



Quantitative Theorems for a Rich Class of Novel Miheşan-type Approximation Operators Incorporating the Boas-Buck Polynomials

Dhruv Bhatnagar

Received: 28 August 2021 / Accepted: 29 December 2021 / Published online: 17 January 2022
 © The Indian National Science Academy 2022

Abstract In this work, new summation-integral approximation operators based on a versatile generalization of the classic Szász-Mirakjan type operators, and incorporating the Boas-Buck polynomials are considered. We show how the proposed operators can get reduced to a multitude of operators involving classic approximation operators studied over past many decades. We nomenclate the individual cases hybrid generalizations of Bernstein, Baskakov, Lupaş and Szász-Mirakjan operators, each incorporating the Boas-Buck, Brenke, Sheffer and Appell polynomials. Indispensable properties of the proposed operators based on first and second order modulus of continuity are derived. Approximation on weighted space is also considered. In addition, quantitative Voronovskaja-type theorems have very recently been acknowledged as valuable properties for approximating functions. These form a noteworthy part of the present work.

Keywords Weighted modulus of continuity · Boas-Buck polynomials · Steklov mean

1 Introduction

Approximation of functions is of vital significance in engineering mathematics, and also as a mathematical field in its own right. The classic Szász-Mirakjan approximation operators [37] are defined, for $g : [0, \infty) \rightarrow \mathbb{R}$ for which the series is convergent, as:

$$S_n(g; y) = \frac{1}{e^{ny}} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} g\left(\frac{k}{n}\right). \quad (1)$$

In 2008, Miheşan [32] obtained the following generalized Szász-Mirakjan operators $\mathfrak{M}_n^{(\rho)}$ for $\rho \in \mathbb{R}$, $\rho + ny > 0$, by applying Gamma transform to the Szász-Mirakjan operators, which are given by:

$$\mathfrak{M}_n^{(\rho)}(g; y) = \sum_{k=0}^{\infty} \mathfrak{r}_{n,k}^{(\rho)}(y) g\left(\frac{k}{n}\right), \quad y \in [0, \infty) \quad (2)$$

where

$$\mathfrak{r}_{n,k}^{(\rho)}(y) = \frac{(\rho)_k}{k!} \frac{\left(\frac{ny}{\rho}\right)^k}{\left(1 + \frac{ny}{\rho}\right)^{\rho+k}} \quad (3)$$

Communicated by B V Rajarama Bhat.

D. Bhatnagar (✉)
 Applied Physics, Delft University of Technology (TU Delft), Delft, The Netherlands
 E-mail: dhruvbhatnagar1@gmail.com

and $(\rho)_k$ denotes the Pochhammer symbol for rising factorial of ρ , given by

$$(\rho)_k = (\rho)(\rho + 1) \dots (\rho + k - 1), (\rho)_0 = 1. \quad (4)$$

Note that $\rho \neq 0$. The operators $\mathfrak{M}_n^{(\rho)}$ were shown to be guaranteed to converge [24].

Remark 1 These operators are a landmark work because they reproduce important classical operators, studied over past many decades, in particular cases [27], [20]:

1. If $\rho = -n$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Bernstein operators [14],
2. If $\rho = n$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Baskakov operators [13],
3. If $\rho \rightarrow \infty$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Szász-Mirakjan operators [37],
4. If $\rho = ny$, $y > 0$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Lupaş operators [31] [9].

Due to their versatility, the operators $\mathfrak{M}_n^{(\rho)}$ (2) have been examined comprehensively, like [23], [27]. In [24], several indispensable results for (2) have been established. Modifications of Szász-Mirakjan operators have recently been studied in [25], [1], [23], [27], [6].

In terms of fresh developments, approximation operators which reproduce well-known classic operators have gained importance in frontier research, for e.g. [10]. In the same spirit, in [20], [19], it has been demonstrated how the operators based on generalized Szász-Mirakjan operators due to Miheşan (2) (considered in the present work) get reduced to a multitude of well-known operators studied over past many decades.

Summation-integral type operators have also been intensively studied recently, and Voronovskaya type theorems were obtained in quantitative forms. Some recent comprehensive literature on these is [2], [4], [26], [7], [30], [16]

Let us turn our attention to approximation involving the Appell polynomials and their generalizations like Brenke polynomials and Boas-Buck polynomials. Let $U(\mu)$, $V(\mu)$ and $W(\mu)$ be analytic functions of the form

$$U(\mu) = \sum_{j=0}^{\infty} a_j \mu^j, a_0 \neq 0, \quad (5)$$

$$V(\mu) = \sum_{j=0}^{\infty} b_j \mu^j, b_j \neq 0, \quad (6)$$

$$W(\mu) = \sum_{j=0}^{\infty} h_j \mu^j, h_1 \neq 0. \quad (7)$$

Jakimovski and Leviatan's classic work [22] introduced generalized Szász-Mirakjan-Appell operators, given as:

$$P_n(\mu, y) = \frac{e^{-ny}}{U(1)} \sum_{j=0}^{\infty} \mathfrak{H}_j(ny) \mu \left(\frac{k}{n} \right), \quad (8)$$

where $\mathfrak{H}_j(y)$ are the Appell polynomials, defined as $U(\mu)e^{y\mu} = \sum_{j=0}^{\infty} \mathfrak{H}_j(y) \mu^j$ and $U(\mu)$ is as defined before.

Varma in [41] put forth an operator inspired by Szász-Mirakjan operators and incorporating the Brenke polynomials, $\mathfrak{v}_j(y)$:

$$L_n(g; y) := \frac{1}{U(1)V(ny)} \sum_{k=0}^{\infty} \mathfrak{v}_k(ny) g \left(\frac{k}{n} \right), \quad y \geq 0, n \in \mathbb{N} \quad (9)$$

where the Brenke polynomials $\sum_{j=0}^{\infty} \mathfrak{v}_j(y) \mu^j = U(\mu)V(y\mu)$ et al.

In a very recent development [40], Sucu et al. proposed operators formulated from Szász-Mirakjan operators involving the Boas-Buck polynomials:



$$B_n(g; y) = \frac{1}{U(1)V(nyW(1))} \sum_{k=0}^{\infty} p_k(ny) g\left(\frac{k}{n}\right), \quad y > 0, n \in \mathbb{N} \quad (10)$$

where the generating relation, $p_j(y)$, is given by

$$U(\mu)V(yW(\mu)) = \sum_{j=0}^{\infty} p_j(y)\mu^j \quad (11)$$

with U , V and W as specified in (5)–(7).

Remark 2 For convergence, the operators (10) were assumed to satisfy:

1. $U(1) \neq 0$, $W^{(1)} = 1$, $p_j(y) \geq 0$, $j = 0, 1, 2, \dots$,
2. $V : \mathbb{R} \rightarrow (0, \infty)$,
3. (11), (5)–(7) converge for $|\mu| < R$, $R > 1$.

Similar to the work of [40], Sidharth et al. [38] recently proposed and investigated the properties of a Szász-Mirakjan-Durrmeyer operator involving the Boas-Buck polynomials:

$$M_n(g; y) = \frac{1}{U(1)V(nyW(1))} \sum_{k=1}^{\infty} \frac{p_k(ny)}{\beta(k, n+1)} \int_0^{\infty} \frac{\mu^{k-1}}{(1+\mu)^{n+k+1}} g(\mu) d\mu \\ + \frac{a_0 b_0}{U(1)V(nyW(1))} f(0), \quad (12)$$

where $\beta(k, n+1)$ is the beta function and $y > 0$, $n \in \mathbb{N}$. For additional significant and recent work involving approximation via these polynomial classes, consult [39], [33], [11], [41], [42], [12], [36], [15], [35], [34].

Remark 3 It is important to note that in special cases, the Boas-Buck polynomials get reduced to well-studied polynomials as follows:

1. In (11), let $W(\mu) = \mu$. We get the Brenke polynomials.
2. In (11), let $V(\mu) = e^{\mu}$. We get the Sheffer polynomials.
3. In (11), let $V(\mu) = e^{\mu}$, $W(\mu) = \mu$. We get the Appell polynomials.

Thus, approximation operators involving the Boas-Buck polynomials form a rich class, and the results derived for these can be easily get reduced to the results for operators based on the aforementioned polynomials as well.

Therefore, motivated by [40], [19], [20], [38], we propose a new summation-integral operator formulated by Durrmeyer-type modification of (2) and the Boas-Buck polynomials. The main contribution of this study is that the proposed operators can reproduce a large number of approximation operators, based on functions studied in past decades. Thus, the properties of the proposed operators can serve as a general results for these special cases, for which these results can be derived with ease. We dedicate a discussion to this later. The remaining paper contains important results for the proposed operator on uniform convergence, Voronovskaja-type theorem, results involving the usual modulus of continuity, and approximation on weighted space. Further, quantitative Voronovskaja-type theorems have very recently been acknowledged as valuable properties for approximating functions [8]. These form a noteworthy part of this work.

2 Theoretical Framework

2.1 Construction of the Proposed Operator

Let $\gamma > 0$, $C_{\gamma}[0, \infty)$ be the space $\{g \in C[0, \infty) : |g(\mu)| \leq M(1 + \mu^{\gamma}) \text{ for some } M > 0\}$ equipped with the norm

$$\|g\|_{\gamma} = \sup_{\mu \in [0, \infty)} \frac{|g(\mu)|}{1 + \mu^{\gamma}}.$$



Then, for $g \in C_\gamma[0, \infty)$, we propose the following novel approximation operator:

$$\begin{aligned} \mathfrak{R}_n^{(\rho)}(g; y) = & \frac{n(\rho-1)}{\rho} \frac{1}{U(1)V(nyW(1))} \sum_{k=1}^{\infty} p_k(ny) \int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) g(\mu) d\mu \\ & + \frac{1}{U(1)V(nyW(1))} p_0(ny) g(0). \end{aligned} \quad (13)$$

Remark 4 The proposed operators (13) are a generalization of the operators in [22], [41], [38]. Further, using Remark 1 and Remark 3, we have the following particular cases for $\mathfrak{R}_n^{(\rho)}$:

1. In (13), let $\rho = -n$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Boas-Buck polynomials.
2. In (13), let $\rho = n$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Boas-Buck polynomials.
3. In (13), let $\rho \rightarrow \infty$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Boas-Buck polynomials.
4. In (13), let $\rho = ny$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Boas-Buck polynomials.
5. In (13), let $\rho = -n$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Brenke polynomials.
6. In (13), let $\rho = n$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Brenke polynomials.
7. In (13), let $\rho \rightarrow \infty$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Brenke polynomials.
8. In (13), let $\rho = ny$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Brenke polynomials.
9. In (13), let $\rho = -n$ and $V(\mu) = e^\mu$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Sheffer polynomials.
10. In (13), let $\rho = n$ and $V(\mu) = e^\mu$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Sheffer polynomials.
11. In (13), let $\rho \rightarrow \infty$ and $V(\mu) = e^\mu$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Sheffer polynomials.
12. In (13), let $\rho = ny$ and $V(\mu) = e^\mu$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Sheffer polynomials.
13. In (13), let $\rho = -n$ and $V(\mu) = e^\mu$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Appell polynomials.
14. In (13), let $\rho = n$ and $V(\mu) = e^\mu$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Appell polynomials.
15. In (13), let $\rho \rightarrow \infty$ and $V(\mu) = e^\mu$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Appell polynomials.
16. In (13), let $\rho = ny$ and $V(\mu) = e^\mu$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Appell polynomials.

2.2 Some Auxiliary Results

Lemma 1 [40], [38] For the Boas-Buck polynomials (11), we have the following useful results :

1. $\sum_{j=0}^{\infty} p_j(ny) = U(1)V(nyW(1))$
2. $\sum_{j=0}^{\infty} j p_j(ny) = [U^{(1)}(1)V(nyW(1))] + ny[U(1)V^{(1)}(nyW(1))]$



$$\begin{aligned}
3. \quad \sum_{j=0}^{\infty} j^2 p_j(ny) &= [U^{(2)}(1) + U^{(1)}(1)]V(nyW(1)) \\
&\quad + [2U^{(1)}(1) + U(1) + U(1)W^{(2)}(1)]V^{(1)}(nyW(1))(ny) + U(1)V^{(2)}(nyW(1))(ny)^2 \\
4. \quad \sum_{j=0}^{\infty} j^3 p_j(ny) &= [4U^{(2)}(1) + U^{(1)}(1)]V(nyW(1)) + [6U^{(1)}(1) + U(1) \\
&\quad + 3U(1)W^{(2)}(1) + 3U^{(2)}(1) + 3U^{(1)}(1)W^{(2)}(1) + U(1)W^{(3)}(1)]V^{(1)}(nyW(1))(ny) \\
&\quad + [3U(1) + 3U^{(1)}(1) + 3U(1)W^{(2)}(1)]V^{(2)}(nyW(1))(ny)^2 + [U(1)]V^{(3)}(nyW(1))(ny)^3 \\
5. \quad \sum_{j=0}^{\infty} j^4 p_j(ny) &= [13U^{(2)}(1) + U^{(1)}(1) + U^{(4)}(1)]V(nyW(1)) + [4U^{(3)}(1) + 6U^{(2)}(1)W^{(2)}(1) \\
&\quad + 4U^{(1)}(1)W^{(3)}(1) + U(1)W^{(4)}(1) + 36U^{(1)}(1) + U(1) + 7U(1)W^{(2)}(1) \\
&\quad + 18U^{(2)}(1) + 18U^{(1)}W^{(2)}(1) + 6U(1)W^{(3)}(1) - 22U^{(1)}(1)]V^{(1)}(nyW(1))(ny) \\
&\quad + [6U^{(2)}(1) + 12W^{(2)}(1) + U^{(1)}(1) \\
&\quad + 4U(1)W^{(3)}(1) + 3U(1)[W^{(2)}(1)]^2 + 7U(1) + 18U^{(1)}(1) \\
&\quad + 18U(1)W^{(2)}(1)]V^{(2)}(nyW(1))(ny)^2 \\
&\quad + [4U^{(1)}(1) + 6U(1)W^{(2)}(1) + 6U(1)]V^{(3)}(nyW(1))(ny)^3 \\
&\quad + [U(1)]V^{(4)}(nyW(1))(ny)^4.
\end{aligned}$$

Lemma 2 For the moments of the form

$$\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^r d\mu, \quad r = 0, 1, \dots, 4,$$

where $\mathfrak{r}_{n,k-1}^{(\rho)}(\mu)$ is as in (3), we have the following results:

$$\begin{aligned}
1. \quad \int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) d\mu &= \frac{\rho}{n(\rho-1)} \\
2. \quad \int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu d\mu &= \frac{\rho^2 k}{n^2(\rho-1)(\rho-2)} \\
3. \quad \int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^2 d\mu &= \frac{\rho^3 k(k+1)}{n^3(\rho-1)(\rho-2)(\rho-3)} \\
4. \quad \int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^3 d\mu &= \frac{\rho^4 k(k+1)(k+2)}{n^4(\rho-1)(\rho-2)(\rho-3)(\rho-4)} \\
5. \quad \int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^4 d\mu &= \frac{\rho^5 k(k+1)(k+2)(k+3)}{n^5(\rho-1)(\rho-2)(\rho-3)(\rho-4)(\rho-5)}.
\end{aligned}$$

Proof All parts follow a simple and direct computation. Part (3.) will be proved, and other parts follow likewise. Consider

$$\begin{aligned}
\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^2 d\mu &= \int_0^{\infty} \frac{(\rho)_{k-1}}{(k-1)!} \frac{\left(\frac{n\mu}{\rho}\right)^{k-1}}{\left(1 + \frac{n\mu}{\rho}\right)^{\rho+k-1}} \mu^2 d\mu \\
&= \frac{(\rho)_{k-1}}{(k-1)!} \frac{\rho^2}{n^2} \int_0^{\infty} \frac{\left(\frac{n\mu}{\rho}\right)^{k+1}}{\left(1 + \frac{n\mu}{\rho}\right)^{\rho+k-1}} d\mu.
\end{aligned} \tag{14}$$

Let $\mathfrak{l} = \frac{n\mu}{\rho}$, $d\mu = \frac{\rho}{n} d\mathfrak{l}$. Then

$$\begin{aligned}
\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^2 d\mu &= \frac{(\rho)_{k-1}}{(k-1)!} \frac{\rho^2}{n^2} \frac{\rho}{n} \int_0^{\infty} \frac{(\mathfrak{l})^{k+1}}{(1 + \mathfrak{l})^{\rho+k-1}} d\mathfrak{l} \\
&= \frac{(\rho)_{k-1}}{(k-1)!} \frac{\rho^3}{n^3} \beta(k+2, \rho-3),
\end{aligned} \tag{15}$$

where $\beta(m, n)$ is the beta function of second kind defined as:

$$\int_0^{\infty} \frac{\mathfrak{z}^{m-1}}{(1 + \mathfrak{z})^{m+n}} d\mathfrak{z} = \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{16}$$



Then, substituting for $\beta(k+2, \rho-3)$ in (15), some simplification leads to

$$\int_0^\infty \mathfrak{r}_{n,k-1}^{(\rho)}(\mu) \mu^2 d\mu = \frac{\rho^3}{n^3} \frac{(k+1)(k)}{(\rho-1)(\rho-2)(\rho-3)}. \quad (17)$$

□

2.3 Results on Moments

Using Lemmas 1 and 2, we present the results on moments of (13).

Lemma 3 For the operators $\mathfrak{R}_n^{(\rho)}(g; y)$, the moments are given by:

1. $\mathfrak{R}_n^{(\rho)}(1; y) = 1$
2. $\mathfrak{R}_n^{(\rho)}(\mu^1; y) = \frac{\rho}{n(\rho-2)} \left[\frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + \frac{U^{(1)}(1)}{U(1)} \right]$
3. $\mathfrak{R}_n^{(\rho)}(\mu^2; y) = \frac{\rho^2}{n^2(\rho-2)(\rho-3)} \left[\frac{V^{(2)}(nyW(1))}{V(nyW(1))}(ny)^2 + \left(2\frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + 2\frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)} \right]$
4. $\mathfrak{R}_n^{(\rho)}(\mu^3; y) = \frac{\rho^3}{n^3(\rho-2)(\rho-3)(\rho-4)} \left[\frac{V^{(3)}(nyW(1))}{V(nyW(1))}(ny)^3 + \left(3\frac{U^{(1)}(1)}{U(1)} + 6 + 3W^{(2)}(1) \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + \frac{V^{(2)}(nyW(1))}{V(nyW(1))}(ny)^2 + \left(12\frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 3\frac{U^{(1)}(1)}{U(1)}W^{(2)}(1) + W^{(3)}(1) + 4 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + 7\frac{U^{(2)}(1)}{U(1)} + 6\frac{U^{(1)}(1)}{U(1)} \right]$
5. $\mathfrak{R}_n^{(\rho)}(\mu^4; y) = \frac{\rho^4}{n^4(\rho-2)(\rho-3)(\rho-4)(\rho-5)} \left[\frac{V^{(4)}(nyW(1))}{V(nyW(1))}(ny)^4 + \left(4\frac{U^{(1)}(1)}{U(1)} + 6W^{(2)}(1) + 12 \right) \frac{V^{(3)}(nyW(1))}{V(nyW(1))}(ny)^3 + \left(6\frac{U^{(2)}(1)}{U(1)} + 12\frac{W^{(2)}(1)}{U(1)}U^{(1)}(1) + 21\frac{U^{(1)}(1)}{U(1)} + 3\frac{W^{(2)}(1)}{U(1)} + 4W^{(3)}(1) + 18W^{(2)}(1) + 3[W^{(2)}(1)]^2 + 21 \right) \frac{V^{(2)}(nyW(1))}{V(nyW(1))}(ny)^2 + \left(4\frac{U^{(2)}(1)}{U(1)} + 6\frac{U^{(2)}(1)}{U(1)}W^{(1)}(1) + 36\frac{U^{(2)}(1)}{U(1)}W^{(2)}(1) + 42\frac{U^{(1)}(1)}{U(1)} + 4\frac{U^{(1)}(1)}{U(1)}W^{(3)}(1) + 36\frac{U^{(2)}(1)}{U(1)} + W^{(4)}(1) + 12W^{(3)}(1) + 36W^{(2)}(1) + 24 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + \frac{U^{(4)}(1)}{U(1)} + 48\frac{U^{(2)}(1)}{U(1)} + 13\frac{U^{(1)}(1)}{U(1)} + 11 \right].$

Proof In view of the results in 2, 1, all parts are the result of a direct calculation, thus omitted. □



2.4 Results on Central Moments

Lemma 4 Let $\mathfrak{T}_{n,r}^{(\rho)}(y) := \mathfrak{R}_n^{(\rho)}((\mu - y)^r; y)$ nomenclate the r^{th} central moment, $r = 0, 1, \dots, 4$. Using the fact that the operator $\mathfrak{R}_n^{(\rho)}$ is a positive linear operator, from Lemma 3, the following results on central moments can be obtained:

1. $\mathfrak{T}_{n,0}^{(\rho)}(y) = 1$
2. $\mathfrak{T}_{n,1}^{(\rho)}(y) = \left[\frac{\rho}{(\rho-2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 1 \right] (y) + \frac{\rho}{n(\rho-2)} \frac{U^{(1)}(1)}{U(1)}$
3. $\mathfrak{T}_{n,2}^{(\rho)}(y) = \left[\frac{\rho^2}{(\rho-2)(\rho-3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{(\rho-2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1 \right] (y)^2$
 $+ \left[\frac{\rho^2}{n(\rho-2)(\rho-3)} \left(\frac{2U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{n(\rho-2)} \frac{U^{(1)}(1)}{U(1)} \right] (y)$
 $+ \left[\frac{\rho^2}{n^2(\rho-2)(\rho-3)} \left(2 \frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)} \right) \right]$
4. $\mathfrak{T}_{n,4}^{(\rho)}(y) = (y)^4 \left[\frac{\rho^4}{(\rho-2)(\rho-3)(\rho-4)(\rho-5)} \frac{V^{(4)}(nyW(1))}{V(nyW(1))} \right.$
 $- 4 \frac{\rho^3}{(\rho-2)(\rho-3)(\rho-4)} \frac{V^{(3)}(nyW(1))}{V(nyW(1))}$
 $+ 6 \frac{\rho^2}{(\rho-2)(\rho-3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - 4 \frac{\rho}{(\rho-2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1 \left. \right]$
 $+ (y)^3 \left[\frac{\rho^4}{n(\rho-2)(\rho-3)(\rho-4)(\rho-5)} \left(4 \frac{U^{(1)}(1)}{U(1)} + 6W^{(2)}(1) + 12 \right) \frac{V^{(3)}(nyW(1))}{V(nyW(1))} \right.$
 $- 4 \frac{\rho^3}{n(\rho-2)(\rho-3)(\rho-4)} \left(3 \frac{U^{(1)}(1)}{U(1)} + 6 + 3W^{(2)}(1) \right) \frac{V^{(2)}(nyW(1))}{V(nyW(1))}$
 $+ 6 \frac{\rho^2}{n(\rho-2)(\rho-3)} \left(2 \frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 4 \frac{\rho}{n(\rho-2)} \frac{U^{(1)}(1)}{U(1)} \left. \right]$
 $+ (y)^2 \left[\frac{\rho^4}{n^2(\rho-2)(\rho-3)(\rho-4)(\rho-5)} \left(6 \frac{U^{(2)}(1)}{U(1)} + 12 \frac{U^{(1)}(1)}{U(1)} W^{(2)}(1) + 21 \frac{U^{(1)}(1)}{U(1)} \right. \right.$
 $+ 3 \frac{W^{(2)}(1)}{U(1)} + 4W^{(3)}(1) + 18W^{(2)}(1) + 3[W^{(2)}(1)]^2 + 21 \left. \right) \frac{V^{(2)}(nyW(1))}{V(nyW(1))}$
 $- 4 \frac{\rho^3}{n^2(\rho-2)(\rho-3)(\rho-4)} \left(12 \frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 3 \frac{U^{(1)}(1)}{U(1)} W^{(2)}(1) + W^{(3)}(1) + 4 \right)$
 $\cdot \frac{V^{(1)}(nyW(1))}{V(nyW(1))} \left. \right]$
 $+ (y) \left[\frac{\rho^4}{n^3(\rho-2)(\rho-3)(\rho-4)(\rho-5)} \left(4 \frac{U^{(2)}(1)}{U(1)} + 6 \frac{U^{(2)}(1)}{U(1)} W^{(1)}(1) \right. \right.$
 $+ 36 \frac{U^{(2)}(1)}{U(1)} W^{(2)}(1) + 42 \frac{U^{(1)}(1)}{U(1)} + 4 \frac{U^{(1)}(1)}{U(1)} W^{(3)}(1) + 36 \frac{U^{(2)}(1)}{U(1)} + W^{(4)}(1)$
 $+ 12W^{(3)}(1) + 36W^{(2)}(1) + 24 \left. \right) - 4 \frac{\rho^3}{n^3(\rho-2)(\rho-3)(\rho-4)}$



$$\cdot \left(7 \frac{U^2(1)}{U(1)} + 6U^1(1)U(1) \right) \Bigg] \\ + \left[\frac{\rho^4}{n^4 (\rho - 2) (\rho - 3) (\rho - 4) (\rho - 5)} \left(\frac{U^{(4)}(1)}{U(1)} + 48 \frac{U^{(2)}(1)}{U(1)} + 13 \frac{U^{(1)}(1)}{U(1)} + 11 \right) \right].$$

The result for $\mathfrak{R}_n^{(\rho)}((\mu - y)^6; y)$, which is omitted as it is quite complicated and lengthy, will be needed to establish the quantitative Voronovskaja-type theorem later.

Remark 5 For the purpose of the results presented in the paper, we make the following assumptions:

1. $\rho = \rho(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \frac{n}{\rho} = q \in \mathbb{R}$
2. $\lim_{\mu \rightarrow \infty} \frac{V^{(k)}(\mu)}{V(\mu)} = 1, k \in \mathbb{N}, k \geq 1$
3. $\lim_{n \rightarrow \infty} n \left[\frac{\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 1 \right] = m_1(y)$
4. $\lim_{n \rightarrow \infty} n \left[\frac{\rho^2}{(\rho - 2)(\rho - 3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1 \right] = m_2(y)$
5. $\lim_{n \rightarrow \infty} n^2 \left[\frac{\rho^4}{n(\rho - 2)(\rho - 3)(\rho - 4)(\rho - 5)} \left(4 \frac{U^{(1)}(1)}{U(1)} + 6W^{(2)}(1) + 12 \right) \frac{V^{(3)}(nyW(1))}{V(nyW(1))} \right. \\ \left. - 4 \frac{\rho^3}{n(\rho - 2)(\rho - 3)(\rho - 4)} \left(3 \frac{U^{(1)}(1)}{U(1)} + 6 + 3W^{(2)}(1) \right) \frac{V^{(2)}(nyW(1))}{V(nyW(1))} \right. \\ \left. + 6 \frac{\rho^2}{n(\rho - 2)(\rho - 3)} \left(2 \frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 4 \frac{\rho}{n(\rho - 2)} \frac{U^{(1)}(1)}{U(1)} \right] = m_3(y)$
6. $\lim_{n \rightarrow \infty} n^2 \left[\frac{\rho^4}{(\rho - 2)(\rho - 3)(\rho - 4)(\rho - 5)} \frac{V^{(4)}(nyW(1))}{V(nyW(1))} - 4 \frac{\rho^3}{(\rho - 2)(\rho - 3)(\rho - 4)} \frac{V^{(3)}(nyW(1))}{V(nyW(1))} \right. \\ \left. + 6 \frac{\rho^2}{(\rho - 2)(\rho - 3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - 4 \frac{\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1 \right] = m_4(y).$

Under these assumptions, we have

1. $\lim_{n \rightarrow \infty} n \mathfrak{T}_{n,1}^{(\rho)}(y) = m_1(y)y + \frac{U^{(1)}(1)}{U(1)}$
2. $\lim_{n \rightarrow \infty} n \mathfrak{T}_{n,2}^{(\rho)}(y) = m_2(y)y^2 + (W^{(2)}(1) + 2)y = v_1(y),$
say.
3. $\lim_{n \rightarrow \infty} n^2 \mathfrak{T}_{n,4}^{(\rho)}(y)$
 $= m_4(y)y^4 + m_3(y)y^3 + \left(6 \frac{U^{(2)}(1)}{U(1)} + 14W^{(2)}(1) \right. \\ \left. + 3[W^{(2)}(1)]^2 - 27 \frac{U^{(1)}(1)}{U(1)} + \frac{W^{(2)}(1)}{U(1)} + 5 \right) y^2 = v_2(y),$
say.



3 Direct Results

The following can be immediately established due to the foregoing results.

3.1 Uniform Convergence

Theorem 1 Let $g \in C_K[0, \infty)$ and $\rho = \rho(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} \mathfrak{R}_n^{(\rho)}(g; y) = g(y), \quad (18)$$

uniformly in each compact subset of $[0, \infty)$.

Proof From lemma 3 and using the assumptions in the Remark 2, as $n \rightarrow \infty$, $\mathfrak{R}_n^{(\rho)}(1; y) = 1$, $\mathfrak{R}_n^{(\rho)}(\mu; y) \rightarrow y$, $\mathfrak{R}_n^{(\rho)}(\mu^2; y) \rightarrow y^2$ uniformly in each compact subset of $[0, \infty)$. Thus, from the Bohman-Korovkin theorem, $\mathfrak{R}_n^{(\rho)}(g; y) \rightarrow g(y)$ as $n \rightarrow \infty$ for any g , uniformly in each compact subset of $[0, \infty)$. $\square$

3.2 Voronovskaja-type Theorem

Theorem 2 Let $g \in C_K[0, \infty)$ and $\rho = \rho(n) \rightarrow \infty$ as $n \rightarrow \infty$. If $g^{(2)}$ exists at $y \in [0, \infty)$ and

$$\lim_{n \rightarrow \infty} \frac{n}{\rho(n)} = q \in \mathbb{R},$$

then the following holds:

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \left[\mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right] \\ &= \left[m_1(y)y + \frac{U^{(1)}(1)}{U(1)} \right] g^{(1)}(y) + \left[m_2(y)y^2 + \left(W^{(2)}(1) + 2 \right) y \right] \frac{g^{(2)}(y)}{2}. \end{aligned} \quad (19)$$

Proof Consider Taylor's expansion in the form

$$g(\mu) = \sum_{r=0}^2 \frac{1}{r!} g^{(r)}(y) (\mu - y)^r + \xi(\mu, y) (\mu - y)^2, \quad (20)$$

where $\xi(\mu, y)$ is a function such that $\lim_{\mu \rightarrow y} \xi(\mu, y) = 0$. Therefore, operating by $\mathfrak{R}_n^{(\rho)}$ on (20),

$$\mathfrak{R}_n^{(\rho)}(g(\mu); y) - g(y) = \sum_{r=1}^2 \frac{1}{r!} g^{(r)}(y) \mathfrak{T}_{n,r}^{(\rho)}(y) + \mathfrak{R}_n^{(\rho)}(\xi(\mu, y)(\mu - y)^2; y). \quad (21)$$

From the Cauchy-Schwarz inequality,

$$n \mathfrak{R}_n^{(\rho)}(\xi(\mu, y)(\mu - y)^2; y) \leq \sqrt{\mathfrak{R}_n^{(\rho)}(\xi^2(\mu, y); y)} \sqrt{n^2 \mathfrak{T}_{n,4}^{(\rho)}(y)}. \quad (22)$$

By Theorem 1,

$$\lim_{n \rightarrow \infty} \mathfrak{R}_n^{(\rho)}(\xi^2(\mu, y); y) = 0. \quad (23)$$

Using Remark 5 and (23) above, in (22), we get:

$$\lim_{n \rightarrow \infty} n \mathfrak{R}_n^{(\rho)}(\xi(\mu, y)(\mu - y)^2; y) = 0. \quad (24)$$

Hence, substituting the values of central moments from lemma 4, we get:

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \left[\mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right] \\ &= \left[m_1(y)y + \frac{U^{(1)}(1)}{U(1)} \right] g^{(1)}(y) + \left[m_2(y)y^2 + \left(W^{(2)}(1) + 2 \right) y \right] \frac{g^{(2)}(y)}{2}. \end{aligned} \quad (25)$$

$\square$



3.3 Local Approximation Properties

Preliminaries Let $\tilde{C}_B[0, \infty)$ depict the space of all real valued, bounded, and uniformly continuous functions g on $[0, \infty)$, with

$$\|g\|_{\tilde{C}_B[0, \infty)} = \sup_{y \in [0, \infty)} |g(y)| \quad (26)$$

being the norm on $\tilde{C}_B[0, \infty)$. The modulus of continuity of $g \in \tilde{C}_B[0, \infty)$ is defined as

$$\omega(g; \mathfrak{d}) = \sup_{y, \mathfrak{a}, \mathfrak{b} \geq 0} \sup_{|\mathfrak{a} - \mathfrak{b}| \leq \mathfrak{d}} |g(y + \mathfrak{a}) - g(y + \mathfrak{b})|, \mathfrak{d} \geq 0 \quad (27)$$

and the second order modulus of continuity is defined as

$$\omega_2(g; \mathfrak{d}) = \sup_{y, \mathfrak{a}, \mathfrak{b} \geq 0} \sup_{|\mathfrak{a} - \mathfrak{b}| \leq \mathfrak{d}} |g(y + 2\mathfrak{a}) - 2g(y + \mathfrak{a} + \mathfrak{b}) + g(y + 2\mathfrak{b})|, \mathfrak{d} \geq 0. \quad (28)$$

For $g \in \tilde{C}_B[0, \infty)$, the Steklov mean is defined as [18]:

$$g_{\mathfrak{h}}(y) = \frac{4}{\mathfrak{h}^2} \int_0^{\frac{\mathfrak{h}}{2}} \int_0^{\frac{\mathfrak{h}}{2}} [2g(y + \mathfrak{a} + \mathfrak{b}) - g(y + 2(\mathfrak{a} + \mathfrak{b}))] d\mathfrak{a} d\mathfrak{b}. \quad (29)$$

The following properties related to the Steklov mean can be observed, [18]:

1. $\|g_{\mathfrak{h}} - g\|_{\tilde{C}_B[0, \infty)} \leq \omega_2(g; \mathfrak{h})$
2. $g_{\mathfrak{h}}^{(1)}, g_{\mathfrak{h}}^{(2)} \in \tilde{C}_B[0, \infty); \|g_{\mathfrak{h}}^{(1)}\|_{\tilde{C}_B[0, \infty)} \leq \frac{5}{\mathfrak{h}} \omega(g; \mathfrak{h});$
 $\|g_{\mathfrak{h}}^{(2)}\|_{\tilde{C}_B[0, \infty)} \leq \frac{9}{\mathfrak{h}^2} \omega_2(g; \mathfrak{h})$

Theorem 3 Let $g \in \tilde{C}_B[0, \infty)$. Then, for every $y \geq 0$, the following inequality holds:

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \leq 5\omega \left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \right) + \frac{13}{2} \omega_2 \left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \right) \quad (30)$$

Proof Using the definition of Steklov mean from (29), the following can be written:

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \leq \mathfrak{R}_n^{(\rho)}(|g - g_{\mathfrak{h}}|; y) + \left| \mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}} - g_{\mathfrak{h}}(y); y) \right| + |g_{\mathfrak{h}}(y) - g(y)| \quad (31)$$

Also, for every $g \in \tilde{C}_B[0, \infty)$, we have

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) \right| \leq \|g\|_{\tilde{C}_B[0, \infty)} \quad (32)$$

Consider the first term in (31). Using (32) and then property (1.) of Steklov mean, we get

$$\begin{aligned} \mathfrak{R}_n^{(\rho)}(|g - g_{\mathfrak{h}}|; y) &\leq \|\mathfrak{R}_n^{(\rho)}(g - g_{\mathfrak{h}}; y)\|_{\tilde{C}_B[0, \infty)} \\ &\leq \|g - g_{\mathfrak{h}}\|_{\tilde{C}_B[0, \infty)} \\ &\leq \omega_2(g; h) \end{aligned} \quad (33)$$

Now, consider the second term in (31). Expanding $g_{\mathfrak{h}}(\mu)$ as a Taylor series upto second derivative term, we get

$$g_{\mathfrak{h}}(\mu) = \sum_{r=0}^2 g_{\mathfrak{h}}^{(r)} \frac{(\mu - y)^r}{r!} + \xi(\mu, y)((\mu - y)^2) \quad (34)$$

Therefore



$$\begin{aligned} \left| \mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}} - g_{\mathfrak{h}}(y); y) \right| &\approx \left| \mathfrak{R}_n^{(\rho)}((\mu - y)g_{\mathfrak{h}}^{(1)}(y); y) + \mathfrak{R}_n^{(\rho)}\left(\frac{(\mu - y)^2}{2}g_{\mathfrak{h}}^{(2)}(y); y\right) \right| \\ &\leq \left| \mathfrak{R}_n^{(\rho)}((\mu - y)g_{\mathfrak{h}}^{(1)}(y); y) \right| + \left| \mathfrak{R}_n^{(\rho)}\left(\frac{(\mu - y)^2}{2}g_{\mathfrak{h}}^{(2)}(y); y\right) \right| \end{aligned} \quad (35)$$

By the definition of supremum norm and linearity of $\mathfrak{R}_n^{(\rho)}(g; y)$, we can write:

$$\left| \mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}}(\mu) - g_{\mathfrak{h}}(y); y) \right| \leq \|g_{\mathfrak{h}}^{(1)}\|_{\tilde{C}_B[0, \infty)} \left| \mathfrak{T}_{n,1}^{(\rho)}(y) \right| + \frac{1}{2} \|g_{\mathfrak{h}}^{(2)}\|_{\tilde{C}_B[0, \infty)} \left| \mathfrak{T}_{n,2}^{(\rho)}(y) \right| \quad (36)$$

Using Cauchy-Schwarz inequality on the first term, we get:

$$\left| \mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}} - g_{\mathfrak{h}}(y); y) \right| \leq \|g_{\mathfrak{h}}^{(1)}\| \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} + \frac{1}{2} \|g_{\mathfrak{h}}^{(2)}\| \mathfrak{T}_{n,2}^{(\rho)}(y) \quad (37)$$

Now, consider the third term in (31). Using property (1.) of Steklov mean:

$$\left| g_{\mathfrak{h}}(y) - g(y) \right| \leq \|g_{\mathfrak{h}} - g\|_{\tilde{C}_B[0, \infty)} \leq \mathfrak{w}_2(g; \mathfrak{h}) \quad (38)$$

Using (33), (37), (38) in (31), and choosing $\mathfrak{h} = \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$ gives:

$$\begin{aligned} \left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| &\leq \mathfrak{w}_2\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) + \|g_{\mathfrak{h}}^{(1)}\| \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \\ &\quad + \frac{1}{2} \|g_{\mathfrak{h}}^{(2)}\| \mathfrak{T}_{n,2}^{(\rho)}(y) + \mathfrak{w}_2\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) \end{aligned}$$

Finally, using property (2.) of Steklov mean gives

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \leq 5\mathfrak{w}\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) + \frac{13}{2}\mathfrak{w}_2\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

□

Theorem 4 For any $g \in \tilde{C}_B^1[0, \infty)$ and $y \in \mathbb{R}_+ \cup \{0\}$, we have

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \leq 2\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \mathfrak{w}\left(g^{(1)}; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

Proof Using

$$\int_y^\mu \left(g^{(1)}(\mathfrak{a}) - g^{(1)}(y) \right) d\mathfrak{a} = g(\mu) - g(y) - g^{(1)}(y)(\mu - y)$$

we get

$$g(\mu) - g(y) = g^{(1)}(y)(\mu - y) + \int_y^\mu \left(g^{(1)}(\mathfrak{a}) - g^{(1)}(y) \right) d\mathfrak{a} \quad (39)$$

Operate $\mathfrak{R}_n^{(\rho)}$ on both sides

$$\mathfrak{R}_n^{(\rho)}((g(\mu) - g(y)); y) = g^{(1)}(y)\mathfrak{T}_{n,1}^{(\rho)}(y) + \mathfrak{R}_n^{(\rho)}\left(\int_y^\mu \left(g^{(1)}(\mathfrak{a}) - g^{(1)}(y) \right) d\mathfrak{a}; y\right)$$

from properties of modulus of continuity,

$$|g(\mu) - g(y)| \leq \mathfrak{w}(g; \mathfrak{d}) \left(\frac{|\mu - y|}{\mathfrak{d}} + 1 \right), \quad \mathfrak{d} > 0$$

in the form

$$\left| g^{(1)}(\mathfrak{a}) - g^{(1)}(y) \right| \leq \mathfrak{w}(g^{(1)}; \mathfrak{d}) \left(\frac{|\mathfrak{a} - y|}{\mathfrak{d}} + 1 \right), \quad \mathfrak{d} > 0$$



after integrating both sides and simplifying, we obtain:

$$\left| \int_y^\mu (g^{(1)}(a) - g^{(1)}(y)) du \right| \leq \mathfrak{w}(g^{(1)}; \mathfrak{d}) \left(\frac{|\mu - y|^2}{\mathfrak{d}} + |\mu - y| \right)$$

Therefore

$$\begin{aligned} & \left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \\ & \leq \left| g^{(1)}(y) \right| \left| \mathfrak{T}_{n,1}^{(\rho)}(y) \right| + \mathfrak{R}_n^{(\rho)} \left(\mathfrak{w}(g^{(1)}; \mathfrak{d}) \left(\frac{|\mu - y|^2}{\mathfrak{d}} + |\mu - y| \right); y \right) \end{aligned}$$

that is,

$$\begin{aligned} & \left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \\ & \leq \left| g^{(1)}(y) \right| \left| \mathfrak{T}_{n,1}^{(\rho)}(y) \right| + \mathfrak{w}(g^{(1)}; \mathfrak{d}) \left(\frac{1}{\mathfrak{d}} \mathfrak{R}_n^{(\rho)}(|\mu - y|^2; y) + \mathfrak{R}_n^{(\rho)}(|\mu - y|; y) \right) \end{aligned}$$

Using the Cauchy-Schwarz inequality at this stage,

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \leq \left| g^{(1)}(y) \right| \left| \mathfrak{T}_{n,1}^{(\rho)}(y) \right| + \mathfrak{w}(g^{(1)}; \mathfrak{d}) \left(\frac{1}{\mathfrak{d}} \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} + 1 \right) \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$$

Selecting $\mathfrak{d} = \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$, and in the limit of large enough n and ρ , we get the stated result. $\square$

Let us depict by $\mathcal{H}_\zeta[0, \infty)$ the space of all real valued functions on $[0, \infty)$ which satisfy $|g(y)| \leq A_g \zeta(y)$, where A_g is a positive constant dependent on g , and $\zeta(y) = 1 + y^2$ is a weight function.

Let $C_\zeta[0, \infty)$ depict the space of all continuous functions in $\mathcal{H}_\zeta[0, \infty)$, equipped with the norm

$$\|g\|_\zeta := \sup_{y \in [0, \infty)} \frac{|g(y)|}{\zeta(y)} \quad (40)$$

Also, let $C_\zeta^*[0, \infty)$ depict the space of all functions $g \in C_\zeta[0, \infty)$ for which the limit $\lim_{y \rightarrow \infty} \frac{|g(y)|}{\zeta(y)}$ exists and is finite.

The usual modulus of continuity of g on $[0, \lambda]$ is defined as

$$\mathfrak{w}_\lambda(g; \mathfrak{d}) = \sup_{0 \leq |\mu - y| \leq \mathfrak{d}} \sup_{y, \mu \in [0, \lambda]} |g(\mu) - g(y)| \quad (41)$$

Theorem 5 Let $g \in C_\zeta[0, \infty)$. Then the following result holds:

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \leq 4A_g(1 + y^2)\mathfrak{T}_{n,2}^{(\rho)}(y) + 2\mathfrak{w}_{\lambda+1} \left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \right)$$

Proof Referring to [21], [23], for $y \in [0, \lambda]$ and $t \geq 0$, we have

$$|g(\mu) - g(y)| \leq 4A_g(1 + y^2)(\mu - y)^2 + \left(1 + \frac{|\mu - y|}{\mathfrak{d}} \mathfrak{w}_{\lambda+1}(g; \mathfrak{d}) \right), \quad \mathfrak{d} > 0 \quad (42)$$

Therefore

$$\left| \mathfrak{R}_n^{(\rho)}(g(\mu); y) - g(y) \right| \leq 4A_g(1 + y^2)\mathfrak{T}_{n,2}^{(\rho)}(y) + \mathfrak{w}_{\lambda+1}(g; \mathfrak{d}) \left(1 + \frac{1}{\mathfrak{d}} \left| \mathfrak{T}_{n,1}^{(\rho)}(y) \right| \right)$$

Using the Cauchy-Schwarz inequality,

$$\left| \mathfrak{R}_n^{(\rho)}(g(\mu); y) - g(y) \right| \leq 4A_g(1 + y^2)\mathfrak{T}_{n,2}^{(\rho)}(y) + \mathfrak{w}_{\lambda+1}(g; \mathfrak{d}) \left(1 + \frac{1}{\mathfrak{d}} \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \right)$$

Choosing $\mathfrak{d} = \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$, we get

$$\left| \mathfrak{R}_n^{(\rho)}(g(\mu); y) - g(y) \right| \leq 4A_g(1 + y^2)\mathfrak{T}_{n,2}^{(\rho)}(y) + 2\mathfrak{w}_{\lambda+1} \left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \right)$$

$\square$



4 Weighted Approximation Properties

Theorem 6 Let $g \in C_\zeta^*[0, \infty)$ and $\rho = \rho(n)$ be such that as $n \rightarrow \infty$, $\rho(n) \rightarrow \infty$. Then, the following holds:

$$\lim_{n \rightarrow \infty} \left\| \mathfrak{R}_n^{(\rho)}(g(\mu)) - g \right\|_\zeta = 0. \quad (43)$$

Proof To demonstrate this result, it is sufficient to establish the following three relations [17]:

$$\lim_{n \rightarrow \infty} \left\| \mathfrak{R}_n^{(\rho)}(\mu^r) - y^r \right\|_\zeta = 0, \quad r = 0, 1, 2. \quad (44)$$

Because $\mathfrak{R}_n^{(\rho)}(1; y) = 1$ due to Lemma 3, the condition in (44) holds true for $r = 0$.

Using Lemma 3, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\| \mathfrak{R}_n^{(\rho)}(\mu) - y \right\|_\zeta \\ &= \lim_{n \rightarrow \infty} \left\| \left(\frac{\rho}{n(\rho - 2)} \left(\frac{V^{(1)}(nyW(1))}{V(nyW(1))} (ny) + \frac{U^{(1)}(1)}{U(1)} \right) \right) - y \right\|_\zeta \end{aligned} \quad (45)$$

$$= 0. \quad (46)$$

Therefore, $\lim_{n \rightarrow \infty} \left\| \mathfrak{R}_n^{(\rho)}(\mu) - y \right\|_\zeta = 0$.

$$\lim_{n \rightarrow \infty} \left\| \mathfrak{R}_n^{(\rho)}(\mu^2) - y^2 \right\|_\zeta$$

also turns out to be equal to 0 under the assumptions in Remark 2, leading to $\lim_{n \rightarrow \infty} \left\| \mathfrak{R}_n^{(\rho)}(\mu^2) - y^2 \right\|_\zeta = 0$. Hence the stated result follows. $\square$

We invoke the definition of the weighted modulus of continuity $\mathfrak{W}(g; \mathfrak{d})$ defined on $[0, \infty)$ (see [43]) as follows:

$$\mathfrak{W}(g; \mathfrak{d}) = \sup_{|m| < \mathfrak{d}, y \in [0, \infty)} \frac{|g(y + m) - g(y)|}{(1 + m^2)(1 + y^2)} \quad \text{for } g \in C_\zeta[0, \infty) \quad (47)$$

Lemma 5 [43] Let $g \in C_\zeta^*[0, \infty)$, then the following hold:

1. $\mathfrak{W}(g; \mathfrak{d})$ is monotone increasing function in $\mathfrak{d}$;
2. $\lim_{\mathfrak{d} \rightarrow 0^+} \mathfrak{W}(g; \mathfrak{d}) = 0$;
3. for each $\phi \in \mathbb{N}$, $\mathfrak{W}(g; \phi \mathfrak{d}) \leq \phi \mathfrak{W}(g; \mathfrak{d})$;
4. for each $\vartheta \in [0, \infty)$, $\mathfrak{W}(g; \vartheta \mathfrak{d}) \leq (1 + \vartheta) \mathfrak{W}(g; \mathfrak{d})$.

Theorem 7 Let $g \in C_\zeta^*[0, \infty)$ and $\rho = \rho(n)$ be such that as $n \rightarrow \infty$, $\rho(n) \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} \frac{n}{\rho} = q \in \mathbb{R},$$

then there exists $m_0 \in \mathbb{N}$ and a constant $Q(q) \in \mathbb{R}^+$ that depends on q , such that:

$$\sup_{y \in \mathbb{R}^+} \frac{\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right|}{(1 + y^2)^{5/2}} \leq Q(q) \mathfrak{W}\left(g; n^{-1/2}\right), \quad \text{for } n > m_0. \quad (48)$$

Proof For $\mu > 0$, $y \in \mathbb{R}^+$, $\mathfrak{d} > 0$, by using the definition of $\mathfrak{W}(g; \mathfrak{d})$ and the associated Lemma 5, we can write



$$\begin{aligned}
|g(\mu) - g(y)| &\leq (1 + (\mu - y)^2) (1 + y^2) \mathfrak{W}(g; |\mu - y|) \\
&\leq (1 + y^2) (1 + (\mu - y)^2) \left(1 + \frac{|\mu - y|}{\mathfrak{d}}\right) \mathfrak{W}(g; \mathfrak{d}) \\
&\leq (1 + y^2) \mathfrak{W}(g; \mathfrak{d}) \left(1 + (\mu - y)^2 + (1 + (\mu - y)^2) \frac{|\mu - y|}{\mathfrak{d}}\right)
\end{aligned} \quad (49)$$

Because $\mathfrak{R}_n^{(\rho)}$ is a positive linear operator,

$$\begin{aligned}
&\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \\
&\leq (1 + y^2) \mathfrak{W}(g; \mathfrak{d}) \left\{ 1 + \mathfrak{T}_{n,2}^{(\rho)}(y) + \mathfrak{R}_n^{(\rho)} \left(\left(1 + (\mu - y)^2\right) \frac{|\mu - y|}{\mathfrak{d}}; y \right) \right\}
\end{aligned} \quad (50)$$

Using Cauchy-Schwarz inequality, we write

$$\mathfrak{R}_n^{(\rho)} \left(\left(1 + (\mu - y)^2\right) \frac{|\mu - y|}{\mathfrak{d}}; y \right) \leq \frac{1}{\mathfrak{d}} \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} + \frac{1}{\mathfrak{d}} \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} \sqrt{\mathfrak{T}_{n,4}^{(\rho)}(y)} \quad (51)$$

Using the Remark 5, it can be said that there is an $\mathfrak{m}_1 \in \mathbb{N}$ s.t.

$$\mathfrak{T}_{n,2}^{(\rho)}(y) \leq Q_1(q) \frac{1 + y^2}{n}, \text{ for } n > \mathfrak{m}_1 \quad (52)$$

and an $\mathfrak{m}_2 \in \mathbb{N}$ s.t.

$$\sqrt{\mathfrak{T}_{n,4}^{(\rho)}(y)} \leq Q_2(q) \frac{\sqrt{1 + y^2}}{n}, \text{ for } n > \mathfrak{m}_2 \quad (53)$$

where $Q_1(q), Q_2(q) \in \mathbb{R}^+$ are constants that depends on q . Let $\mathfrak{m}_0 = \max\{\mathfrak{m}_1, \mathfrak{m}_2\}$. Combining the above results from (49)-(53), and with the choice $\mathfrak{d} = n^{-1/2}$, for $y > \mathfrak{m}_0$, we obtain the stated result. $\square$

5 Quantitative Voronovskaja-type theorem

In this section, we establish a quantitative Voronovskaja-type theorem for the operators $\mathfrak{R}_n^{(\rho)}$ by using the weighted modulus of continuity, $\mathfrak{W}(g; \mathfrak{d})$ 5. Recent important works in this direction are [34] [5] [11] [29].

Theorem 8 Let $g \in C_\zeta^*[0, \infty)$ such that $g^{(1)}$ and $g^{(2)} \in C_\zeta^*[0, \infty)$, and $y \geq 0$. Then, the following holds:

$$\begin{aligned}
&\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) \right| \\
&- \left[\left[\frac{\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 1 \right] (y) + \frac{\rho}{n(\rho - 2)} \frac{U^{(1)}(1)}{U(1)} \right] g^{(1)}(y) \\
&- \left(\left[\frac{\rho^2}{(\rho - 2)(\rho - 3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1 \right] (y)^2 \right. \\
&+ \left[\frac{\rho^2}{n(\rho - 2)(\rho - 3)} \left(\frac{2U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))} \right. \\
&- \left. \left. \frac{2\rho}{n(\rho - 2)} \frac{U^{(1)}(1)}{U(1)} \right] (y) + \left[\frac{\rho^2}{n^2(\rho - 2)(\rho - 3)} \left(2 \frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)} \right) \right] \right) g^{(2)}(y) \Big| \\
&\leq 8(1 + y^2) O\left(\frac{1}{n}\right) \mathfrak{W}\left(g^{(2)}; n^{-1/2}\right)
\end{aligned}$$



Proof Let $y, \mu \geq 0$. Using Taylor's expansion, we can write

$$g(\mu) = g(y) + g^{(1)}(y)(\mu - y) + \frac{g^{(2)}(y)}{2!}(\mu - y)^2 + \mathfrak{E}(\mu, y),$$

where $\mathfrak{E}(\mu, y) = \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!}(\mu - y)^2$ and $\mathfrak{p}$ lies between μ and y . Operating by $\mathfrak{R}_n^{(\rho)}$ gives

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) - g^{(1)}(y)\mathfrak{T}_{n,1}^{(\rho)}(y) - g^{(2)}(y)\mathfrak{T}_{n,2}^{(\rho)}(y) \right| \leq \mathfrak{R}_n^{(\rho)}(|\mathfrak{E}(\mu, y)|; y) \quad (54)$$

Substituting the central moments from Lemma 4 gives

$$\begin{aligned} & \left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) - \left[\left[\frac{\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 1 \right] (y) + \frac{\rho}{n(\rho - 2)} \frac{U^{(1)}(1)}{U(1)} \right] g^{(1)}(y) \right. \\ & - \left[\left[\frac{\rho^2}{(\rho - 2)(\rho - 3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{(\rho - 2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1 \right] (y)^2 \right. \\ & + \left[\frac{\rho^2}{n(\rho - 2)(\rho - 3)} \left(\frac{2U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2 \right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - \right. \\ & \left. \left. \frac{2\rho}{n(\rho - 2)} \frac{U^{(1)}(1)}{U(1)} \right] (y) + \left[\frac{\rho^2}{n^2(\rho - 2)(\rho - 3)} \left(2 \frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)} \right) \right] \right] g^{(2)}(y) \left. \right| \\ & \leq \mathfrak{R}_n^{(\rho)}(|\mathfrak{E}(\mu, y)|; y) \end{aligned} \quad (55)$$

Using the properties of $\mathfrak{W}(g; \mathfrak{d})$ from Lemma 5, we can write

$$\begin{aligned} & \left| \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!} \right| \\ & \leq \frac{1}{2} \mathfrak{W}(g^{(2)}; |\mathfrak{p} - y|) (1 + (\mathfrak{p} - y)^2) (1 + y^2) \\ & \leq \frac{1}{2} \mathfrak{W}(g^{(2)}; |\mu - y|) (1 + (\mu - y)^2) (1 + y^2) \\ & \leq \left(1 + \frac{|\mu - y|}{\mathfrak{d}} \right) (1 + \mathfrak{d}^2) \mathfrak{W}(g^{(2)}; \mathfrak{d}) (1 + (\mu - y)^2) (1 + y^2) \end{aligned}$$

Also [28]

$$\left| \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!} \right| \leq \begin{cases} 2(1 + \mathfrak{d}^2)^2(1 + y^2) \mathfrak{W}(g^{(2)}; \mathfrak{d}), & |\mu - y| \leq \mathfrak{d} \\ 2(1 + \mathfrak{d}^2)^2(1 + y^2) \frac{(\mu - y)^2}{\mathfrak{d}^4} \mathfrak{W}(g^{(2)}; \mathfrak{d}), & |\mu - y| \geq \mathfrak{d} \end{cases} \quad (56)$$

Therefore, for $0 \leq \mathfrak{d} \leq 1$, we get

$$\begin{aligned} \left| \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!} \right| & \leq 2(1 + y^2) \left(1 + \frac{(\mu - y)^4}{\mathfrak{d}^4} \right) (1 + \mathfrak{d}^2)^2 \mathfrak{W}(g^{(2)}; \mathfrak{d}) \\ & \leq 8(1 + y^2) \left(1 + \frac{(\mu - y)^4}{\mathfrak{d}^4} \right) \mathfrak{W}(g^{(2)}; \mathfrak{d}) \end{aligned}$$

Therefore

$$|\mathfrak{E}(\mu, y)| = \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!}(\mu - y)^2 \leq$$



$$8 \left(1 + y^2\right) \left((\mu - y)^2 + \frac{(\mu - y)^6}{\mathfrak{d}^4} \right) \mathfrak{W} \left(g^{(2)}; \mathfrak{d} \right) \quad (57)$$

Using the linearity and positivity of $\mathfrak{R}_n^{(\rho)}$, and using the results from central moments from Lemma 4,

$$\begin{aligned} \mathfrak{R}_n^{(\rho)}(|\mathfrak{E}(\mu, y)|; y) &\leq 8 \left(1 + y^2\right) \left\{ \mathfrak{T}_{n,2}^{(\rho)}(y) + \frac{1}{\mathfrak{d}^4} \mathfrak{T}_{n,6}^{(\rho)}(y) \right\} \mathfrak{W} \left(g^{(2)}; \mathfrak{d} \right) \\ &\leq 8 \left(1 + y^2\right) \left\{ O\left(\frac{1}{n}\right) + \frac{1}{\mathfrak{d}^4} O\left(\frac{1}{n^3}\right) \right\} \mathfrak{W} \left(g^{(2)}; \mathfrak{d} \right) \end{aligned}$$

Choosing $\mathfrak{d} = n^{-1/2}$ gives

$$\mathfrak{R}_n^{(\rho)}(|\mathfrak{E}(\mu, y)|; y) \leq 8 \left(1 + y^2\right) O\left(\frac{1}{n}\right) \mathfrak{W} \left(g^{(2)}; n^{-1/2} \right) \quad (58)$$

Combining equations (55) and (58) leads to the required result. $\square$

6 Grüss-Voronovskaja-type theorem

The following result brings out the non-multiplicativity of $\mathfrak{R}_n^{(\rho)}$. Similar recent studies include [3], [34], [5], [11], [29].

Theorem 9 Let $g(y), u(y) \in C_\zeta^*[0, \infty)$ such that $g^{(1)}(y), g^{(2)}(y), u^{(1)}(y), u^{(2)}(y), (gu)^{(1)}(y), (gu)^{(2)}(y) \in C_\zeta^*[0, \infty)$, and $y, \mu \geq 0$. Also, let $\rho = \rho(n)$ be such that as $n \rightarrow \infty$, $\rho \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \frac{n}{\rho(n)} = q \in \mathbb{R}$$

then:

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left[\mathfrak{R}_n^{(\rho)}((gu)(\mu); y) - \mathfrak{R}_n^{(\rho)}(g(\mu); y) \mathfrak{R}_n^{(\rho)}(u(\mu); y) \right] \\ = (v_1(y)) g^{(1)}(y) u^{(1)}(y). \end{aligned}$$

Proof Consider the expression

$$n \left[\mathfrak{R}_n^{(\rho)}((gu)(\mu); y) - \mathfrak{R}_n^{(\rho)}(g(\mu); y) \mathfrak{R}_n^{(\rho)}(u(\mu); y) \right] \quad (59)$$

Using $(gu)^{(1)}(y) = g^{(1)}(y)u(y) + g(y)u^{(1)}(y)$ and $(gu)^{(2)}(y) = g^{(2)}(y)u(y) + 2g^{(1)}(y)u^{(1)}(y) + g(y)u^{(2)}(y)$, it is easy to verify that the following expression is equivalent to the expression (59):

$$\begin{aligned} n \left\{ \Phi_1 - u(y) \{\Phi_2\} - \mathfrak{R}_n^{(\rho)}(g(\mu); y) \{\Phi_3\} + \mathfrak{R}_n^{(\rho)}(\mu - y; y) u^{(1)}(y) \{\Phi_4\} \right. \\ \left. + \frac{1}{2!} \mathfrak{R}_n^{(\rho)}((\mu - y)^2; y) \left\{ u^{(2)}(y) \{\Phi_4\} + \Phi_5 \right\} \right\} \quad (60) \end{aligned}$$

where

$$\begin{aligned} \Phi_1 &= \mathfrak{R}_n^{(\rho)}((gu)(\mu); y) - (gu)(y) - (gu)^{(1)}(y) \mathfrak{R}_n^{(\rho)}(\mu - y; y) \\ &\quad - \frac{(gu)^{(2)}(y)}{2!} \mathfrak{R}_n^{(\rho)}((\mu - y)^2; y) \end{aligned} \quad (61)$$

$$\Phi_2 = \mathfrak{R}_n^{(\rho)}(g(\mu); y) - g(y) - g^{(1)}(y) \mathfrak{R}_n^{(\rho)}(\mu - y; y) - \frac{g^{(2)}(y)}{2!} \mathfrak{R}_n^{(\rho)}((\mu - y)^2; y) \quad (62)$$

$$\Phi_3 = \mathfrak{R}_n^{(\rho)}(u(\mu); y) - u(y) - u^{(1)}(y) \mathfrak{R}_n^{(\rho)}(\mu - y; y) - \frac{u^{(2)}(y)}{2!} \mathfrak{R}_n^{(\rho)}((\mu - y)^2; y) \quad (63)$$



$$\Phi_4 = g(y) - \mathfrak{R}_n^{(\rho)}(g(\mu); y) \quad (64)$$

and

$$\Phi_5 = 2g^{(1)}(y)u^{(1)}(y). \quad (65)$$

Consider the terms Φ_1 , Φ_2 and Φ_3 . In the limit as $n \rightarrow \infty$, by Theorem 8 and Lemma 5, for any $h \in C_\zeta^*[0, \infty)$, we have

$$\lim_{n \rightarrow \infty} n \left[\mathfrak{R}_n^{(\rho)}(h(\mu); y) - h(y) - h^{(1)}(y)\mathfrak{R}_n^{(\rho)}(\mu - y; y) - \frac{h^{(2)}(y)}{2!}\mathfrak{R}_n^{(\rho)}((\mu - y)^2; y) \right] = 0 \quad (66)$$

(for Φ_1 this can be observed by considering $(gu)(\mu) = h(\mu) \forall \mu \geq 0$). Further, for Φ_4 , using Theorem 1, we have, $\forall y \geq 0$, $\mathfrak{R}_n^{(\rho)}(g(\mu); y) \rightarrow g(y)$. Therefore, we can conclude that

$$\lim_{n \rightarrow \infty} n\Phi_r = 0, \quad r = 1, 2, 3, 4. \quad (67)$$

Combining these results(67), and using Remark 5 in expressions (59) and (60), we get:

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \left[\mathfrak{R}((gu)(\mu); y) - \mathfrak{R}(g(\mu); y) \mathfrak{R}(u(\mu); y) \right] \\ &= \lim_{n \rightarrow \infty} n \frac{1}{2!} \mathfrak{R}_n^{(\rho)}((\mu - y)^2; y) \{\Phi_5\} \\ &= \lim_{n \rightarrow \infty} n \mathfrak{T}_{n,2}^{(\rho)}(y) g^{(1)}(y) u^{(1)}(y) \\ &= (v_1(y)) g^{(1)}(y) u^{(1)}(y), \end{aligned} \quad (68)$$

which is the required result. $\square$

7 Conclusion

In this paper, we have presented a rich class of positive, linear approximation operators based on Miheřan's generalization of the Szász-Mirakjan operators, and incorporating the Boas-Buck polynomials. The proposed operators form an important link in the field, as they reproduce several types of operators. Further, we have shown how some essential properties based on modulus of continuity, as well as the recently-acknowledged Quantitative Voronovskaja-type approximation theorems hold true for our operator.

Declarations

Authors' Contributions All authors contributed significantly to the manuscript.

Funding Information Not applicable, as no funding was involved.

Availability of data and material Not applicable.

Conflict of interest The authors declare that they have no conflict of interest.

Code Availability Not applicable.



References

1. Acar, T., Gupta, V., Aral, A. (2011). Rate of convergence for generalized şzász operators. *Bull Math Sci*, 1(1), 99–113.
2. Acar, T. (2015). Asymptotic Formulas for Generalized Szász-Mirakyan Operators, *Appl. Math. Comp.*, 263, 223–239.
3. Acar, T. (2016). Quantitative q-Voronovskaya and q-Grüss-Voronovskaya-type results for q-Szász operators. *Georgian Mathematical Journal*, 23(4), 459–468.
4. Acar, T., Ulusoy, G. (2016). Approximation by modified Szász-Durrmeyer operators. *Period. Math. Hung.* 72(1), 64–75.
5. Acar, T., Aral, A., Rasa, I. (2016). The new forms of voronovskaya's theorems in weighted spaces. *Positivity*, 20, 25–40.
6. Acar, T., Aral, A., Cárdenas-Morales, D., Garrancho, P. (2017). Szász-mirakyan type operators which fix exponentials. *Results Math*, 72(3), 1393–1404.
7. Acar, T., Aral, A., Rasa, I. (2019). Positive linear operators preserving τ and τ^2 , *Constr. Math. Anal.*, 2 (3), 98–102.
8. Acu, A., Gonska, H., Rasa, I. (2011). Grüss-type and ostrowski-type inequalities in approximation theory. *Ukrainian Mathematical Journal*, 63, 843–864.
9. Agratini, O. (1999). On a sequence of linear and positive operators. *Facta Universitatis (Niš) Ser Math Inform*, 14, 41–48.
10. Agratini, O. (2021). Approximation properties of a family of integral type operators. *Positivity*, 25, 97–108.
11. Agrawal, P., Baxhaku, B., Chauhan, R. (2018). Quantitative voronovskaya- and grüss-voronovskaya-type theorems by the blending variant of şzász operators including brenke-type polynomials. *Turkish Journal of Mathematics*, 42, 1610–1629.
12. Atakut, C., Büyükyazıcı, I. (2016). Approximation by kantorovich-şzász type operators based on brenke type polynomials. *Numerical Functional Analysis and Optimization*, 37(12), 1488–1502.
13. Baskakov, V. (1957). A sequence of linear positive operators in the space of continuous functions. *Dokl Acad Nauk SSSR*, 113, 249–251.
14. Bernstein, S. (1912-13). Demonstration du theoreme de weierstrass fondee sur le calcul de probabilités. *Commun Soc Math Kharkow*, 13(2), 1–2.
15. Braha, N., Kadak, U. (2019). Approximation properties of the generalized szasz operators by multiple appell polynomials via power summability method. *Mathematical Methods in the Applied Sciences*, 1–20.
16. Bustamante, J., Flores de Jesús, L. (2020). Strong Converse Inequalities and Quantitative Voronovskaya-Type Theorems for Trigonometric Fejér Sums, *Constr. Math. Anal.*, 3 (2), 53–63.
17. Gadjiev, A. (1976). On p. p. korovkin type theorems. *Math Zametki*, 20(5), 781–786.
18. Gupta, V., Agarwal, R. (2014). *Convergence Estimates in Approximation Theory*, Springer.
19. Gupta, V. (2019). A large family of linear positive operators. *Rendiconti del Circolo Matematico di Palermo Series II*, 69(3), 1–9.
20. Gupta, V. (2020). A generalized class of integral operators. *Carpathian Journal of Mathematics*, 36(3), 423–431.
21. Ibikli, E., Gadjieva, E. (1995). The order of approximation of some unbounded functions by the sequences of positive linear operators. *Turkish Journal of Mathematics*, 19(3), 331–337.
22. Jakimovski, A., Leviatan, D. (1969). Generalized şzász operators for the approximation in the infinite interval. *Mathematica*, 11, 97–103.
23. Kajla, A. (2017). Direct estimates of certain miheşan-durrmeyer type operators. *Adv Oper Theory*, 2(2), 162–178.
24. Kajla, A. (2018). Approximation properties of generalized şzász-type operators. *Acta Math Vietnam*, 43, 549–563.
25. Kajla, A., Acar, T. (2018). A new modification of durrmeyer type mixed hybrid operators. *Carpathian J. Math.*, 34(1), 47–56.
26. Kajla, A., Acar, T. (2018). Blending type approximation by generalized Bernstein-Durrmeyer type operators. *Miskolc Mathematical Notes*, 19 (1), 2018, 319–326.
27. Kajla, A., Araci, S., Goyal, M., Acikgoz, M. (2019). Generalized şzász-kantorovich type operators. *Communications in Mathematics and Applications*, 10(3), 403–413.
28. Kajla, A., Deshwal, S., Agrawal, P.N. (2019). Quantitative Voronovskaya and Grüss-Voronovskaya type theorems for Jain-Durrmeyer operators of blending type., *Anal. Math. Phys.*, 9, 1241–1263.
29. Kajla, A., Mohiuddine, S., Alotaibi, A., Goyal, M., Singh, K. (2020). Approximation by ϑ -baskakov-durrmeyer-type hybrid operators. *Iran J Sci Technol Trans Sci*, 44, 1111–1118.
30. Kajla, A., Acar, T. (2020). Bézier-Bernstein-Durrmeyer type operators. *RACSAM (Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas)*, 114, 31.
31. Lupaş, A. (1995). The approximation by means of some linear positive operators. *Approximation Theory, Proceedings of the International Dortmund Meeting on Approximation Theory, Berlin, Germany*, pp. 201–229, (M.W. Müller, M. Felten and D.H. Mache Eds.), Akademik Verlag, Berlin.
32. Miheşan, V. (2008). Gamma approximating operators. *Creative Math Inf*, 17, 466–472.
33. Mursaleen, M., Al-Abied, A., Acu, A. (2018). Approximation by chlodowsky type of şzász operators based on boas-buck-type polynomials. *Turkish Journal of Mathematics*, 42(5), 2243–2259.
34. Neer, T., Agrawal, P. (2017). Quantitative-voronovskaya and grüss-voronovskaya type theorems for şzász-durrmeyer type operators blended with multiple appell polynomials. *J Inequal Appl*, 2017(244).
35. Neer, T., Acu, A., Agrawal, P. (2019). Baskakov–durrmeyer type operators involving generalized appell polynomials. *Mathematical Methods in the Applied Sciences*, 43, 2911–2923.
36. Özarslan, M. (2020). Approximation properties of jain–appell operators. *Applicable Analysis and Discrete Mathematics*, 14(3), 654–669.
37. Szász, O. (1950). Generalization of s. bernstein's polynomials to the infinite interval. *J Res Nat Bur Standards* 45, 239–245.
38. Sidharth, M., Agrawal, P., Araci, S. (2017). Szász-durrmeyer operators involving boas-buck polynomials of blending type. *Journal of Inequalities and Applications*, 2017(122).
39. Sucu, S., Varma, S. (2015). Generalization of jakimovski–leviatan type şzász operators. *Applied Mathematics and Computation*, 270, 977–983.
40. Sucu, S., Varma, S. (2019). Approximation by sequence of operators involving analytic functions. *Mathematics*, 17 (2), 188.



41. Varma, S., Sucu, S., İçöz, G. (2012). Generalization of şász operators involving brenke type polynomials. *Computers and Mathematics with Applications*, 64, 121–127.
42. Yilik, ÖÖ., Garg, T., Agrawal, P. (2020). Convergence rate of şász operators involving boas-buck-type polynomials. *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, <https://doi.org/10.1007/s40010-020-00663-3>.
43. Yüksel, I., Ispir, N. (2006). Weighted approximation by a certain family of summation integral-type operators. *Comput. Math. Appl.*, 52 (10-11), 1463–1470.

