} m(m-1) + \frac{3(c-s)}{4} m\cos^2\theta + 2sm - \\
&\quad - \sum_{i\neq j}^{m+s} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) + \sum_{i\neq j}^{m+s} g(\sigma(e_i, e_i), \sigma(e_j, e_j).
\end{aligned} \tag{24}$$

Now, from (11), (12) and (20), we can easily obtain the terms corresponding to the structure fields ξ_α, for each $\alpha = 1, \ldots, s$:

$$\begin{aligned}
&-2\sum_{i=1}^{m} g(\sigma(e_i, \xi_\alpha), \sigma(e_i, \xi_\alpha)) + 2\sum_{j=1}^{m} g(\sigma(\xi_\alpha, \xi_\alpha), \sigma(e_j, e_j) = \\
&-2\sum_{i=1}^{m} g(Ne_i, Ne_i) = -2m\,\mathrm{sen}^2\theta.
\end{aligned} \tag{25}$$

Thus, from (23), (24) and (25), we can check that

$$\begin{aligned}
&(m+s)^2\|H\|^2 - \frac{2(m+2)}{m-1}\tau = \\
&-\frac{m+2}{m-1}\left[m(m-1)\frac{c+3s}{4} + 3m\cos^2\theta\frac{c-s}{4} + 2sm\cos^2\theta\right] \\
&+\frac{6(m+2)}{m-1}\sum_{i<j<k}(\sigma_{jk}^i)^2 + \frac{3}{m-1}\sum_{i\neq j,k}\sum_{j<k}(\sigma_{jj}^i - \sigma_{kk}^i)^2 \\
&+\frac{2(m+2)}{m-1}\sum_{j<k}(\sigma_{jk}^j)^2 - \frac{3}{m-1}\sum_{i\neq j,k}\sum_{j<k}((\sigma_{jj}^i)^2 + (\sigma_{kk}^i)^2) \\
&-\frac{6}{m-1}\sum_{j\neq k}\left(\sigma_{jj}^j\sigma_{kk}^j + (\sigma_{jk}^k)^2\right) + \sum_{i\neq j}(\sigma_{jj}^i)^2 + \sum_j(\sigma_{jj}^j)^2.
\end{aligned} \tag{26}$$

Now, if we group (26) using the Lemma 2 and taking into account that

$$\frac{1}{m-1}\sum_{j\neq k}(\sigma_{jj}^j - 3\sigma_{kk}^j)^2 = \frac{1}{m-1}((m-1)\sum_j(\sigma_{jj}^j)^2 + 9\sum_{j\neq k}(\sigma_{kk}^j)^2 - 6\sum_{j\neq k}\sigma_{jj}^j\sigma_{kk}^j), \tag{27}$$

we obtain:

$$\begin{aligned}
&(m+s)^2\|H\|^2 - \frac{2(m+2)}{m-1}\tau = \\
&-\frac{m+2}{m-1}\left[m(m-1)\frac{c+3s}{4} + 3m\cos^2\theta\frac{c-s}{4} + 2sm\cos^2\theta\right] \\
&+\frac{6(m+2)}{m-1}\sum_{i<j<k}(\sigma^i_{jk})^2 + \frac{3}{m-1}\sum_{i\neq j,k}\sum_{j<k}(\sigma^i_{jj}-\sigma^i_{kk})^2 \\
&+\frac{1}{m-1}\sum_{j\neq k}(\sigma^j_{jj}-3\sigma^j_{kk})^2,
\end{aligned} \tag{28}$$

so we have

$$(m+s)^2\|H\|^2 - \frac{2(m+2)}{m-1}\tau \geq -\frac{m+2}{m-1}\left[m(m-1)\frac{c+3s}{4} + 3m\cos^2\theta\frac{c-s}{4} + 2sm\cos^2\theta\right],$$

and we can get the desired inequality. □

Note 1 We can observe that, for $s = 1$ the inequality (22) coincides with (1), taking into account in this case that $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ and $\alpha = 1$.

Now, we analyze the equality case of (22). From (28), we can establish the following conditions to verify the equality case:

$$\begin{aligned}
\sigma^i_{jk} &= 0, \quad i<j<k, \\
\sigma^i_{jj}-\sigma^i_{kk} &= 0, \quad i\neq j,k, \quad j<k, \\
\sigma^j_{jj}-3\sigma^j_{kk} &= 0, \quad j\neq k.
\end{aligned} \tag{29}$$

Moreover, considering an adapted slant frame of $\widetilde{M}^{2m+s}$,

$$\{e_1,\dots,e_m,\xi_1,\dots,\xi_s,e_{1^*},\dots,e_{m^*}\},$$

using the procedure detailed in the Lemma 1, we have:

$$H = \frac{1}{m+s}\sum_{i,k}\sigma^k_{ii}e_{k^*} = \|H\|e_{1^*},$$

and then

$$\sum_i \sigma^1_{ii} = (m+s)\|H\|, \qquad \sum_i \sigma^k_{ii} = 0,$$

for $k > 1$. Now, from (29), we can write

$$\sum_{i=1}^m \sigma^1_{ii} = \sigma^1_{11} + \sum_{i=2}^m \sigma^1_{ii} = \sigma^1_{11} + (m-1)\sigma^1_{22} = \frac{m+2}{3}\sigma^1_{11},$$

so

$$\sigma^1_{11} = \frac{3(m+s)}{m+2}\|H\|, \qquad \sigma^1_{22} = \sigma^1_{ii} = \frac{m+s}{m+2}\|H\|,$$

for $i > 2$. Moreover, if $k > 1$, we have

$$\sigma^k_{kk} = 3\sigma^k_{11}, \quad \sigma^k_{11} = \sigma^k_{ii},$$

for $i > 1$, $i \neq k$. Thus, again taking into account (29), we can write

$$0 = \sum_{i=1}^m \sigma^k_{ii} = \sigma^k_{11} + \sum_{i=2}^k \sigma^k_{ii} = (m+2)\sigma^k_{11},$$

so, if $k > 1$, we have

$$\sigma_{ii}^k = 0, \qquad i = 1, \ldots, m.$$

As a result of this, we can establish the following corollary:

Corollary 1 *Let M^{m+s} be a slant submanifold ($m \geq 2$) tangent to the structure fields $\xi_1, \ldots, \xi_s$ of a S-space-form $\widetilde{M}^{2m+s}(c)$. Then, M satisfies the equality case of (22) at each point if and only if it exists a real function λ defined on M such that the second fundamental form of M verifies*

$$\begin{array}{lll} \sigma(e_1, e_1) = 3\lambda e_{1^*},\ \sigma(e_i, e_i) = \lambda e_{1^*}, & i = 2, \ldots, m, & \\ \sigma(e_1, e_i) = \lambda e_{i^*},\ \sigma(e_i, e_j) = 0, & 2 \leq i \neq j \leq m, & \\ \sigma(\xi_\alpha, \xi_\beta) = 0, \quad \sigma(\xi_\alpha, e_i) = -\sin\theta e_{i^*}, & i = 2, \ldots, m, & \alpha, \beta = 1, \ldots, s, \end{array} \tag{30}$$

respect to certain adapted slant frame $e_1, \ldots, e_m, \xi_1, \ldots, \xi_s$, where $e_{i^*} = \csc\theta N e_i$, where N is the normal component of f for tangent fields and $\lambda = \frac{m+s}{m+2}\|H\|$.

In addition, as a final result, we can prove the following theorem in which we establish another condition to satisfy the equality case of (22), in this case giving an expression for the second fundamental form of M.

Theorem 2 *Let M^{m+s} be a slant submanifold ($m \geq 2$) tangent to the structure fields $\xi_1, \ldots, \xi_s$ of a S-space-form $\widetilde{M}^{2m+s}(c)$. Then, M satisfies the equality case of (22) at each point if and only if*

$$\begin{aligned} \sigma(X, Y) = \frac{m+s}{m+2} \Bigg\{ & \left(g(X, Y) - \sum_{\alpha=1}^{t} \eta_\alpha(X)\eta_\alpha(Y) \right) H \\ & + \left(\frac{1}{\sin^2\theta} g(fX, H) - \frac{m+2}{m+s} \sum_{\alpha=1}^{t} \eta_\alpha(X) \right) NY \\ & + \left(\frac{1}{\sin^2\theta} g(fY, H) - \frac{m+2}{m+s} \sum_{\alpha=1}^{t} \eta_\alpha(Y) \right) NX \Bigg\}. \end{aligned} \tag{31}$$

Proof Let's start again considering an adapted slant frame of $\widetilde{M}^{2m+s}$,

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1^*}, \ldots, e_{m^*}\},$$

as described in Lemma 1. Then, we know that $H = \|H\| e_{1^*}$, that together with $\lambda = \frac{m+s}{m+2}\|H\|$, gives us

$$H = \frac{m+2}{m+s} \lambda e_{1^*}. \tag{32}$$

Moreover, from (11), (30) and (32), we can obtain

$$\begin{aligned} \sigma(X, Y) &= \sigma\left(\sum_{i=1}^{m+1} X^i e_i, \sum_{j=1}^{m+1} Y^j e_j \right) = \sum_{i,j=1}^{m+1} X^i Y^j \sigma(e_i, e_j) \\ &= X^1 Y^1 3\lambda e_{1^*} + \sum_{i=2}^{m} X^i Y^i \lambda e_{1^*} + \sum_{j=2}^{m} X^1 Y^j \lambda e_{j^*} + \sum_{i=2}^{m} X^i Y^1 \lambda e_{i^*} \\ &\quad + \sum_{\alpha=1}^{s} \eta_\alpha(X)\sigma(\xi_\alpha, Y) + \sum_{\alpha=1}^{s} \eta_\alpha(Y)\sigma(X, \xi_\alpha) - \sum_{\alpha,\beta=1}^{s} \eta_\alpha(X)\eta_\beta(Y)\sigma(\xi_\alpha, \xi_\beta) \\ &= \left(g(X, Y) - \sum_{\alpha,\beta=1}^{s} \eta_\alpha(X)\eta_\beta(Y) \right) \lambda e_{1^*} + 2X^1 Y^1 \lambda e_{1^*} \\ &\quad + \sum_{j=2}^{m} X^1 Y^j \lambda e_{j^*} + \sum_{i=2}^{m} X^i Y^1 \lambda e_{i^*} - \sum_{\alpha=1}^{s} \eta_\alpha(X) NY - \sum_{\alpha=1}^{s} \eta_\alpha(Y) NX, \end{aligned} \tag{33}$$

for any two tangent fields X, Y of M. Moreover, using

$$e_{i^*} = \frac{1}{\sin\theta} N e_i,$$

we can write

$$NY = \sum_{i=1}^{m+1} Y^i N e_i = \sin\theta \sum_{i=1}^{m+1} Y^i e_{i^*}. \tag{34}$$

Moreover, we have

$$\begin{aligned} g(\phi X, H) =& g\left(\sum_{i=1}^{m+1} X^i \phi e_i, H\right) = g\left(\sum_{i=1}^{m+1} X^i N e_i, H\right) = g\left(\sum_{i=1}^{m+1} X^i \sin\theta e_{i^*}, H\right) = \\ &= \sin\theta g\left(\sum_{i=1}^{m+1} X^i e_{i^*}, \frac{m+2}{m+1}\lambda e_{1^*}\right) = X^1 \frac{m+2}{m+1}\lambda \sin\theta. \end{aligned} \tag{35}$$

Thus, the expression (31) can be immediately obtained from (33), (34) and (35). Conversely, from (31) it is easy to see that (30) is satisfied and then also the equality case of (22). □

Note 2 We can observe that if $s = 1$, the expression (31) is the same as the one established in [1]:

$$\begin{aligned} \sigma(X, Y) \ =& \frac{m+1}{m+2}\Big\{(g(X, Y) - \eta(X)\eta(Y))\, H \\ &+ \left(\frac{1}{\sin^2\theta} g(\phi X, H) - \frac{m+2}{m+1}\eta(X)\right) NY \\ &+ \left(\frac{1}{\sin^2\theta} g(\phi Y, H) - \frac{m+2}{m+1}\eta(Y)\right) NX\Big\}, \end{aligned}$$

for the Sasakian case. Moreover, submanifolds whose second fundamental form is given by (31), that is, those submanifolds that satisfy de equality case of (22), were called $*$-*slant submanifolds* in [1].

Note 3 We can extend the Theorem 1 to the anti-invariant case, because if we consider an adapted anti-invariant frame

$$\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1^*}, \ldots, e_{m^*}\},$$

that is, a local orthonormal base of $\widetilde{M}$, with $e_{i^*} = f e_i$, for $i = 1, \ldots, m$, and following the same steps as in the Theorem 1, we can obtain the inequality:

$$\|H\|^2 \geq \frac{2(m+2)}{(m+s)^2(m-1)} \left[\tau - \frac{1}{2}m(m-1)\frac{c+3s}{4}\right]. \tag{36}$$

Now, if we consider $\mathbb{R}^{2m+s}$ as ambient space endowed with its standard f-structure, so $c = -3s$, we can establish directly from (36) that

$$\|H\|^2 \geq \frac{2(m+2)}{(m+s)^2(m-1)}\tau, \tag{37}$$

which corresponds to the inequality obtained by L. M. Fernández and A. Prieto-Martín in [11].

On the other hand, from Theorem 1, we can directly obtain the following corollary whose proof is immediate.

Corollary 2 *Let M^{m+s} be a slant submanifold $(m \geq 2)$ of $\mathbb{R}^{2m+s}$ endowed with its standard f-structure. Then, its squared mean curvature $\|H\|^2$ and its scalar curvature τ satisfy at each point the following inequality:*

$$\|H\|^2 \geq \frac{2(m+2)}{(m+s)^2(m-1)} \left[\tau + \frac{1}{2} sm \cos^2\theta\right]. \tag{38}$$

Now, we can present some examples of submanifolds that satisfy de equality case of (22) on $\mathbb{R}^{2m+s}$:

Example 1 We follow the construction carried out in [11]. We consider the Riemannian submersion $\pi : \mathbb{R}^{2m+s} \rightarrow \mathbb{C}^m$ given by

$$\pi(x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_s) = \frac{1}{2}(y_1, \ldots, y_m, x_1, \ldots, x_m),$$

and a Lagrangian submanifold $\widehat{M}$ of $\mathbb{C}^m$, such that the following diagram is commutative

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{R}^{2m+s} \\ \downarrow & & \downarrow\ \pi \\ \widehat{M} & \longrightarrow & \mathbb{C}^m, \end{array}$$

where M is the fiber bundle on $\widehat{M}$. Then, the anti-invariant submanifold M locally isometric to the Riemannian product of a portion of a Whitney sphere and $\mathbb{R}^s$ is a slant submanifold which satisfies the equality case of (37), according to [11, Theorem 5.4].

Finally, we present two examples of slant submanifolds, verifying the equality case of the Corollary 2.

Example 2 In [12], the following slant submanifold is considered:

$$x(u, v, t_1, \ldots, t_s) = 2(ucos\theta, usin\theta, v, 0, t_1, \ldots, t_s).$$

This submanifold defines a minimal $(2+s)$-dimensional slant submanifold with slant angle θ in $\mathbb{R}^{4+s}$ endowed with its standard Sasakian structure.

Example 3 On the other hand, in [12], is considered this other slant submanifold:

$$x(u, v, w, r, t_1, \ldots, t_s) = 2(u, 0, w, 0, vcos\theta, vsin\theta, rcos\theta, rsin\theta, t_1, \ldots, t_s).$$

It is a minimal $(4+s)$-dimensional slant submanifold with slant angle θ in $\mathbb{R}^{8+s}$ with its usual Sasakian structure.

4 The Ricci Curvature

Now, we study the relationship between another of the main intrinsic invariants, the Ricci curvature, Ric, and the main extrinsic invariant, H. In this case, we focus our study on slant submanifolds that satisfy the equality case of (22) in a S-space-form.

First, we give the expression of Ric based on H, for those submanifolds that satisfy de equality case of (22) and in the successive corollaries we establish lower and upper bounds for the Ricci curvature.

Theorem 3 *Let M^{m+s} be a slant submanifold ($m \geq 2$) tangent to the structure fields $\xi_1, \ldots, \xi_s$ of a S-space-form $\widetilde{M}^{2m+s}(c)$ that satisfies de equality case of (22). Then, its squared mean curvature, $\|H\|^2$, and its Ricci curvature, Ric, for any unit tangent vector field, X, orthogonal to $\xi_1, \ldots, \xi_s$, are related in the following expression:*

$$\begin{aligned} Ric(X) \ = & \left(\frac{m+s}{m+2}\right)^2 \left[m\|H\|^2 + \frac{m-2}{\sin^2\theta} g^2(NX, H)\right] \\ & + m\frac{c+3s}{4} + 3\cos^2\theta\frac{c-s}{4} + s\cos^2\theta. \end{aligned} \tag{39}$$

Proof Let an adapted slant frame $\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s, e_{1^*}, \ldots, e_{m^*}\}$, and X a unit tangent vector field, X, orthogonal to $\xi_1, \ldots, \xi_s$. Then, from (8) and (14), we can write de Ricci curvature in the following way:

$$Ric(X) = \sum_i^{m+s} \widetilde{R}(X, e_i; e_i, X) - \sum_i^{m+s} g(\sigma(X, e_i), \sigma(X, e_i)) + g(\sigma(X, X), (m+s)H). \tag{40}$$

Now, from (7), we can establish that

$$\begin{aligned}
&\sum_{i=1}^{m+s} \widetilde{R}(X, e_i; e_i, X)\\
=&\sum_{i=1}^{m+s}\Big(\frac{c+3s}{4}\{fg(e_i, fe_i)g(fX, fX) - g(fX, fe_i)g(fe_i, fX)\}\\
&+\frac{c-s}{4}\{g(X, fX)g(e_i, fe_i) - g(X, fe_i)g(e_i, fX) - 2g(X, fe_i)g(e_i, fX)\}\\
&+\sum_{\alpha,\beta=1}^{s}\{g(fX, fX)\eta_\alpha(e_i)\eta_\beta(e_i) - g(fX, fe_i)\eta_\alpha(e_i)\eta_\beta(X)\\
&+g(fe_i, fe_i)\eta_\alpha(X)\eta_\beta(X) - g(fe_i, fX)\eta_\alpha(X)\eta_\beta(e_i)\}\Big)\\
=&\frac{c+3s}{4}\left[\sum_{i=1}^{m+s} 1 - \sum_{i=1}^{m+s} g^2(X, e_i)\right] + 3\frac{c-s}{4}\sum_{i=1}^{m+s} g^2(fX, e_i) + \sum_{\alpha,\beta=1}^{s}\sum_{i=1}^{m+s}\eta_\alpha^2(e_i)g(X, X)\\
=&m\frac{c+3s}{4} + 3\cos^2\theta\frac{c-s}{4} + s.
\end{aligned} \tag{41}$$

On the other hand, as M satisfies de equality case of (22), using (31) and taking into account that X is a unit vector field, we can obtain:

$$\begin{aligned}
\sum_{i=1}^{m} g(\sigma(X, e_i), \sigma(X, e_i)) &= \left(\frac{m+s}{m+2}\right)^2\Big\{\|H\|^2 + \frac{m}{\sin^2\theta}g^2(NX, H) + \|H\|^2\\
&\quad + \frac{2}{\sin^2\theta}g^2(NX, H) + \frac{2}{\sin^2\theta}g^2(NX, H) + \frac{2}{\sin^2\theta}g^2(NX, H)\Big\}\\
&= \left(\frac{m+s}{m+2}\right)^2\left\{2\|H\|^2 + \frac{m+6}{\sin^2\theta}g^2(NX, H)\right\},
\end{aligned} \tag{42}$$

for any tangent vector field $e_i \neq \xi_1, \dots, \xi_s$ in the adapted slant frame.

Now, for the terms in which $e_i = \xi_\alpha, \quad \alpha = 1, \dots, s$, taking into account (11), we have:

$$\sum_{\alpha=1}^{s} g(\sigma(X, \xi_\alpha), \sigma(X, \xi_\alpha)) = g(-NX, -NX) = s\sin^2\theta. \tag{43}$$

Moreover, using again (31), we have:

$$g(\sigma(X, X), (m+s)H) = \frac{(m+s)^2}{m+2}\left(\|H\|^2 + \frac{2}{\sin^2\theta}g^2(NX, H)\right). \tag{44}$$

Then, using (41), (42), (43) and (44) in (40), we obtain the desired expression. □

Finally, in the following corollaries, we establish lower and upper bounds for the Ricci curvature, which are obtained directly from Theorem 3.

Corollary 3 *Let M^{m+s} be a slant submanifold ($m \geq 2$) tangent to the structure fields $\xi_1, \dots, \xi_s$ of a S-space-form $\widetilde{M}^{2m+s}(c)$ that satisfies de equality case of (22). Then, its squared mean curvature, $\|H\|^2$, and its Ricci* *curvature, Ric, for any unit tangent vector field, X, orthogonal to $\xi_1, \ldots, \xi_s$, verify the following inequalities:*

$$\left(\frac{m+s}{m+2}\right)^2 m\|H\|^2 + m\frac{c+3s}{4} + 3\cos^2\theta\frac{c-s}{4} + s\cos^2\theta \leq Ric(X) \leq$$
$$\leq \left(\frac{m+s}{m+2}\right)^2 (2m-2)\|H\|^2 + m\frac{c+3s}{4} + 3\cos^2\theta\frac{c-s}{4} + s\cos^2\theta.$$

Corollary 4 *Let M^{m+s} be a slant submanifold ($m \geq 2$) tangent to the structure fields $\xi_1, \ldots, \xi_s$ of $\mathbb{R}^{2m+s}$ endowed with its standard f-structure that satisfies de equality case of (22). Then, its squared mean curvature, $\|H\|^2$, and its Ricci curvature, Ric, for any unit tangent vector field, X, orthogonal to $\xi_1, \ldots, \xi_s$, verify the following inequalities:*

$$\left(\frac{m+s}{m+2}\right)^2 m\|H\|^2 - 2s\cos^2\theta \leq Ric(X) \leq$$
$$\leq \left(\frac{m+s}{m+2}\right)^2 (2m-2)\|H\|^2 - 2s\cos^2\theta.$$

We can observe that, from Corollary 4, we easily obtain the Ricci curvature of those submanifolds introduced in Examples 2 and 3. Thus, for any tangent vector field X, $Ric(X) = -2s\cos^2\theta$.

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