



On independent cliques and linear complementarity problems

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Abstract In recent work (Pandit and Kulkarni [Discrete Applied Mathematics, 244 (2018), pp. 155–169]), the independence number of a graph was characterized as the maximum of the ℓ_1 norm of solutions of a Linear Complementarity Problem (LCP) defined suitably using parameters of the graph. Solutions of this LCP have another relation, namely, that they corresponded to Nash equilibria of a public goods game. Thus the maximum ℓ_1 norm has an important economic interpretation as the maximum total investment over all equilibria of this game. Motivated by this, we consider a perturbation of this LCP that corresponds to a public goods game with imperfect substitutability. We identify the combinatorial structures on the graph that correspond to the maximum ℓ_1 norm of solutions (equivalently, investment maximizing equilibria) of the new LCP. We show that these solutions correspond to “independent cliques” – collections of cliques such that no two vertices from two distinct cliques are adjacent. When the perturbation becomes null, these cliques collapse to singletons and we recover the earlier relation to maximum independent sets. Independent cliques have been studied before as a generalization of independent sets. Our work establishes an intricate connection between independent cliques, LCPs and public goods games.

Keywords Linear Complementarity Problems · Independent Sets · Dominating Sets · Cliques

1 Introduction

An undirected graph G is defined as $G = (V, E)$, where V is a finite set of vertices and E is a set of unordered pairs of vertices called *edges*. Two vertices $i, j \in V$ are said to be *adjacent* to each other if they have an edge between them, i.e., $(i, j) \in E$. The *neighborhood* of a vertex is the set of all vertices adjacent to it. A set of vertices such that none of them is adjacent to each other is called an *independent set*. An independent set of maximum cardinality is called a *maximum independent set* and its cardinality is denoted as $\alpha(G)$, called the independence number of G . An independent set is said to be *maximal* if it not a strict subset of another independent set. If weights $w = (w_1, w_2, \dots, w_{|V|}) \geq 0$ are assigned to each vertex, then a weighted maximum independent set is an independent set which maximizes $\sum_{i \in S} w_i$ over all independent sets S and the value of this maximum is denoted by $\alpha_w(G)$, called the w -weighted independence number of G .

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This paper is about a relation between graphs and a class of continuous optimization problems called Linear Complementarity Problems (LCPs). Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, $\text{LCP}(M, q)$ is the following problem:

$$\text{Find } x \in \mathbb{R}^n \text{ such that } x \geq 0, y = Mx + q \geq 0, y^\top x = 0.$$

Any such x is called a solution of $\text{LCP}(M, q)$. For a graph G on n vertices, let $A \in \{0, 1\}^{n \times n}$ denote its adjacency matrix: i.e., $A(i, j) = 1$ if $(i, j) \in E$, else it is 0. Now consider the LCP given by $\text{LCP}(I + \delta A, -\mathbf{e}) =: \text{LCP}_\delta(G)$ where I is the $n \times n$ identity matrix, A is the adjacency matrix of G and $\mathbf{e}$ is a vector of 1's and $\delta < 1$. Denote the solutions to $\text{LCP}_\delta(G)$ by $\text{SOL}_\delta(G)$. We consider the optimization problem:

$$\begin{array}{ll} \max \text{SOL}_\delta(G) & \text{maximize } \mathbf{e}^\top x \\ & \text{subject to } x \in \text{SOL}_\delta(G). \end{array}$$

We show that the solution to the above problem is same as maximizing over a smaller set which we call the set of *independent clique solutions* (ICS) of $\text{LCP}_\delta(G)$. These are vectors in $\text{SOL}_\delta(G)$ such that their supports form a union of *independent cliques*, where two cliques are said to be *independent* if no vertex of one clique has any vertex of the other clique as its neighbor.

Our work is motivated by results in [16] which established an LCP based characterization of $\alpha_w(G)$. They consider the LCP given by $\text{LCP}_1(G) := \text{LCP}(I + A, -\mathbf{e})$ (denoted by $\text{LCP}(G)$ hereon) and show that the maximum weighted ℓ_1 norm amongst the solutions of $\text{LCP}(G)$ equals the weighted independence number of G . Formally,

$$\alpha_w(G) = \max\{w^\top x \mid x \text{ is a solution of } \text{LCP}(I + A, -\mathbf{e})\}. \quad (1)$$

It easy to argue that the characteristic vector of any maximal independent set in G solves $\text{LCP}(G)$. Consequently, one trivially has that $\alpha_w(G) \leq w^\top x$ for all x that solve $\text{LCP}(G)$. The above result is nontrivial because it shows that no fractional solution x of $\text{LCP}(G)$ can attain a strictly greater value for $w^\top x$ than that attained by the characteristic vector of a w -weighted maximum independent set.

We note that the independence number has also been related to quadratic programs before. For instance, Motzkin and Strauss [14] showed that for a graph G with n vertices,

$$\frac{1}{\alpha(G)} = \min\{x^\top (I + A)x \mid x \geq 0; \mathbf{e}^\top x = 1\}. \quad (2)$$

However, it is important to note that (1) is not an obvious corollary of (2). Specifically, the KKT conditions of the quadratic program in (2) can be denoted by an LCP, but it is a different LCP than $\text{LCP}(G)$. Moreover, (1) applies to the weighted independence number, whereas (2) applies only to the unweighted one. $\text{LCP}(G)$ is also distinct from other LCPs that have been studied in the context of independence numbers [12], [13]; more details on these subtleties can be found in [16].

Indeed there is a third connection which concerns the interpretation of solutions of $\text{LCP}(G)$ in terms of Nash equilibria of a *public goods game* defined on a network G . In the model considered in Bramoullé and Kranton [3], each vertex i of the graph G represents an agent who exerts a scalar effort $x_i \geq 0$ and its utility for the effort profile $x = (x_1, x_2, \dots, x_{|V|})$ is

$$u_i(x) = b\left(x_i + \sum_{j \in N_G(i)} x_j\right) - cx_i,$$

where $N_G(i)$ denotes the neighborhood of vertex i in graph G , $b(\cdot)$ is a differentiable strictly concave increasing benefit function and c is the marginal cost of exerting unit effort. Thus each agent benefits from its own effort and the effort exerted by its neighbors (more details on this model can be found in Bramoullé and Kranton [3]). Assume for simplicity that $b'^{-1}(c) = 1$, i.e., marginal benefit equals marginal cost at unit effort. It is shown in [17] that the Nash equilibria of the above public goods game are given by solutions of $\text{LCP}(G)$. Moreover, each *maximal independent set* corresponds to an equilibrium of the above game wherein each vertex in the maximal independent set exerts unit effort and all other vertices exert no effort. It follows that for a given vector of non-negative weights w , the maximum weighted effort amongst all equilibria, which is the maximum weighted ℓ_1 norm amongst the solutions of $\text{LCP}(G)$, is achieved by the equilibrium corresponding to the w -weighted maximum independent set. These results illustrate how Nash equilibria of this game are intimately related to combinatorial structures on the underlying graph.



In this paper, we are concerned with perturbations of the above model. Specifically, we ask, what happens if the argument of the benefit function is $x_i + \delta \sum_{j \in N_G(i)} x_j$ instead of $x_i + \sum_{j \in N_G(i)} x_j$? Thus the utility function now is

$$u_i(x) = b\left(x_i + \delta \sum_{j \in N_G(i)} x_j\right) - cx_i.$$

Here $\delta > 0$ is a substitutability factor which captures the case where, of the effort of neighbors of agent i and fraction δ of it benefits agent i . Thus from the point of view of agent i , unit effort of agent i is substitutable with a total effort of $\frac{1}{\delta}$ by its neighbors. We view δ as a small perturbation from unity, i.e., δ is close to unity, but may not be exactly unity. Thus we think of $\delta = 1$ as the idealized case where efforts of neighbors substitute exactly for the agent's own effort, and $\delta \neq 1$ can be thought of as arising due to small losses or misspecifications from this idealized situation. The existence of equilibria in this case has been studied in [4]. It is easy to argue that an equilibrium exists and all the equilibria of this new game are given by solutions of $\text{LCP}(I + \delta A, -\mathbf{e}) =: \text{LCP}_\delta(G)$. With this motivation, in this paper, we characterize ℓ_1 norm maximizing solutions of $\text{LCP}_\delta(G)$ for $\delta \neq 1$, but close to 1.

Our effort is to relate these solutions to combinatorial structures on the underlying graph. Before we mention our contributions, we make a few observations about $\text{LCP}_\delta(G)$. First, we observe that small perturbations δ around unity have a nontrivial effect on the combinatorial structure of solutions. In particular, for $\delta < 1$, it is not true that the characteristic vector of every maximal independent set solves $\text{LCP}_\delta(G)$. In fact, a binary vector solves $\text{LCP}_\delta(G)$ if and only if it is the characteristic vector of a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set, an object which may not exist in G . In the case when a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set does not exist, identifying combinatorial structures that support solutions and maximize the ℓ_1 norm becomes challenging and forces one to expand the search space of combinatorial structures that can be identified with solutions of $\text{LCP}_\delta(G)$.

In this paper we do precisely this. We introduce a new concept called *independent cliques solutions* (ICS) which are defined as solutions of $\text{LCP}_\delta(G)$ whose support is a union of independent cliques. Two cliques in a graph are said to be *independent* if no vertex of one clique has any vertex of the other clique as its neighbor. Independent cliques can be thought of as a generalization of independent sets since when each clique is degenerate (a single vertex), a union of independent cliques is an independent set. Such combinatorial structures have been studied in the literature ([1], [6], [2], [8]) under different names. [1] and [6] call them *independent union of cliques* (IUCs) and show that the characteristic vectors of these structures are the local maxima of a continuous optimization problem. They study the complexity of finding the IUC with maximum cardinality for planar graphs, claw graphs and bipartite graphs. They also provide a linear time algorithm to find the maximum cardinality IUC for trees. The authors of [2] study an equivalent formulation of the problem called *cluster vertex deletion* where the goal is to delete as few vertices as possible so that the remaining graph is an independent union of cliques. In [8], the authors study the properties of the polyhedron formed by the independent union of cliques. The problem we consider is different from the maximum IUC problem because our objective is to maximize the ℓ_1 -norm amongst solutions of $\text{LCP}_\delta(G)$ which may not be binary (characteristic) vectors. Different solutions imply different weights on vertices and hence this problem cannot be stated as a weighted maximum IUC problem.

We prove that the maximum ℓ_1 norm amongst all solutions of $\text{LCP}_\delta(G)$ is achieved by an ICS with $\alpha(G)$ cliques for $\delta \in [\eta(G), 1)$, where

$$\eta(G) = \max \left\{ \frac{\alpha(G)(\omega(G) - 1) - \omega(G)}{\alpha(G)(\omega(G) - 1)}, \frac{\omega(G) - 3 + \sqrt{(\omega(G) - 3)^2 + 4(\omega(G) - 1)}}{2(\omega(G) - 1)} \right\},$$

and $\omega(G)$ is the size of the largest clique in G . We define the optimization problem :

$\max \text{ICS}_\delta(G)$	maximize $\mathbf{e}^\top x$
subject to $x \in \text{ICS}_\delta(G)$	

Formally, our main result is stated below as Theorem 1:

Theorem 1 *Any solution of $\max\text{ICS}_\delta(G)$ is also a solution of $\max\text{SOL}_\delta(G)$ for any $\delta \in [\eta(G), 1)$.*

Thus, while ℓ_1 norm maximizing solutions of $\text{LCP}(G)$ include those supported by a maximum independent set, i.e. $\alpha(G)$ degenerate cliques, for $\delta \in [\eta(G), 1)$ the corresponding solutions of $\text{LCP}_\delta(G)$ comprise of $\alpha(G)$ not-necessarily-degenerate cliques. Moreover, in the case when a unique maximum independent set exists in a graph G the characteristic vector of the unique maximum independent set is the ℓ_1 norm maximizing solution of $\text{LCP}_\delta(G)$ for $\delta \in [\eta(G), 1)$. Lastly, we show that for $\delta \geq 1$, the results of [16] continue to hold, i.e., the maximum weighted ℓ_1 norm amongst the solutions of $\text{LCP}_\delta(G)$ is the weighted independence number achieved by the characteristic vector of a w -weighted maximum independent set.

These results are proved as follows. We show the existence of ICSs via an algorithm (Algorithm 1) that constructs an ICS for any graph. We prove that Algorithm 1 outputs a vector, as a function of δ with support as a union of independent cliques. Moreover, the support does not depend on δ . From this we show that these independent cliques support an ICS for all $\delta \in [\gamma(G), 1)$, where

$$\gamma(G) = \frac{\omega(G) - 3 + \sqrt{(\omega(G) - 3)^2 + 4(\omega(G) - 1)}}{2(\omega(G) - 1)}.$$

We also prove that the lower bound on δ , namely $\gamma(G)$, is tight by showing via examples that when this $\delta < \gamma(G)$, an ICS need not exist. Next, we prove that the ICS of $\text{LCP}_\delta(G)$ which achieves the maximum ℓ_1 norm amongst all the ICSs also achieves the maximum ℓ_1 norm amongst all solutions of $\text{LCP}_\delta(G)$ for $\delta \in [\eta(G), 1)$.

The problem of characterizing ℓ_1 norm maximizing solutions of $\text{LCP}_\delta(G)$ is highly complex since the solution set of the LCP is not convex (it is a union of polyhedra [5]) and no known graph structures directly provide solutions to $\text{LCP}_\delta(G)$. Our results show that for $\delta \geq \eta(G)$, an ICS always exists and is also ℓ_1 norm maximizing amongst all solutions of $\text{LCP}_\delta(G)$. It is also easy to show that for $\delta < -\frac{1}{\lambda_{\min}(A)}$, $\text{LCP}_\delta(G)$ admits a *unique* solution that is related to centrality notions on graphs (see [4]). It would be fascinating to ascertain the combinatorial structures that characterize ℓ_1 norm maximizing solutions of $\text{LCP}_\delta(G)$ for the entire range of δ from $-\frac{1}{\lambda_{\min}(A)}$ to unity. Our results take a first step in this direction by addressing the regime when δ is close to unity. We believe that in this range they provide interesting relations between the structural properties of graphs and solutions of $\text{LCP}_\delta(G)$.

Before discussing the related work, we distinguish ourselves from some past work, particularly [16,17]. In [16,17], the authors use the well-known concept of maximum independent sets to characterize the ℓ_1 norm maximizing solutions of the LCP they consider. On the other hand, in the case we consider, the characteristic vectors of maximum independent sets may not even be solutions of the LCP we consider. Hence, we introduce and study the concept of independent clique solutions, prove their existence and show that they are ℓ_1 norm maximizing solutions to the generalized LCP we consider. Lemmas 1,2,3 are simple generalizations of the corresponding lemmas in [16], but the main result Theorem 1, although seems similar to the main theorem of [16] in its statement, is quite non-trivial to prove since it involves the introduction of a new graph theoretic concept.

1.1 Related Work

Our results, in effect, give a characterization of the solutions to a special class of Linear Programs with Complementary Constraints (LPCC). In its most general form, an LPCC is defined as

LPCC $\begin{aligned} &\text{maximize}_{x,y} && c^\top x + d^\top y \\ &\text{subject to} && Bx + Cy \geq b \\ & && Mx + Ny + q \geq 0 \\ & && x \geq 0 \\ & && x^\top (Mx + Ny + q) = 0 \end{aligned}$
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When we take B, C, b, N, d to be 0 in the LPCC, and take $c = \mathbf{e}$, $M = I + \delta A$ and $q = -\mathbf{e}$, the LPCC reduces to the problem we consider. LPCCs provide a generalization to problem classes such as linear programming and finding sparse (minimum ℓ_0 norm) solutions of linear equations ([9,10]). The results in [16] show that it is hard to find approximate solutions of an LPCC. While the LPCC is a newly explored topic, LCPs are deeply studied subjects, book-length treatments of which can be found in [5,15].



The idea that Nash equilibria of games can be related to solutions of LCPs is not new. Consider a simultaneous move game with two players (I, II), where player I has m possible actions and the player II has n possible actions. The cost matrices $A_I, A_{II} \in \mathbb{R}^{m \times n}$ are such that when player I chooses action i and player II chooses action j , they incur costs $A_I(i, j)$ and $A_{II}(i, j)$ respectively. Players can also choose to play mixed strategies which are vectors defined over the probability simplex in an m -dimensional space for player I and n dimensional space for player II. A Nash equilibrium in mixed strategies in this game is defined as a pair of vectors $x^* \in \Delta^n, y^* \in \Delta^m$ such that

$$(x^*)^\top A_I y^* \leq x^\top A_I y^* \quad \forall x \in \Delta^n \quad \text{and} \quad (x^*)^\top A_{II} y^* \leq (x^*)^\top A_{II} y \quad \forall y \in \Delta^m,$$

where Δ^k is a probability simplex in $\mathbb{R}^k$, $\Delta^k := \{x \in \mathbb{R}^k \mid \sum_i x_i = 1, x_i \geq 0\}$. Assuming A_I and A_{II} are entrywise positive matrices, we define $(\tilde{x}, \tilde{y})$ as

$$\tilde{x} = \frac{x^*}{(x^*)^\top A_{II} y^*} \quad \text{and} \quad \tilde{y} = \frac{y^*}{(x^*)^\top A_I y^*}.$$

It can be shown ([5]) that $(\tilde{x}, \tilde{y})$ satisfy $\text{LCP}(M, q)$ with

$$M = \begin{bmatrix} 0 & A_I \\ A_{II}^\top & 0 \end{bmatrix} \quad \text{and} \quad q = -\mathbf{e}.$$

More generally, certain equilibria of games involving coupled constraints [11] also reduce to LCPs.

There has been prior effort at relating LCPs with independent sets ([12, 13]). Particularly, the authors of [13] show that x is the characteristic vector of a maximal independent set if and only if $(\frac{x}{\|x\|_1}, y_1, y_2)$ solves $\text{LCP}(M_G, q_G)$ with

$$M = \begin{bmatrix} A + I & -\mathbf{e} \mathbf{e}^\top \\ \mathbf{e}^\top & 0 & 0 \\ -\mathbf{e}^\top & 0 & 0 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

On the contrary, our work is to characterize ℓ_1 norm maximizing solutions of $\text{LCP}_\delta(G)$, solutions to which lie in a different space ($\mathbb{R}^n$) compared to those of $\text{LCP}(M_G, q_G)$ ($\mathbb{R}^{n+2}$). It is shown in [16] that the characteristic vectors of maximal independent sets are solutions of $\text{LCP}(G)$, but the same is not the case with solutions of $\text{LCP}_\delta(G)$ for general δ . Our work is distinct from both [16] and [13] since we consider a different class of LCPs and relate their equilibria to combinatorial structures in the graph.

Generalizations of independent sets have been studied in other contexts. The authors of [18] generalize independent sets to k -independent sets where a set I of vertices of G is said to be k -independent if I is independent and every independent subset J of G with $|J| \geq |I| - (k - 1)$ is a subset of I . It is easy to note that 0-independent sets are independent sets and 1-independent sets are unique maximum independent sets. This generalization is more restrictive, and k -independent sets need not even exist for all graphs for all $k \geq 1$. In contrast, we provide a more inclusive generalization to a union of independent cliques which includes all independent sets as a special case. We also study the special case when unique maximum independent sets exist. These sets need not always exist for a graph. The authors of [7] provide sufficient conditions on the graph under which such independent sets exist.

1.2 Organization of the paper

The rest of the paper is organized as follows. In Section 2, we provide the preliminaries and the notation used throughout the paper along with an introduction to LCPs. In Section 3, we discuss properties of $\text{LCP}_\delta(G)$ and its solutions. In Section 4, we introduce the notion of Independent Clique Solutions (ICS), provide an algorithm to find them and prove that they achieve the maximum ℓ_1 norm amongst all $\text{LCP}_\delta(G)$ solutions for $\delta < 1$. In Section 5, we extend the results of [16] to the case when $\delta \geq 1$. The paper concludes in Section 6.



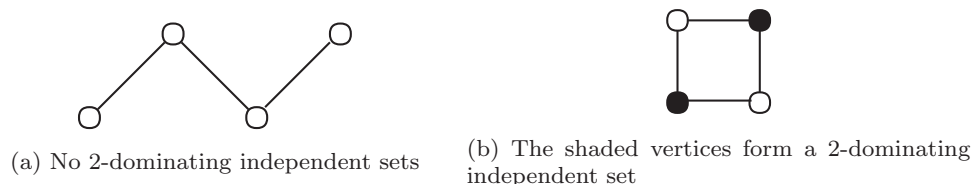


Fig. 1 Examples of existence and non-existence of 2-dominating independent sets

2 Preliminaries and Notation

For a graph G , we denote by $V(G)$ and $E(G)$ its vertex and edge sets respectively. For a vertex $i \in V(G)$, let $N_G(i)$ denote the neighborhood of i in G , i.e. $N_G(i) = \{j \in V(G) | (i, j) \in E(G)\}$. For a set of vertices $K \subset V(G)$, $N_G(K) = \bigcup_{i \in K} N_G(i) \setminus K$. Also, let $\bar{N}_G(K) = N_G(K) \cup K$ denote the closed neighborhood of K in G . For $S \subset V(G)$, let G_S denote the graph restricted to the vertex set S and for a vector x indexed by $V(G)$, let x_S denote the subvector of x with components indexed by the vertex set S . For a vector x indexed by $V(G)$, let $\sigma(x)$ be its support, i.e.

$$\sigma(x) := \{i \in V(G) | x_i > 0\}.$$

Let $\mathbf{1}_S$ denote the characteristic vector of set S , i.e.

$$\mathbf{1}_S(i) := (\mathbf{1}_S)_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

We use the standard notation $\|\cdot\|_1$ for the ℓ_1 norm. Let A denote the adjacency matrix of a graph G given by

$$A(i, j) = a_{ij} = \begin{cases} 1, & \text{if } (i, j) \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

An *independent set* of a graph is defined as a set of vertices such that none of them are neighbors. A *maximal independent set* is defined as an independent set S such that all the vertices which are not in S have at least one neighbor in S , i.e. $|N_G(i) \cap S| \geq 1 \forall i \notin S$. A *maximum independent set* is an independent set which has the highest cardinality amongst all independent sets. The cardinality of the maximum independent set is denoted by $\alpha(G)$ and is also called the independence number of the graph G . Let $\alpha_w(G)$ denote the weighted independence number for $w \geq 0$ which is the maximum sum of weights amongst all independent sets, i.e. $\alpha_w(G) := \max\{\sum_{i \in S} w_i | S \subset V(G) \text{ independent}\}$. A *dominating set* (D) is a set such that $N_G(D) \cup D = V(G)$. Thus, an independent set which is also dominating is a maximal independent set. A *k-dominating set* is defined as a dominating set D such that $|N_G(i) \cap D| \geq k \forall i \in V(G) \setminus D$. A set which is both k -dominating and independent is called a *k-dominating independent set*. It can be observed that a k -dominating independent set is also a m -dominating independent set for all $m < k$. For $k = 1$, these are equivalent to maximal independent sets and they always exist. For $k \geq 2$, they may or may not exist depending on the graph. Examples of both the cases is shown in Fig. 1.

3 $\text{LCP}_\delta(G)$

For a graph G with adjacency matrix A recall the problem $\text{LCP}(I + \delta A, -\mathbf{e}) = \text{LCP}_\delta(G)$ where δ is a positive parameter. In this section we prove some properties of $\text{LCP}_\delta(G)$ that will be used later in the paper.

Define $\mathcal{C}(x) := (I + \delta A)x$ and denote by $\mathcal{C}_i(x)$ the i^{th} component of $\mathcal{C}(x)$. $\mathcal{C}_i(x)$ is called the *discounted sum of the closed neighborhood* of i with respect to x ,

$$\mathcal{C}_i(x) = x_i + \delta \sum_{j \in V(G)} a_{ij} x_j = x_i + \delta \sum_{j \in N_G(i)} x_j. \quad (3)$$



We denote the set of solutions of $\text{LCP}_\delta(G)$ by $\text{SOL}_\delta(G)$. Clearly $x \in \text{SOL}_\delta(G)$ if and only if,

$$\begin{aligned} x_i &\geq 0, & (4) \\ \mathcal{C}_i(x) &\geq 1, & (5) \\ \text{and } x_i(\mathcal{C}_i(x) - 1) &= 0 & (6) \end{aligned}$$

$\forall i \in V(G)$. First, we establish some basic properties of $\text{SOL}_\delta(G)$ and how its elements relate to the graph G .

Lemma 1 Consider the $\text{LCP}_\delta(G) = \text{LCP}(I + \delta A, -e)$. Then,

- (a) $0 \notin \text{SOL}_\delta(G)$,
- (b) $C(x) \geq x \ \forall x \in \text{SOL}_\delta(G)$,
- (c) $\text{SOL}_\delta(G) \subset [0, 1]^n$,
- (d) If a graph G is a disjoint union of graphs G_1 and G_2 , then $\text{SOL}_\delta(G) = \text{SOL}_\delta(G_1) \times \text{SOL}_\delta(G_2)$,
- (e) For a graph G , if $x \in \text{SOL}_\delta(G)$, then $\sigma(x)$ is a $\lceil \frac{1}{\delta} \rceil$ -dominating set of G ,
- (f) For a graph G , if $x \in \text{SOL}_\delta(G)$, then $\tilde{x} = x_{\sigma(x)} \in \text{SOL}_\delta(G_{\sigma(x)})$ and $\sigma(\tilde{x}) = V(G_{\sigma(x)})$.

Proof See Section A.1. □

The following lemma characterizes integer (binary) solutions of $\text{LCP}_\delta(G)$.

Lemma 2 For a graph G and $\delta \in (0, \infty)$, x is an integer solution of $\text{LCP}_\delta(G)$ if and only if it is a characteristic vector of a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set of G

Proof See Section A.2. □

Consider the following optimization problem which gives the ℓ_1 norm maximizing solution amongst $\text{SOL}_\delta(G)$:

$\begin{aligned} \max \text{SOL}_\delta(G) \quad & \text{maximize } \mathbf{e}^\top x \\ \text{subject to } & x \in \text{SOL}_\delta(G). \end{aligned}$
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We next show that if a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set S is contained in the support of a solution x then $\mathbf{1}_S$ has ℓ_1 -norm no lesser than that of x .

Lemma 3 For a graph G and $\delta \in (0, \infty)$, if x is a solution of $\text{LCP}_\delta(G)$ and $S \subset \sigma(x)$ is a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set of G , then $\|x\|_1 \leq |S|$.

Proof See Section A.3 □

Consequently, if G is such that every solution x of $\text{LCP}_\delta(G)$ contains a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set in its support, then the solution of $\max \text{SOL}_\delta(G)$ would be attained at a characteristic vector of a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set. Clearly, such a property does not hold for all graphs G since $\lceil \frac{1}{\delta} \rceil$ -dominating independent sets do not exist in every graph. This requires us to analyze the solutions of $\max \text{SOL}_\delta(G)$ more carefully.

We now introduce a function called the *potential function*, the stationary points of which are the solutions of $\text{LCP}_\delta(G)$. This result appeared in [4] in the context of games on networks and is known more generally in the LCP literature [5]. Define,

$$\phi(x; \delta, G) := x^\top \mathbf{e} - \frac{1}{2} x^\top (I + \delta A) x \quad \forall x \in \mathbb{R}^n \quad (7)$$

where $n = |V|$.

Lemma 4 $\text{SOL}_\delta(G)$ equals the set of stationary points of $\phi(x; \delta, G)$ in $\mathbb{R}_+^n$.

Proof See Section A.4 □

We note that, when $\delta < -\frac{1}{\lambda_{\min}(A)}$, where $\lambda_{\min}(A)$ represents the minimum eigenvalue of the adjacency matrix of the graph G , the potential function is a strictly concave function. Thus, in this case, $\text{SOL}_\delta(G)$ is a singleton.



4 Solutions of $\text{LCP}_\delta(G)$ for $\delta < 1$

In this section, we assume throughout that $\delta < 1$. Our goal is to prove that the solution of $\text{maxSOL}_\delta(G)$ is achieved by a member of the class of solutions that we call independent clique solution (ICS). To prove this, we first formally define independent clique solutions and prove their existence constructively (Section 4.1). We also prove some results relating this special class of solutions to independent sets. We then show that the maximum ℓ_1 norm amongst the solutions in this class is monotonically increasing under graph inclusion (4.2). Using this, we inductively argue that the solution of $\text{maxSOL}_\delta(G)$ is achieved by an independent clique solution (4.3).

4.1 Independent Clique Solutions

First, we note the following definition:

Definition 1 Two cliques in a graph are said to be *independent* if no vertex of one clique has any vertex of the other clique as its neighbor.

Note that the above definition is a generalization of the definition of independence of vertices which are in fact K_1 's.

Definition 2 Let G be a graph and $\delta > 0$. An *independent clique solution (ICS)* is a solution of $\text{LCP}_\delta(G)$ whose support is a union of independent cliques.

We denote the set of independent clique solutions of $\text{LCP}_\delta(G)$ as $\text{ICS}_\delta(G)$. Consider the following optimization problem which gives us the ℓ_1 norm maximizing solution amongst $\text{ICS}_\delta(G)$:

$\begin{aligned} \text{maxICS}_\delta(G) \quad & \text{maximize} \quad \mathbf{e}^\top x \\ & \text{subject to} \quad x \in \text{ICS}_\delta(G) \end{aligned}$

We now show that for any graph G an ICS exists if δ is greater than a threshold $\gamma(G) \in (0, 1)$. We give a constructive proof. Algorithm 1 gives a method to construct an ICS using maximum independent sets in any graph. The justification of why this algorithm gives an ICS under suitable conditions on δ is given in Theorem 2.

Algorithm 1 Construct an Independent Clique solution

Input: $G(V, E)$
Output: x

- 1: Initialize $\bar{V} = V(G)$, $\bar{G} = G$, $\bar{S} = S$ a maximum independent set of $\bar{G}$ and $x = \mathbf{1}_S$
- 2: Set $\bar{S} = \{1, 2, \dots, |\bar{S}|\}$ and define $L = \{l \in V \mid l \text{ has only one neighbor in } \bar{S}\}$
- 3: **for** $i \in \{1, 2, \dots, |\bar{S}|\}$ **do**
- 4: Define $C_i = N_{\bar{G}}(i) \cap L$ and $\bar{C}_i = C_i \cup \{i\}$
- 5: **for all** $j \in \bar{C}_i$ **do**
- 6: $x_j = \frac{1}{1 + (|\bar{C}_i| - 1)\delta}$
- 7: **end for**
- 8: Define $V' := \bar{V} \setminus (N_{\bar{G}}(\bar{C}_i))$
- 9: Define $S' := \bar{S} \setminus \{i\}$
- 10: Set $\bar{V} \leftarrow V'$, $\bar{G} \leftarrow \bar{G}_{V'}$
- 11: Set $\bar{S} \leftarrow S'$
- 12: **end for**
- 13: **return** x

We will show that this algorithm returns x which is an ICS supported on $\bar{C}_1 \cup \bar{C}_2 \cdots \cup \bar{C}_{|\bar{S}|}$ for a maximum independent set S in G . Below we argue in Lemma 6 that these sets $\{\bar{C}_i\}$ are independent cliques. In Theorem 2, we show that the x returned in Algorithm 1 solves the LCP. We first note a few remarks.

Remark 1 In Algorithm 1, for each i , $N_{\bar{G}}(\bar{C}_i) \cap S = \{i\}$. To see this, note that $N_{\bar{G}}(\bar{C}_i) = \{i\} \cup N_{\bar{G}}(i) \cup N_{\bar{G}}(C_i)$. Since $i \in S$, $N_{\bar{G}}(i) \cap S = \emptyset$. Since $C_i \subset L$, $N_{\bar{G}}(C_i) \cap S = \emptyset$. Thus, no vertex in $N_{\bar{G}}(\bar{C}_i) \setminus \{i\}$ is in S .



Remark 2 In Algorithm 1, at the return step, $\bar{V} = \emptyset$, i.e., we eventually remove all vertices from the graph. To see this, the set of removed nodes in each step, say R_i , is such that $\bar{N}_G(k) \subseteq \bigcup_{i=1}^k R_i \subseteq V$. Taking union on both sides from $k = 1$ to $|S|$, we get $\bigcup_{k=1}^{|S|} \bar{N}_G(k) \subseteq \bigcup_{i=1}^{|S|} R_i \subseteq V$. S being a maximal independent set, $\bigcup_{i=1}^k \bar{N}_G(k) = V$ and hence $\bigcup_{i=1}^{|S|} R_i = V$.

Now, we prove a few lemmata about the algorithm that help us prove its validity. First, we show that after each iteration, the updated values S' and V' defined in Line 9 and Line 8 satisfy the conditions satisfied by $\bar{S}$ and $\bar{V}$ respectively before the iteration.

Lemma 5 *Let $\bar{V}$ and $\bar{S}$ be updated as in Line 10 and Line 11 in Algorithm 1 respectively. Then, after any iteration of the loop Line 3 to Line 12,*

- (a) $\bar{S} \subset \bar{V}$.
- (b) $\bar{S}$ is a maximum independent set of $\bar{G}$.

Proof See Section B.1 □

This lemma is useful because it gives us a way to look at each iteration of the loop in Algorithm 1 as discarding the vertices $\bar{N}_{\bar{G}}(\bar{C}_i)$ and dealing with the remaining graph. For this point of view to hold, it is necessary that the resulting $\bar{S}$ forms a maximum independent set of $\bar{V}$, which is shown by Lemma 5. We now address the first step of showing that Algorithm 1 returns an ICS, namely, showing $\{\bar{C}_i\}$ are independent cliques.

Lemma 6 $\bar{C}_i$, as defined in Line 4 of Algorithm 1, forms a clique. Moreover, any two cliques $\bar{C}_i$ and $\bar{C}_j$ for $i \neq j$ and $i, j \in S$ are independent.

Proof See Section B.2. □

To prove that Algorithm 1 outputs an ICS, we need to show that x returned at the end of the algorithm satisfies $\text{LCP}_\delta(G)$ conditions (4), (5) and (6) for all $i \in V$ and that $\sigma(x)$ is a union of independent cliques. The fact that $\sigma(x)$ is a union of independent cliques is evident from Lemma 6. To show that the $\text{LCP}_\delta(G)$ conditions are satisfied, we show for the x generated when we exit from the algorithm, these conditions are satisfied for $\delta \in [\gamma(G), 1)$ in Theorem 2.

Before we discuss Theorem 2, we have a lemma describing the conditions on the neighborhood of a vertex j so that with such a neighborhood and corresponding discounted sum of closed neighborhood, $x_j = 0$ for the $\text{LCP}_\delta(G)$ conditions (4), (5) and (6) to be satisfied for j .

Lemma 7 (Two Clique Lemma) *Consider a vertex i and cliques $C_1 = K_n$ and $C_2 = K_m$ of graph G such that $C_1 \subset N_G(i)$ and there exists another clique $C_2 = K_m$ for some m with $C_1 \cap C_2 = \emptyset$ such that $V(C_2) \cap N_G(i) \neq \emptyset$. Let $x \in \mathbb{R}^n$ such that*

$$x_j = \begin{cases} \frac{1}{1+(n-1)\delta} & \text{if } j \in C_1, \\ \frac{1}{1+(m-1)\delta} & \text{if } j \in C_2, \\ 0 & \text{if } j = i. \end{cases}$$

and $x_j \geq 0 \forall j \in N_G(i)$. Then, if $\delta \in [\gamma(G), 1)$, we have that for i the $\text{LCP}_\delta(G)$ conditions (4), (5) and (6) are satisfied by the above x , where

$$\gamma(G) = \frac{\omega(G) - 3 + \sqrt{(\omega(G) - 3)^2 + 4(\omega(G) - 1)}}{2(\omega(G) - 1)}, \quad (8)$$

and $\omega(G)$ represents the size of the largest clique in the graph G .

Proof See Section B.3. □

Remark 3 Note that for any graph with at least one edge, the value of $\omega(G)$ is at least 2 and hence the value of $\gamma(G)$ is at least $\frac{\sqrt{5}-1}{2}$, which is the golden ratio. For trees G , $\gamma(G) = \frac{\sqrt{5}-1}{2}$. We do not know of any deeper significance or interpretations of the appearance of the golden ratio in this problem.

We say that vertex i is *fully connected* to a clique C if all the vertices of C are neighbors of i , i.e. $j \in N_G(i) \forall j \in C$.



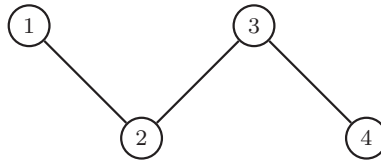


Fig. 2 An example graph showing tightness of $\gamma(G)$

Theorem 2 For $\text{LCP}_\delta(G)$ with $\delta \in [\gamma(G), 1)$ where $\gamma(G)$ is defined as in (8), Algorithm 1 returns an ICS whose support is a union of $\alpha(G)$ independent cliques.

Proof See Section B.4 □

Example 1 In this example, we discuss the tightness of the bound $\delta \geq \gamma(G)$ in Theorem 2. We will show that for a path of length 4 as shown in Fig. 2, the vector returned by Algorithm 1 is not a solution when $\delta < \gamma(G) = \frac{\sqrt{5}-1}{2}$. Algorithm 1 returns $x = \left(1, 0, \frac{1}{1+\delta}, \frac{1}{1+\delta}\right)$. Now, $\sigma(x)$ is a union of independent cliques whose vertex sets are given by $\{1\}$ and $\{3, 4\}$ respectively. For x to be an ICS, $\mathcal{C}_2(x) \geq 1$. Now, $\mathcal{C}_2(x) = \delta \left(1 + \frac{1}{1+\delta}\right) < 1$ for $0 < \delta < \frac{\sqrt{5}-1}{2}$. This example shows that there are cases where $\delta \in [\gamma(G), 1)$ is not only sufficient but also necessary.

Theorem 2 proves that for a given $\delta \in [\gamma(G), 1)$, an ICS (say x_δ) exists. We now show that there exists an ICS of $\text{LCP}_{\delta'}(G)$ where $\delta' \geq \delta$, such that its support is same as that of x_δ . To see this, we note that given a set of independent cliques $\mathcal{K} = \{C_1, \dots, C_n\}$, the only possible ICS for some $\delta \in [\gamma(G), 1)$ with its support as $\mathcal{K}$ is $x^\mathcal{K}(\delta)$ given by

$$x_i^\mathcal{K}(\delta) = \begin{cases} \frac{1}{1+(|C_j|-1)\delta}, & \text{if } i \in C_j, C_j \in \mathcal{K} \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

This gives us the following result.

Corollary 1 For a given set of independent cliques $\mathcal{K} = \{C_1, \dots, C_n\}$, if $x^\mathcal{K}$ given by (9) is an ICS for some $\delta > 0$, then there exists an ICS with the same support $\mathcal{K}$ for any $\delta' \in [\delta, 1)$.

Proof See Section B.5 □

Remark 4 Consider a maximum independent set $S = \{1, \dots, |S|\}$ and a union of independent cliques $\mathcal{K} = \{\bar{C}_1, \dots, \bar{C}_{|S|}\}$ obtained from Algorithm 1 and $x^\mathcal{K}(\cdot) : [\gamma(G), 1) \rightarrow [0, 1]^{|V|}$ as defined in (9). Then, $x^\mathcal{K}(\delta) \in \text{SOL}_\delta(G)$ for $\delta \in [\gamma(G), 1)$. This follows since Theorem 2 shows that the output of Algorithm 1 viewed as a function of δ is a valid ICS of $\text{LCP}_\delta(G)$ for any $\delta \in [\gamma(G), 1)$.

We now show that there exists a set of independent cliques $\mathcal{K}$ with $|\mathcal{K}| = \alpha(G)$, such that $x^\mathcal{K}(\delta)$ is a solution of $\text{maxICS}_\delta(G)$ for $\delta \in [\eta(G), 1)$. Here, $\eta(G)$ is defined as,

$$\eta(G) = \max \left\{ \frac{\alpha(G)(\omega(G) - 1) - \omega(G)}{\alpha(G)(\omega(G) - 1)}, \frac{\omega(G) - 3 + \sqrt{(\omega(G) - 3)^2 + 4(\omega(G) - 1)}}{2(\omega(G) - 1)} \right\}, \quad (10)$$

Proposition 1 For $\delta \in [\eta(G), 1)$, there exists a union of independent cliques $\mathcal{K}$ and a solution $x^\mathcal{K}(\delta)$ of $\text{maxICS}_\delta(G)$ such that $|\mathcal{K}| = \alpha(G)$.

Proof See Section B.6 □



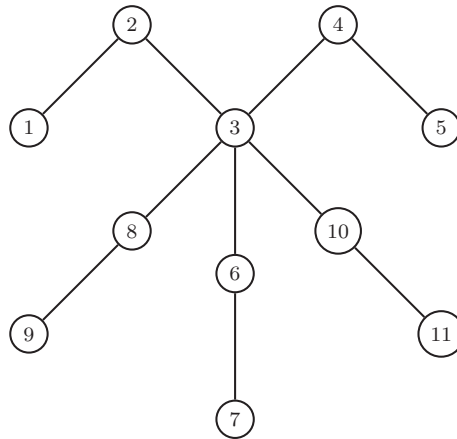


Fig. 3 An example graph showing tightness of $\eta(G)$

Example 2 In this example, we discuss the tightness of the bound $\delta \geq \eta(G)$. Note that $\eta(G)$ is defined as the maximum amongst two terms, the first of which is $\gamma(G)$. This term comes in because we assume the existence of a set of independent cliques $\mathcal{K}_1$ such that $|\mathcal{K}_1| = \alpha(G)$ for which we refer to Theorem 2 which requires $\delta \in [\gamma(G), 1)$ and is tight for certain graphs. The second term in the maximum function in the definition of $\eta(G)$ is $\kappa(G)$. We now show the tightness of this term with the graph in Figure 3 as an example. Note that $\omega(G) = 2$ and $\alpha(G) = 6$, hence $\gamma(G) = \frac{\sqrt{5}-1}{2}$ and $\kappa(G) = \frac{2}{3}$ giving $\eta(G) = \kappa(G)$. The maximum independent set is $\{1, 3, 5, 7, 9, 11\}$. Also, $\mathcal{K}_1 = \{1, 3, 5, 7, 9, 11\}$ and consider $\mathcal{K}_2 = \{\{1, 2\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}\}$. $\|x_{\delta}^{\mathcal{K}_1}(G)\|_1 = 6$ and the solution is valid for $\delta \geq \frac{1}{2}$. Also, $\|x_{\delta}^{\mathcal{K}_2}(G)\|_1 = \frac{10}{1+\delta}$ and it is valid for $\delta \geq \frac{1}{4}$. Now, for $\delta < \frac{2}{3} = \frac{\alpha(G)(\omega(G)-1)-\omega(G)}{\alpha(G)(\omega(G)-1)}$, we have $\frac{10}{1+\delta} > 6$ and hence $\|x_{\delta}^{\mathcal{K}_2}(G)\|_1 > \|x_{\delta}^{\mathcal{K}_1}(G)\|_1$. But we want $\|x_{\delta}^{\mathcal{K}_2}(G)\|_1 \leq \|x_{\delta}^{\mathcal{K}_1}(G)\|_1$ in Proposition 1 for which it is not only sufficient but also necessary to have $\delta \geq \frac{\alpha(G)(\omega(G)-1)-\omega(G)}{\alpha(G)(\omega(G)-1)}$ in this example. Thus, since the both the terms over which the maximization takes place in the definition of $\eta(G)$ is tight, it is a tight bound for Proposition 1.

4.2 Monotonicity of the Maximum ℓ_1 Norm amongst ICSs

In this section, we show that the solutions of $\max\text{ICS}_{\delta}(G)$ for some $\delta \in [\eta(G), 1)$ are monotonic with respect to the induced subgraph relation. We first prove a lemma which gives conditions under which a graph G' formed by adding a vertex to a graph G admits an ICS $(x^{\mathcal{K}}(\delta), 0)$ where $x^{\mathcal{K}}(\delta)$ is an ICS of G and $\mathcal{K}$ is a set of independent cliques in G . In other words, an ICS of G' can be formed by appending a 0 corresponding to the new vertex to an ICS of G .

Lemma 8 Let $x^{\mathcal{K}}(\delta) \in \text{ICS}_{\delta}(G)$ where $\mathcal{K}$ are a set of independent cliques in G . Let $\delta \in [\gamma(G'), 1)$, where G' is such that

1. $V(G') = V(G) \cup \{i\}$,
2. $\exists C \in \mathcal{K}$ s.t. $(i, j) \in E(G') \forall j \in C$,
3. $\exists C' \in \mathcal{K}, C' \neq C$ s.t. $(i, j) \in E(G')$ for some $j \in C'$.

Then $x^{\mathcal{K}}(\delta)$ defined as

$$x_j^{\mathcal{K}}(\delta) = \begin{cases} x_j^{\mathcal{K}}(\delta) & \text{if } j \in V(G), \\ 0 & \text{if } j = i. \end{cases}$$

is an ICS of $\text{LCP}_{\delta}(G')$.

Proof In G' , i is a vertex such that $C \subset N_{G'}(i)$ and $C' \cap N_{G'}(i) \neq \emptyset$. Thus, using Two Clique Lemma (Lemma 7), i satisfies the $\text{LCP}_{\delta}(G)$ conditions (4),(5) and (6) for $\delta \in [\gamma(G'), 1)$. $\square$



We now show the main result of this subsection showing the monotonicity of maximum ℓ_1 norm amongst Independent Clique Solutions. In the following lemma, we have a graph G' which is constructed by adding a vertex i to G and it is shown that the maximum ℓ_1 norm amongst Independent Clique Solutions of G' is greater than that amongst G . Recall that Proposition 1 showed that there always exists a union of independent cliques $\mathcal{K}$ of G with $|\mathcal{K}| = \alpha(G)$ such that $x_{\max}^{\mathcal{K}}(\delta)$ as defined in (9) is a solution of $\max\text{ICS}_{\delta}(G)$.

Lemma 9 Consider two graphs $G(V, E)$ and $G'(V', E')$, where $V' = V \cup \{i\}$ and $G'_V = G$. Let $x^{\mathcal{K}}(\delta)$ and $x^{\mathcal{K}'}(\delta)$ be solutions of $\max\text{ICS}_{\delta}(G)$ and $\max\text{ICS}_{\delta}(G')$ respectively, where $\mathcal{K}$ and $\mathcal{K}'$ are sets of independent cliques of G and G' such that $|\mathcal{K}| = \alpha(G)$ for some $\delta \in [\eta(G'), 1)$ with $\eta(\cdot)$ is defined as in (10). Then, $\|x^{\mathcal{K}}(\delta)\|_1 \leq \|x^{\mathcal{K}'}(\delta)\|_1$.

Proof See Section B.7 □

4.3 Proof of Main Results

We have shown the existence of ICSs in any graph and the monotonicity of the solution of $\max\text{ICS}_{\delta}(G)$ with respect to the number of vertices in the graph. Next, we show that for $\delta \in [\eta(G), 1)$, any solution of $\max\text{ICS}_{\delta}(G)$ is in fact also a solution of $\max\text{SOL}_{\delta}(G)$. To show this, we first show that the ℓ_1 norm of any ICS is greater than that of any full support solution of the $\text{LCP}_{\delta}(G)$.

Proposition 2 Let $\delta > 0$ and consider a graph $G = (V, E)$. The ℓ_1 norm of any ICS is greater than that of any solution of $\text{LCP}_{\delta}(G)$ with support V .

Proof Let $V = \{1, 2, \dots, n\}$. Let $x^{\mathcal{K}}$ be an ICS of $\text{LCP}_{\delta}(G)$ and let x be a solution of $\text{LCP}_{\delta}(G)$ with support V . Clearly, for all $i \in V$,

$$C_i(x) = x_i + \delta \sum_{j=1}^n a_{ij}x_j = 1.$$

Summing over $i \in \sigma(x^{\mathcal{K}})$, we get

$$\sum_{i \in \sigma(x^{\mathcal{K}})} x_i = |\sigma(x^{\mathcal{K}})| - \delta \sum_{i \in \sigma(x^{\mathcal{K}})} \sum_{j=1}^n a_{ij}x_j. \quad (11)$$

Now,

$$\sum_{i \in V} x_i = \sum_{i \in \sigma(x^{\mathcal{K}})} x_i + \sum_{i \in \sigma(x^{\mathcal{K}})^c} x_i,$$

whereby from (11),

$$\begin{aligned} \sum_{i \in V} x_i &= \sum_{i \in \sigma(x^{\mathcal{K}})^c} x_i + |\sigma(x^{\mathcal{K}})| - \delta \sum_{i \in \sigma(x^{\mathcal{K}})} \sum_{j=1}^n a_{ij}x_j \\ &= |\sigma(x^{\mathcal{K}})| - \delta \sum_{i \in \sigma(x^{\mathcal{K}})} \sum_{j \in \sigma(x^{\mathcal{K}})} a_{ij}x_j - \sum_{j \in \sigma(x^{\mathcal{K}})^c} x_j \left(\delta \sum_{i \in \sigma(x^{\mathcal{K}})} a_{ij} - 1 \right). \end{aligned} \quad (12)$$

Let $\mathcal{K} = \{C_j | 1 \leq j \leq m\}$ where C_j 's are disjoint cliques and let $n_j = |C_j|$. Then,

$$\begin{aligned} \sum_{i \in \sigma(x^{\mathcal{K}})} x_i^{\mathcal{K}} &= \sum_{i=1}^m \frac{n_i}{1 + (n_i - 1)\delta}, \\ \text{and } |\sigma(x^{\mathcal{K}})| &= \sum_{i=1}^m n_i. \end{aligned}$$



Also,

$$\delta \sum_{i \in \sigma(x^{\mathcal{K}})} \sum_{j \in \sigma(x^{\mathcal{K}})} a_{ij} x_j = \delta \sum_{i=1}^m \frac{n_i(n_i - 1)}{1 + (n_i - 1)\delta},$$

which follows from the fact that amongst the vertices in $\sigma(x^{\mathcal{K}})$ only the ones which are in the complete networks are neighbors with each other. This gives,

$$\begin{aligned} |\sigma(x^{\mathcal{K}})| - \delta \sum_{i \in \sigma(x^{\mathcal{K}})} \sum_{j \in \sigma(x^{\mathcal{K}})} a_{ij} x_j &= \sum_{i=1}^m n_i - \delta \sum_{i=1}^m \frac{n_i(n_i - 1)}{1 + (n_i - 1)\delta} \\ &= \sum_{i \in \sigma(x^{\mathcal{K}})} x_i^{\mathcal{K}}. \end{aligned}$$

Thus, (12) becomes

$$\sum_{i \in V} x_i + \sum_{j \in \sigma(x^{\mathcal{K}})^c} x_j (\delta \sum_{i \in \sigma(x^{\mathcal{K}})} a_{ij} - 1) = \sum_{i \in \sigma(x^{\mathcal{K}})} x_i^{\mathcal{K}}.$$

Now, consider $(\delta \sum_{i \in \mathcal{K}} a_{ij} - 1)$. Note that each vertex not in $\sigma(x^{\mathcal{K}})$ has at least $\lceil \frac{1}{\delta} \rceil$ neighbors in $\sigma(x^{\mathcal{K}})$, since it is a necessary condition for $x^{\mathcal{K}}$ to be a solution. Thus, $(\delta \sum_{i \in \sigma(x^{\mathcal{K}})} a_{ij} - 1) \geq (\lceil \frac{1}{\delta} \rceil \delta - 1) \geq 0$. Thus the second term in the LHS of the above equation is non-negative. It follows that the ℓ_1 norm of any ICS is greater than that of any full support solution of the $\text{LCP}_\delta(G)$. $\square$

We now provide the proof of our main result (Theorem 1) which we prove by induction using Proposition 2.

Proof (Theorem 1.) First, we note from Proposition 2 that the ℓ_1 norm of any ICS corresponding to maximum independent sets is at least as much as that of any solution with full support. Thus, we only need to prove that the ℓ_1 norm of solution of $\text{maxICS}_\delta(G)$ is at least as much as that of any other solution which does not have full support.

Consider a solution x such that $\sigma(x) \subset V$ and assume that it maximizes the ℓ_1 norm. Then, consider the restricted graph $G_{\sigma(x)}$. We have that in this graph, x is a full support solution. Thus, $\|x\|_1 \leq \|x_{\max}\|_1$, where $x_{\max}$ is a solution of $\text{maxICS}_\delta(G_{\sigma(x)})$. Also, by Lemma 9, we get $\|x_{\max}\|_1 \leq \|x'_{\max}\|_1$, where $x'_{\max}$ is a solution of $\text{maxICS}_\delta(G)$. Thus, the ℓ_1 norm of x is less than or equal to the ℓ_1 norm of any solution of $\text{maxICS}_\delta(G)$ for any $\delta \in [\eta(G), 1)$. $\square$

4.4 Graphs with a Unique Maximum Independent Set

For a graph $G = (V, E)$ set $I \subseteq V$ is said to be the *unique maximum independent set* if there is no other independent set $I' \subseteq V$ with $|I'| \geq |I|$. In this section, we show that when a unique independent set (S) exists for a graph G , then $x = \mathbf{1}_S$ is a solution of $\text{maxSOL}_\delta(G)$ for $\delta \in [\eta(G), 1)$.

Corollary 2 For a graph G , when a unique maximum independent set S exists, the maximum ℓ_1 norm amongst the solutions of $\text{LCP}_\delta(G)$ is achieved by the characteristic vector of S for $\delta \in [\eta(G), 1)$.

Proof From Theorem 1 and Proposition 1, we know that there exists an ICS $x^{\mathcal{K}}(\delta)$ which solves $\text{maxSOL}_\delta(G)$ where $\mathcal{K}$ is a union of disjoint independent cliques and $|\mathcal{K}| = \alpha(G)$. Let $\mathcal{K} = \{C_1, \dots, C_{\alpha(G)}\}$. We claim that all the cliques $\{C_1, \dots, C_{\alpha(G)}\}$ are K_1 's (single vertices). Suppose not, i.e. suppose for some $j \in \{1, \dots, \alpha(G)\}$, C_j has at least two vertices (say j_1 and j_2). Then, we can construct two distinct independent sets (S_1 and S_2) by choosing one vertex from each of C_i for $i \in \{1, \dots, \alpha(G)\} \setminus j$ and choosing j_1 in S_1 and j_2 in S_2 with $|S_1| = |S_2| = \alpha(G)$. But this violates the definition of unique maximum independent sets. Thus, each C_j for $j \in \{1, \dots, \alpha(G)\}$ must have only one vertex and thus $\mathcal{K}$ itself is the unique maximum independent set. Hence, we have that $\mathbf{1}_S$ is a solution of $\text{LCP}_\delta(G)$ which maximizes the ℓ_1 norm. $\square$



5 Solutions of $\text{LCP}_\delta(G)$ for $\delta \geq 1$

The case when $\delta = 1$ has been extensively studied in [16], wherein they show that the weighted independence number of the graph is also the maximum weighted ℓ_1 norm over the solution set of a linear complementarity problem ($\text{LCP}(G)$).

The case when $\delta > 1$ is very similar to the case when $\delta = 1$. Lemma 1 (e) shows that in this case the support of any solution of $\text{LCP}_\delta(G)$ is a 1-dominating set. Also, Lemma 2 shows that x is an integer solution of $\text{LCP}_\delta(G)$ if and only if it is the characteristic vector of a maximal independent set. We state the following theorem similar to (1) for $\delta \geq 1$. The proof follows a similar line of reasoning like Theorem 1 of [16].

Theorem 3 *For the $\text{LCP}_\delta(G)$, $\delta > 1$, the maximum weighted ℓ_1 norm is achieved at the characteristic vector of a maximum weighted independent set, i.e.*

$$\alpha_w(G) = \max\{w^\top x \mid x \text{ solves } \text{LCP}_\delta(G)\},$$

where $\alpha_w(G)$ represents the weighted independence number with weights w .

Proof See Section C.1. □

Thus, we have proved that for the case $\delta \geq 1$, the weighted ℓ_1 norm amongst the set $\text{SOL}_\delta(G)$ is maximized at the characteristic vector of the maximum weighted independent set. Particularly, the ℓ_1 norm is maximized by maximum independent sets and they are also independent clique solutions.

Now, we discuss some generalizations of our setting and illustrate why they are fairly non-trivial and why our current exposition doesn't directly extend to those settings. The case is that of considering different $\delta_{i,j}$ for each edge of the graph instead of a common δ for each interaction. This case can quickly become less interesting since it leads to either unique or no solutions to the constraint set $\text{LCP}_\delta(G)$. To see this, consider the case of graph of a clique with 3 vertices, having a different $\delta_{i,j}$ on each edge would constrain the LCP solutions to be a solution of a linear system with 3 variables with an additional constraint that each coordinate is non-negative. A solution to this problem may not exist for many values of $\delta_{i,j}$ and the constraint on $\delta_{i,j}$ which allows some solutions would be quite convoluted and graph-specific. Even if we were able to find such a constraint to isolate the interesting regime, the solution would be unique rendering the optimization problem itself non-interesting since there would be only 1 feasible solution.

The other possible immediate generalization one can think of is the case when $\delta < \gamma(G)$. The first thing to note in this case is that ICS may not exist. On trying a few examples, one can't really see a common pattern in the solutions that maximize the ℓ_1 norm. One observation is that by considering smaller and smaller values of δ , one quickly moves towards the LCP having a unique solution (an uninteresting case with respect to the optimization problem defined), but this is still an unproved conjecture.

6 Conclusion

In this paper, we identified the ℓ_1 norm maximizing solutions of the $\text{LCP}_\delta(G)$. We introduced a new notion of independent clique solutions. These are generalizations of independent sets with independent sets being the special case in which each clique is degenerate. We showed that for $\delta \geq \eta(G)$, the ℓ_1 norm maximizing ICSs are also the ℓ_1 norm maximizing solutions of $\text{LCP}_\delta(G)$.

The authors of [4] have shown that for $\delta < \frac{1}{|\lambda_{\min}(A)|}$, where $\lambda_{\min}(G)$ is the lowest eigenvalue of the adjacency matrix of graph G , $\text{LCP}_\delta(G)$ has a unique solution. Thus, that solution itself is ℓ_1 norm maximizing. Our results identify certain ℓ_1 norm maximizing solutions for $\delta \geq \eta(G)$. For $\delta \in (\frac{1}{|\lambda_{\min}(A)|}, \eta(G))$, the question of which solutions of $\text{LCP}_\delta(G)$ maximize the ℓ_1 norm remains open.

A Proofs from Section 3

A.1 Proof of Lemma 1

- (a) From (3), $\mathcal{C}_i(0) = 0$ which implies (5) is violated. Thus, $0 \notin \text{SOL}_\delta(G)$.
- (b) From (3), $\mathcal{C}_i(x) = x_i + \delta \sum_{j \in N_G(i)} x_j$. Since $\delta \geq 0$ and $x_i \geq 0 \forall i \in V(G)$ from (4), $\mathcal{C}_i(x) \geq x_i$. Thus, $\mathcal{C}(x) \geq x \forall x \in \text{SOL}_\delta(G)$.



- (c) We know that $x \geq 0$. Let $x_i > 0$ for some i . Then, from (6), $\mathcal{C}_i(x) = 1$. From part (b) of Lemma 1, $x_i \leq \mathcal{C}_i(x) = 1$. Thus, $x_i \in [0, 1] \forall i \in V(G)$.
- (d) First, consider $x^1 \in \text{SOL}_\delta(G_1)$ and $x^2 \in \text{SOL}_\delta(G_2)$, we prove that $x = (x^1, x^2) \in \text{SOL}_\delta(G)$. Since $x^1, x^2 \geq 0$ we have $x \geq 0$. Next, note that since G_1 and G_2 are disjoint, $N_{G_1}(i) = N_G(i) \forall i \in V(G_1)$ and $N_{G_2}(i) = N_G(i) \forall i \in V(G_2)$. Thus, for $i \in V(G_1)$, $\mathcal{C}_i(x) = x_i + \delta \sum_{j \in N_G(i)} x_j = x_i^1 + \delta \sum_{j \in N_{G_1}(i)} x_j^1 = \mathcal{C}_i(x^1)$. Similarly, for $i \in V(G_2)$, $\mathcal{C}_i(x) = \mathcal{C}_i(x^2)$. Thus we get $\mathcal{C}_i(x) \geq 1 \forall i \in V(G)$. Also, $x_i^1(\mathcal{C}_i(x^1) - 1) = 0 \forall i \in V(G_1)$ and $x_i^2(\mathcal{C}_i(x^2) - 1) = 0 \forall i \in V(G_2)$ imply $x_i(\mathcal{C}_i(x) - 1) = 0 \forall i \in V(G)$. Thus, $x \in \text{SOL}_\delta(G)$. Next, given $x \in \text{SOL}_\delta(G)$, we show that $x_{G_1} \in \text{SOL}_\delta(G_1)$ and $x_{G_2} \in \text{SOL}_\delta(G_2)$, where $x_{G_1} \in \mathbb{R}^{|V(G_1)|}$, $x_{G_2} \in \mathbb{R}^{|V(G_2)|}$ and $x = (x_{G_1}, x_{G_2})$. Clearly $x_{G_1} \geq 0$. Using $N_{G_1}(i) = N_G(i) \forall i \in V(G_1)$, we get $\mathcal{C}_i(x) = \mathcal{C}_i(x_{G_1})$ and thus x_{G_1} satisfies both (5) and (6). Hence, $x_{G_1} \in \text{SOL}_\delta(G_1)$. Similarly, $x_{G_2} \in \text{SOL}_\delta(G_2)$.
- (e) From Lemma 1 (c), $0 \leq x_i \leq 1 \forall i \in V(G)$. Thus $\mathbf{1}_{\sigma(x)} \geq x$. Thus, for all $i \in V(G)$, $\mathcal{C}_i(\mathbf{1}_{\sigma(x)}) \geq \mathcal{C}_i(x) \geq 1$. Let $i \notin \sigma(x)$. For this i we have, $\mathcal{C}_i(\mathbf{1}_{\sigma(x)}) = 0 + \delta \sum_{j \in N_G(i)} x_j = \delta |N_G(i) \cap \sigma(x)|$. Thus, $|N_G(i) \cap \sigma(x)| \geq \frac{1}{\delta}$. Since, $|N_G(i) \cap \sigma(x)|$ is a positive integer, $|N_G(i) \cap \sigma(x)| \geq \lceil \frac{1}{\delta} \rceil$. Thus, $\sigma(x)$ is a $\lceil \frac{1}{\delta} \rceil$ -dominating set of G .
- (f) For $i \in V(G_{\sigma(x)})$, $\tilde{x}_i = x_i > 0$. Thus, $\sigma(\tilde{x}) = V(G_{\sigma(x)})$. Note that for a vertex i in $G_{\sigma(x)}$, the discounted sum of the closed neighborhood $\tilde{\mathcal{C}}(\tilde{x}) = \tilde{x}_i + \delta \sum_{j \in N_{G_{\sigma(x)}}(i)} \tilde{x}_j = x_i + \delta \sum_{j \in N_G(i)} x_j = \mathcal{C}_i(x)$ since $x_i = 0$ for $i \notin \sigma(x)$. Hence, $\tilde{x} = x_{\sigma(x)} \in \text{SOL}_\delta(G_{\sigma(x)})$.

A.2 Proof of Lemma 2

Let x be an integer solution of $\text{LCP}_\delta(G)$. Then, by Lemma 1 (c), x is a binary vector, and hence $x = \mathbf{1}_S$ for a subset S of vertices in G . If S is not an independent set, then there exist $i, j \in S$ such that $j \in N_G(i)$, i.e. $x_i = x_j = 1$, whereby $\mathcal{C}_i(x) = x_i + \delta \sum_{j \in N_G(i)} x_j \geq 1 + \delta > 1$. Thus, $\mathcal{C}_i(x) - 1 > 0$ and $x_i > 0$ imply (6) is violated. This gives a contradiction. Hence, S must be an independent set. From Lemma 1 (e) we have that S must also be a $\lceil \frac{1}{\delta} \rceil$ -dominating set, as required.

For the converse, let S be a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set of G . Then, for $i \in S$, $\mathcal{C}_i(x) = 1 + \delta \sum_{j \in N_G(i)} x_j = 1$ since $x_j = 0 \forall j \in N_G(i)$. Also, for $i \notin S$, $\mathcal{C}_i(x) = 0 + \delta \sum_{j \in N_G(i)} x_j \geq \delta \lceil \frac{1}{\delta} \rceil \geq \delta \times \frac{1}{\delta} = 1$. Thus, we have $\mathcal{C}_i(x) \geq 1$ and $x_i(\mathcal{C}_i(x) - 1) = 0$ for all $i \in V(G)$. Hence, if x is a characteristic vector of a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set of G , then it is an integer solution of $\text{LCP}_\delta(G)$.

A.3 Proof of Lemma 3

Let $U = \sigma(x) \setminus S$. From (6), we know that

$$\mathcal{C}_i(x) = x_i + \delta \sum_{j \in V(G)} a_{ij} x_j = 1 \forall i \in \sigma(x). \quad (13)$$

Summing over $i \in \sigma(x)$, we get

$$\sum_{i \in V(G)} x_i + \delta \sum_{j \in \sigma(x)} \sum_{i \in V(G)} a_{ij} x_j = |\sigma(x)| = |S| + |U|.$$

On rearranging,

$$\begin{aligned} |S| - \mathbf{e}^\top x &= \delta \sum_{i \in \sigma(x)} \sum_{j \in V(G)} a_{ij} x_j - |U| \\ &= \delta \sum_{i \in S} \sum_{j \in U} a_{ij} x_j + \delta \sum_{i \in U} \sum_{j \in V(G)} a_{ij} x_j - |U| \end{aligned} \quad (14)$$



$$= \delta \sum_{j \in U} \sum_{i \in S} a_{ij} x_j + \sum_{i \in U} (\delta \sum_{j \in V(G)} a_{ij} x_j - 1) \quad (15)$$

$$\begin{aligned} &= \delta \sum_{j \in U} N_S(j) x_j - \sum_{i \in U} x_i \\ &\geq \sum_{j \in U} x_j - \sum_{i \in U} x_i = 0 \end{aligned} \quad (16)$$

The equality in (14) follows from the fact that $a_{ij} = 0 \forall i, j \in S$ and $x_j = 0 \forall j \notin \sigma(x)$, (15) follows from (13). The inequality (16) follows since S is a $\lceil \frac{1}{\delta} \rceil$ -dominating independent set of G .

A.4 Proof of Lemma 4

We find the Karush Kuhn Tucker (KKT) conditions of the following maximization problem:

$$\max \phi(x; \delta, G) \text{ s.t. } x_i \geq 0.$$

We get x is a stationary point iff $\exists \mu \geq 0$ such that $\mathbf{e} - (I + \delta A)x = -\mu$ and $x^\top \mu = 0$. These conditions are equivalent to the $\text{LCP}_\delta(G)$ conditions in equations (4), (5) and (6).

B Proofs from Section 4

B.1 Proof of Lemma 5

- (a) It suffices to show that $S' \subset V'$ with S' and V' as defined in Line 8 and Line 9. By definition, we have $S' \subset V$. To show that $S' \subset V'$, we need to show that in the vertices we removed from V to form V' , there is no member of S' . Suppose we have executed Line 11 of the k^{th} iteration of the algorithm for some $k \leq |S|$. The set of vertices removed from V at this stage to form V' is $\bigcup_{i=1}^k (\bar{C}_i \cup Q_i)$ where $Q_i = N_{\bar{G}}(\bar{C}_i)$, and we need to show that $\left(\bigcup_{i=1}^k (\bar{C}_i \cup Q_i) \right) \cap S' = \emptyset$. Now, for each i , $(\bar{C}_i \cup Q_i)$ contains i which is a member of S but not S' . The set $(\bar{C}_i \cup Q_i)$ also has neighbors of i which can not be in S' since $S' \subset S$ and S is an independent set. Lastly, $(\bar{C}_i \cup Q_i)$ has neighbors of C_i (defined on Line 4 of Algorithm 1) that are also not in S' since C_i is such that $\forall c \in C_i, N_G(c) \cap S = \{i\}$ and hence $N_G(c) \cap S' = \emptyset$. Since this is true for all $i \in \{1, 2, \dots, k\}$, $\bigcup_{i=1}^k (\bar{C}_i \cup Q_i) \cap S' = \emptyset$, and hence $S' \subset V'$.
- (b) We show this by induction on the iteration number. At the start of the first iteration of the for loop (Line 3 to Line 12 in Algorithm 1), $\bar{S}$ is a maximum independent set of $\bar{G}$. Assume that $\bar{S}$ is a maximum independent set of $\bar{G}$ after $k-1$ iterations (i.e., at the start of iteration k). To show that the claim is true at the start of the $(k+1)^{\text{st}}$ iteration, it suffices to show that S' is a maximum independent set of $\bar{G}_{V'}$ at the end of k^{th} iteration. Note that $|S'| = |\bar{S}| - 1$, since only one element (k) is removed from $\bar{S}$ to form S' . Suppose S' is not a maximum independent set in $\bar{G}_{V'}$. Then there exists an independent set S'' of $\bar{G}_{V'}$ such that $|S''| > |S'| = |\bar{S}| - 1$ and hence $|S''| \geq |\bar{S}|$. Now note that $S' \subset V'$ and since $V' = \bar{V} \setminus (\bar{N}_{\bar{G}}(\bar{C}_k))$ there is no edge between any vertex in V' and $\{k\}$. But then $\hat{S} = S'' \cup \{k\}$ is an independent set of $\bar{G}$ with $|\hat{S}| = |S''| + 1 > |\bar{S}|$, which is a contradiction. Thus, S' is a maximum independent set in $\bar{G}_{V'}$.

B.2 Proof of Lemma 6

First we prove that $\bar{C}_i$ forms a clique for all $i \in \{1, 2, \dots, |S|\}$. At the i^{th} iteration of Algorithm 1 let C_i be as defined in Line 4 and let $\bar{S}$ be a maximum independent set of $\bar{G}$ with $\bar{V} = V(\bar{G})$. To show that $\bar{C}_i$ forms a clique, it suffices to show that C_i forms a clique since all vertices of C_i are neighbors of i . We prove this by contradiction. Suppose C_i is not a clique. Then choose an independent set S_i of C_i such that it has at least two vertices. Consider $\hat{S} = (S \setminus \{i\}) \cup S_i$. We claim that $\hat{S}$ is an independent set. To prove this, it suffices to show that $\forall p \in S \setminus \{i\}, \forall q \in S_i, (p, q)$ is not an edge in G . Now, $S_i \subset L$ whereby each vertex in S_i has only one neighbor in S . Moreover, $S_i \subset N_G(i)$, whereby for all vertices in S_i , their only neighbor in S is i . Consequently, $\hat{S}$ is an



independent set. Moreover, $|\widehat{S}| > |S|$, which is a contradiction since S must be a maximum independent set of G by definition. Hence, C_i and therefore $\bar{C}_i$ must be a clique. Clearly, this holds for $\forall i \in \{1, 2, \dots, |S|\}$.

Now, consider two cliques $\bar{C}_i$ and $\bar{C}_j$ for $i \neq j$. Without loss of generality, we assume $i < j$. Then, $\bar{C}_j \subset V \setminus \bar{N}_G(\bar{C}_i)$. Thus, $\bar{C}_j$ can have no neighbors of $\bar{C}_i$ as its elements. Hence, $\bar{C}_i$ and $\bar{C}_j$ are independent.

B.3 Proof of Two Clique Lemma (Lemma 7)

We will show that for the x given the statement of Lemma 7, $C_i(x) \geq 1$ for the interval $\delta \in [\gamma(G), 1)$. It will then follow that for the given x with $x_i = 0$, the LCP conditions (4),(5) and (6) are satisfied. Since i is connected to all vertices of C_1 and at least one vertex of C_2 ,

$$C_i(x) \geq \delta \left(\frac{n}{1 + (n-1)\delta} + \frac{1}{1 + (m-1)\delta} \right)$$

Now, observe that

$$\begin{aligned} \delta \left(\frac{n}{1 + (n-1)\delta} + \frac{1}{1 + (m-1)\delta} \right) &\geq 1 \\ \iff (m+n-2)\delta^2 + (3-m)\delta - 1 &\geq 0. \end{aligned}$$

This in turn is true in some range $\delta \in (\gamma(G), 1)$ if and only if the positive root, denoted $\gamma(m, n)$, of the above quadratic in δ is less than 1. We know,

$$\gamma(m, n) = \frac{m-3 + \sqrt{(3-m)^2 + 4(m+n-2)}}{2(m+n-2)}$$

Now, it is easy to check that

$$\gamma(m, n) \leq 1 \iff m+n \geq 2,$$

which is true. We have $\gamma(m, n)$ as a decreasing function in n and increasing in m . Thus, the largest value of $\gamma(m, n)$ is attained at $n = 1$ and $m = \omega(G)$. Thus we get the result is true for $\delta \in [\gamma(G), 1)$ where

$$\gamma(G) := \gamma(\omega(G), 1) = \frac{\omega(G) - 3 + \sqrt{(\omega(G) - 3)^2 + 4(\omega(G) - 1)}}{2(\omega(G) - 1)}.$$

This is as required, thereby completing the proof.

B.4 Proof of Theorem 2

Recall from Algorithm 1 that $S = \{1, 2, \dots, |S|\}$ is a maximum independent set of G , $L = \{l \in V \mid l \text{ has only one neighbor in } S\}$. Observe that $C_i = N_{\bar{G}}(i) \cap L$, and $\bar{C}_i = C_i \cup \{i\}$ get defined with respect to $\bar{G}$ at the start of iteration i in Line 4. Let $Q_i = N_{\bar{G}}(\bar{C}_i)$. We will show that after $|S|$ iterations, all the vertices satisfy the $\text{LCP}_\delta(G)$ conditions (4),(5) and (6) for $\delta \in [\gamma(G), 1)$.

We first make a few preliminary observations.

- The set of vertices removed in Algorithm 1 $R = \bigcup_{i=1}^{|S|} (\bar{C}_i \cup Q_i)$ equals V by Remark 2. Thus V can be divided into two disjoint sets, $V = R_C \cup R_Q$ where $R_C := \bigcup_{i=1}^{|S|} \bar{C}_i$ and $R_Q := \bigcup_{i=1}^{|S|} Q_i$. Moreover, Q_i 's are disjoint because if vertex j is chosen after vertex i in the for loop, $Q_j \subseteq V \setminus (Q_i \cup \bar{C}_i)$. Similarly $\bar{C}_i$'s are disjoint.
- We have that $R_Q \cap S = \emptyset$. This can be seen as follows. For some $i \in S$, let $q \in Q_i$. Now, $Q_i \subset N_G(i) \cup N_G(C_i)$. If $q \in N_G(i)$, it cannot be an element of S since S is an independent set. Suppose $q \in N_G(c)$ for some $c \in C_i$. Now recall that $N_G(c) \cap S = \{i\}$ by definition of C_i , whereby $q \notin S$ unless $q = i$, which is not true by definition of Q_i . Hence, $R_Q \cap S = \emptyset$.
- Algorithm 1 returns $x_j = 0 \forall j \in R_Q$. To prove this note that each x_j for $j \in R_Q$ is initialized as 0 since $R_Q \cap S = \emptyset$. Furthermore, in each iteration, we assign non-zero values only to vertices in R_C .



- (d) Let $c \in \bar{C}_i$ for $1 \leq i \leq |S|$. Algorithm 1 returns $x_j = 0 \forall j \in N_G(c) \setminus N_{\bar{C}_i}(c)$. To see this note that $N_G(c) \setminus N_{\bar{C}_i}(c) \subset N_G(\bar{C}_i)$ and since all cliques are independent, we have $N_G(\bar{C}_i) \subset R_Q$. The claim then follows from part (c).
- (e) Consider $q \in Q_i$ for some $1 \leq i \leq |S|$. For the vertex q , define its set of “protective vertices” as $P_q := (N_G(q) \cap S) \setminus \{i\}$. Then P_q is always non-empty. The definition of P_q is valid because Q_j ’s are disjoint. To show $P_q \neq \emptyset$ consider two cases: if $q \in N_G(i)$ then since $q \notin \bar{C}_i$, q must have at least two neighbors in S , one of which is i . On the other hand, if $q \in Q_i \setminus N_G(i)$, it needs to have at least one neighbor in S for S to be a maximum independent set and since q is not a neighbor of i , there must be a neighbor of q in $S \setminus \{i\}$.

Now we claim that, with the returned value of x conditions (4),(5) and (6) are satisfied for all the vertices in V . First we show this for vertices in R_C . For any vertex $c \in \bar{C}_i$, $1 \leq i \leq |S|$, for the x returned by Algorithm 1, we have

$$\mathcal{C}_c(x) = x_c + \delta \sum_{j \in N_G(c)} x_j \quad (17)$$

$$= x_c + \delta \sum_{j \in N_{\bar{C}_i}(c)} x_j + \delta \sum_{j \in N_G(c) \setminus N_{\bar{C}_i}(c)} x_j \quad (18)$$

$$= \frac{1}{1 + (|\bar{C}_i| - 1)\delta} + \frac{\delta(|\bar{C}_i| - 1)}{1 + (|\bar{C}_i| - 1)\delta} + 0 = 1 \quad (19)$$

(19) is justified because of (d) above. Thus, for any vertex $c \in \bar{C}_i$, $\mathcal{C}_c(x) = 1$ and the LCP conditions (4),(5) and (6) are satisfied. It follows that $\text{LCP}_\delta(G)$ conditions hold for all vertices in R_C .

We now show that $\text{LCP}_\delta(G)$ conditions hold for all vertices in R_Q . Let $q \in Q_i$ for $1 \leq i \leq |S|$ and let $P_q = \{p_1, p_2, \dots, p_r\}$ be its set of protective vertices. $P_q \neq \emptyset$ by (e) above. We now have the following cases:
Case 1: $\bar{C}_i \subset N_G(q)$.

Recall that i is not in the set of protective vertices. Since $\bar{C}_i \subset N_G(q)$ we have that q is fully connected to $\bar{C}_i$ and adjacent to at least one protective vertex, say $p_s \in P_q$. Algorithm 1 assigns $x_{p_s} = \frac{1}{1 + (|\bar{C}_{p_s}| - 1)\delta}$ and $x_t = \frac{1}{1 + (|\bar{C}_i| - 1)\delta}$ for all $t \in \bar{C}_i$. Thus using the Two Clique Lemma (Lemma 7) we get that for $\delta \in [\gamma(G), 1)$, x satisfies the LCP conditions (4),(5) and (6) for vertex q .

Case 2: $\bar{C}_{p_s} \subset N_G(q)$ for some $s \in \{1, 2, \dots, r\}$.

Since $\bar{C}_{p_s} \subset N_G(q)$ we have that q is fully connected to $\bar{C}_{p_s}$ and adjacent to some vertex $j \in \bar{C}_i$. Algorithm 1 assigns $x_t = \frac{1}{1 + (|\bar{C}_{p_s}| - 1)\delta} \forall t \in \bar{C}_{p_s}$ and $x_j = \frac{1}{1 + (|\bar{C}_i| - 1)\delta}$. Thus, using the Two Clique Lemma (Lemma 7) for $\delta \in [\gamma(G), 1)$, x satisfies the LCP conditions (4),(5) and (6) for q .

Case 3: $\bar{C}_{p_s} \setminus N_G(q) \neq \emptyset \forall s \in \{1, 2, \dots, r\}$ and $\bar{C}_i \setminus N_G(q) \neq \emptyset$.

We will show that this case is not possible. Define $D_s = \bar{C}_{p_s} \setminus N_G(q) \forall s \in \{1, 2, \dots, r\}$. Note that all $D_s, s = 1, \dots, r$ are independent cliques since $D_s \subset \bar{C}_{p_s}$ and each $\bar{C}_{p_s}$ are independent cliques from Lemma 6.

Now, choose a maximal independent set $D = \{d_1, d_2, \dots, d_r\}$ of $G_{\cup_{s=1}^r D_s}$. Each vertex in D is from a different clique D_s and there is one vertex from each clique; thus, $|D| = r$. Also, let $l \in \bar{C}_i \setminus N_G(q)$. We define

$$S'' := \tilde{S} \cup D \cup \{q, l\}, \quad \tilde{S} := S \setminus (P_q \cup \{i\}),$$

and show that S'' is an independent set with $|S''| > |S|$. To see that S'' is an independent set, we check the independence of each of the pairs of sets in the union above. $\tilde{S}$ and D are independent because $D \subset L \cap N_G(P_q)$ and hence $N_G(D) \cap S \subset P_q$ whereby $N_G(D) \cap \tilde{S} = \emptyset$. $\tilde{S}$ and q are independent because $N_G(q) \cap S \subset P_q \cup \{i\}$. Finally, $N_G(l) \cap S = \{i\}$ by definition whereby $\tilde{S}$ and l are independent. Also, $D \cap N_G(q) = \emptyset$ by definition and $D \cap N_G(l) \subset \emptyset$ since $\bar{C}_t, t = 1, \dots, |S|$ are independent. Finally, $N_G(q) \cap \{l\} = \emptyset$ by definition. Thus, S'' is an independent set. Moreover, $|S''| > |S|$, which gives a contradiction since S was assumed to be a maximum independent set. Hence, *Case 3* is not possible.

Thus, since q was chosen to be an arbitrary vertex in R_Q , the LCP conditions (4),(5) and (6) are satisfied for all vertices in R_Q . Thus, we have exhausted all cases and constructively shown the existence of an independent clique solution in any graph.



B.5 Proof of 1

Consider $x^{\mathcal{K}}(\cdot) : [\delta, 1) \rightarrow [0, 1]^{|V|}$ such that,

$$x_i^{\mathcal{K}}(\delta') = \begin{cases} \frac{1}{1+(|C_j|-1)\delta'}, & \text{if } i \in C_j, C_j \in \mathcal{K} \\ 0, & \text{otherwise,} \end{cases} \quad (20)$$

for $\delta' \in [\delta, 1)$. Then, we claim that for any given $\delta' \in [\delta, 1)$, $x^{\mathcal{K}}(\delta') \in \text{SOL}_{\delta'}(G)$.

We know that for $\delta' = \delta$, $x^{\mathcal{K}}(\delta)$ is same as the one defined in (9) and $x^{\mathcal{K}}(\delta) \in \text{SOL}_{\delta}(G)$. Thus, for $c \in C_i$ for some $C_i \in \mathcal{K}$, $\mathcal{C}_c(x^{\mathcal{K}}(\delta)) = 1$ following the analysis in (17), (18) and (19). Since, the analysis in (17), (18) and (19) is independent of the value of δ , $\mathcal{C}_c(x^{\mathcal{K}}(\delta')) = 1$ for $c \in C_i$ for some $C_i \in \mathcal{K}$ for any $\delta' \in [\delta, 1)$.

For any vertex $j \in V \setminus \mathcal{K}$, let $|N_G(k) \cap C_i| = n_i \forall i$ such that $C_i \in \mathcal{K}$. Then,

$$\mathcal{C}_j(x^{\mathcal{K}}(\delta')) = \delta' \sum_{C_i \in \mathcal{K}} \frac{n_i}{1+(|C_i|-1)\delta'}.$$

We know that $\mathcal{C}_j(x^{\mathcal{K}}(\delta)) \geq 1$ since $x^{\mathcal{K}}(\delta) \in \text{SOL}_{\delta}(G)$. Since, $\mathcal{C}_j(x^{\mathcal{K}}(\delta'))$ is an increasing function in δ' , $\mathcal{C}_j(x^{\mathcal{K}}(\delta')) \geq 1, \forall \delta' > \delta$. Thus, $x^{\mathcal{K}}(\delta') \in \text{SOL}_{\delta'}(G)$ for $\delta' \in [\delta, 1)$.

B.6 Proof of Proposition 1

We show that for any two sets of independent cliques $\mathcal{K}_1 = \{C_1^1, \dots, C_{|\mathcal{K}_1|}^1\}$ and $\mathcal{K}_2 = \{C_1^2, \dots, C_{|\mathcal{K}_2|}^2\}$ such that $|\mathcal{K}_1| = \alpha(G)$ and $|\mathcal{K}_1| > |\mathcal{K}_2|$, we have $\|x^{\mathcal{K}_1}(\delta)\|_1 \geq \|x^{\mathcal{K}_2}(\delta)\|_1$ for all $\delta \in [\eta(G), 1)$. Note that such a $\mathcal{K}_1$ exists, since $\eta(G) \geq \gamma(G)$, by using Remark 4. Also, $\|x^{\mathcal{K}_t}(\delta)\|_1 = \sum_{i=1}^{|\mathcal{K}_t|} \frac{|C_i^t|}{1+(|C_i^t|-1)\delta}$ for $t = 1, 2$. Each term in the sum is increasing with $|C_i^t|$. Thus, the least value of $\|x^{\mathcal{K}_1}(\delta)\|_1$ is when $|C_i^1| = 1, \forall C_i^1 \in \mathcal{K}_1$, which is $\alpha(G)$. Similarly, the maximum value of $\|x^{\mathcal{K}_2}(\delta)\|_1$ is when $|C_i^2| = \omega(G), \forall C_i^2 \in \mathcal{K}_2$ and $|\mathcal{K}_2| = \alpha(G) - 1$, which is $\frac{\omega(G)(\alpha(G)-1)}{1+(\omega(G)-1)\delta}$. It is easy to see that

$$\delta \geq \kappa(G) := \frac{\alpha(G)(\omega(G)-1) - \omega(G)}{\alpha(G)(\omega(G)-1)} \iff \alpha(G) \geq \frac{\omega(G)(\alpha(G)-1)}{1+(\omega(G)-1)\delta}.$$

Thus, $|x_1^{\mathcal{K}}(\delta)| \geq |x_2^{\mathcal{K}}(\delta)|$ for any $\mathcal{K}_2$ with $|\mathcal{K}_2| < \alpha(G)$. Hence, there must exist a solution of $\text{maxICS}_{\delta}(G)$ with its support as a union of $\alpha(G)$ independent cliques.

B.7 Proof of Lemma 9

We divide this proof into three cases depending on the neighborhood of the vertex i which is in G' but not in G . Let $\mathcal{K} = \{C_1, C_2, \dots, C_{|\mathcal{K}|}\}$. The first case is that i is fully connected to some clique $C_j \in \mathcal{K}$ and has at least one more edge connecting at least one other clique $C_k \in \mathcal{K}$ with $k \neq j$. In this case, we explicitly construct a solution of $\text{maxICS}_{\delta}(G')$ with ℓ_1 norm at least as much as $x^{\mathcal{K}}(\delta)$. The second case is when i is not fully connected to any clique $C_i \in \mathcal{K}$. In this case, we show that the size of maximum independent set of G' is bigger than that of G and hence we can construct an ICS of G' using Algorithm 1 which has ℓ_1 norm greater than that of $x^{\mathcal{K}}(\delta)$. The last case is when $N_{G'}(i) \cap \sigma(x^{\mathcal{K}}(\delta))$ is equal to a clique $C_j \in \mathcal{K}$. In this case, we divide into two subcases and construct separate solutions with ℓ_1 norm greater than that of $x^{\mathcal{K}}(\delta)$.

Case 1: i is fully connected to some clique $C_j \in \mathcal{K}$ and has at least one more edge connecting at least one other clique $C_k \in \mathcal{K}$ with $k \neq j$.

Let $y^{\mathcal{K}}(\delta) \in \mathbb{R}^{|V(G')|}$ be defined as

$$y_t^{\mathcal{K}}(\delta) = \begin{cases} x_{\max,t}^{\mathcal{K}}(\delta) & \text{if } t \in V(G) \\ 0 & \text{if } t = i. \end{cases}$$



The $\text{LCP}_\delta(G')$ conditions (4),(5) and (6) are satisfied $y_i^{\mathcal{K}}(\delta)$ for all $j \in V(G)$ since $y_i^{\mathcal{K}}(\delta) = 0$ and they were already satisfied under $x^{\mathcal{K}}(\delta)$ in G . For i , by Lemma 8, since $\delta \geq \eta(G') \geq \gamma(G')$ and since i is fully connected to C_j and has at least one more edge connecting C_k , the $\text{LCP}_\delta(G')$ conditions (4),(5) and (6) are also satisfied for i by the Two Clique Lemma. Thus, in this case, $\|x^{\mathcal{K}}(\delta)\|_1 = \|y^{\mathcal{K}}(\delta)\|_1 \leq \|x^{\mathcal{K}'}(\delta)\|_1$.

Case 2: $C_j \not\subset N_{G'}(i) \cap \sigma(x^{\mathcal{K}}(\delta)) \forall C_j \in \mathcal{K}$, i.e. i is not fully connected to any clique in $\mathcal{K}$.

In this case, we will construct a maximum independent set of the graph G' which has size strictly bigger than the maximum independent set of G thereby arriving at a contradiction. Let S be a maximum independent set of G formed by choosing one node from each independent clique in $\mathcal{K}$. In G' we construct an independent set with size $|S| + 1$ in the following way. For all the cliques which have no vertex as a neighbor of i , choose any vertex to be in the independent set. For the cliques which intersect the neighborhood of i , choose a vertex in the clique which is not a neighbor of i in the independent set. This gives us an independent set of size at least $|S|$. Now, by construction i has no neighbors in the independent set found, hence including i still preserves the independence. Thus we have an independent set, say $\tilde{S}$, of size $|S| + 1$, and the maximum independent set size can increase at most by 1. Hence, $\alpha(G') = |S| + 1$. Next, we construct an ICS of G' using a maximum independent set and show that its ℓ_1 norm is greater than that of $x^{\mathcal{K}}(\delta)$.

Using Algorithm 1 we can find a set of independent cliques $\tilde{\mathcal{K}}$ in G' and a corresponding ICS $x^{\tilde{\mathcal{K}}}(\delta)$ of $\text{LCP}_\delta(G')$ such that $|\tilde{\mathcal{K}}| = \alpha(G') = |S| + 1 > \alpha(G)$. We now find conditions on δ such that $\|x^{\tilde{\mathcal{K}}}(\delta)\|_1 \leq \|x^{\mathcal{K}}(\delta)\|_1$. We consider the least possible value of $\|x^{\tilde{\mathcal{K}}}(\delta)\|_1$ and the maximum possible value of $\|x^{\mathcal{K}}(\delta)\|_1$. Note that we have, $\|x^{\tilde{\mathcal{K}}}(\delta)\|_1 = \sum_{C \in \tilde{\mathcal{K}}} \frac{|C|}{1 + (|C| - 1)\delta}$. Each term in the sum is increasing with $|C|$. Thus, the least value of $\|x^{\tilde{\mathcal{K}}}(\delta)\|_1$ is when $|C| = 1, \forall C \in \tilde{\mathcal{K}}$. In this case $\|x^{\tilde{\mathcal{K}}}(\delta)\|_1 = \alpha(G')$. Similarly, the maximum value of $\|x^{\mathcal{K}}(\delta)\|_1$ is when $|C_i| = \omega(G), \forall i = 1, \dots, |\mathcal{K}|$, which gives $\|x^{\mathcal{K}}(\delta)\|_1 = \frac{\omega(G)(\alpha(G') - 1)}{1 + (\omega(G) - 1)\delta}$. It is easy to see that

$$\delta \geq 1 - \frac{\omega(G)}{\alpha(G')(\omega(G) - 1)} \implies \|x^{\tilde{\mathcal{K}}}(\delta)\|_1 \geq \alpha(G') \geq \frac{\omega(G)(\alpha(G') - 1)}{1 + (\omega(G) - 1)\delta} \geq \|x^{\mathcal{K}}(\delta)\|_1.$$

Thus, since $\delta \in [\eta(G'), 1)$, we have

$$\|x^{\mathcal{K}}(\delta)\|_1 \leq \|x^{\tilde{\mathcal{K}}}(\delta)\|_1 \leq \|x^{\mathcal{K}'}(\delta)\|_1.$$

Case 3: $N_{G'}(i) \cap \sigma(x^{\mathcal{K}}(\delta)) = C_j$ for some $C_j \in \mathcal{K}$, i.e., i is fully connected to exactly one clique $C_j \in \mathcal{K}$ and has no edges to any other cliques $C_t \in \mathcal{K}$ for $t \neq j$.

Define a set of independent cliques of G' , $\hat{\mathcal{K}}$, by replacing C_j in $\mathcal{K}$ by $C_j \cup \{i\}$, i.e., $\hat{\mathcal{K}} = (\mathcal{K} \setminus C_j) \cup \{C_j \cup \{i\}\}$. Let $M = N_{G'}(C_j \cup \{i\})$, which is the neighborhood of the clique $C_j \cup \{i\}$. Note that any $m \in M$ has neighbors in at least one more clique other than C_j . For if not, then $N_G(m) \cap \sigma(x^{\mathcal{K}}(\delta)) \subseteq C_j$ and $\mathcal{C}_m(x^{\mathcal{K}}(\delta)) \leq \frac{|C_j|\delta}{1 + (|C_j| - 1)\delta} < 1$, which is a contradiction since $x^{\mathcal{K}}(\delta)$ is a solution to $\text{LCP}_\delta(G)$.

Define $F = \{m \in M | C_l \subset N_{G'}(m) \cap \sigma(x^{\hat{\mathcal{K}}}(\delta)), \text{ for some } C_l \in \hat{\mathcal{K}}\}$, where $x^{\hat{\mathcal{K}}}(\delta)$ is defined as in (9) for the collection of cliques $\hat{\mathcal{K}}$. F is the set of neighbors of $C_j \cup \{i\}$ such that they are fully connected to at least one clique $C_l \in \hat{\mathcal{K}}$. We now have two cases depending on whether $F = M$.

Case 3a: $F = M$.

In this case, we show that $x^{\hat{\mathcal{K}}}(\delta)$ is a solution of $\text{LCP}_\delta(G')$. To claim this, we divide the vertex set V' as

$$V' = (V' \setminus (F \cup C_j \cup \{i\})) \cup (F) \cup (C_j \cup \{i\}).$$

Consider $v \in V' \setminus (F \cup (C_j \cup \{i\}))$. It follows from (9) that for such a v , we have $x_i^{\hat{\mathcal{K}}}(\delta) = x_i^{\mathcal{K}}(\delta)$ for all $i \in \bar{N}_G(v)$. Since $x^{\mathcal{K}}(\delta)$ is a solution to $\text{LCP}_\delta(G)$, the conditions (4), (5) and (6) for $\text{LCP}_\delta(G')$ are satisfied for all $v \in V' \setminus (F \cup (C_j \cup \{i\}))$ by $x^{\hat{\mathcal{K}}}(\delta)$. Now consider $v \in F$. We note that if v is such that C_l is any clique in $\hat{\mathcal{K}}$ other than $C_j \cup \{i\}$, then v is fully connected to C_l and has at least one neighbor in $C_j \cup \{i\}$. Hence, using Lemma 8, the $\text{LCP}_\delta(G')$ conditions are satisfied with $x^{\hat{\mathcal{K}}}(\delta)$ for v . Next, if $C_l = C_j \cup \{i\}$, since $v \in F = M$, we conclude that v has neighbors in at least one clique other than C_j in $\hat{\mathcal{K}}$. Again, using Lemma 8, LCP conditions are satisfied for v by $x^{\hat{\mathcal{K}}}(\delta)$. Finally for any node in $C_j \cup \{i\}$, the $\text{LCP}_\delta(G')$ conditions are satisfied by definition of $x^{\hat{\mathcal{K}}}(\delta)$. Thus, for any vertex in V' , the $\text{LCP}_\delta(G')$ conditions are satisfied with $x^{\hat{\mathcal{K}}}(\delta)$.

Note that, $\|x^{\hat{\mathcal{K}}}(\delta)\|_1 - \|x^{\mathcal{K}}(\delta)\|_1 = \frac{n+1}{1+n\delta} - \frac{n}{1+(n-1)\delta} \geq 0 \forall \delta < 1$. Thus, $\|x^{\mathcal{K}}(\delta)\|_1 \leq \|x^{\hat{\mathcal{K}}}(\delta)\|_1 \leq \|x^{\mathcal{K}'}(\delta)\|_1$.



Case 3b: $F \neq M$.

We show that in this case we can construct an independent set in G' with size strictly greater than $\alpha(G)$. Let $p \in M \setminus F$. We know that p has neighbors in at least one clique other than C_j , say C_o , and neither clique is fully connected to p since $p \notin F$. Let S be a maximum independent set of G formed by choosing one node from each independent clique in $\mathcal{K}$. Suppose the nodes chosen from C_o and C_j are a and b , respectively. Now, consider $\tilde{S} = (S \setminus \{a, b\}) \cup \{c, d, p\}$ where $c \in C_o \setminus N_{G'}(p)$ and $d \in (C_j \cup \{i\}) \setminus N_{G'}(p)$. It can be observed that $\tilde{S}$ is an independent set and $|\tilde{S}| > |S|$. Thus we have found an independent set of G' of size greater than that of the maximum independent set of G . Using the argument in the second paragraph of Case 2, we have a set of independent cliques in G' , say $\tilde{\mathcal{K}}$ generated from $\tilde{S}$ by Algorithm 1, such that $\|x^{\mathcal{K}}(\delta)\|_1 \leq \|x^{\tilde{\mathcal{K}}}(\delta)\|_1 \leq \|x^{\mathcal{K}'}(\delta)\|_1$.

Thus, having considered all possible cases for the neighborhood of the node i , we have shown that $\|x^{\mathcal{K}}(\delta)\|_1 \leq \|x^{\mathcal{K}'}(\delta)\|_1$.

C Proof from Section 5

C.1 Proof of Theorem 3

Let G be a graph such that $|V(G)| = n$. Let $w = (w_1, w_2, \dots, w_n)$ denote the vector of given weights. We define

$$M_w(\delta, G) = \max\{w^\top x \mid x \text{ solves } \text{LCP}_\delta(G)\}.$$

and prove that $M_w(\delta, G) = \alpha_w(G)$ by showing inequalities in both directions.

Let S be a w -weighted maximum independent set. Then, it is a maximal independent set and $\mathbf{1}_S$ solves $\text{LCP}_\delta(G)$ for $\delta \geq 1$ by Lemma 2. Thus, $M_w(\delta, G) \geq w^\top \mathbf{1}_S = \alpha_w(G)$.

Next, we want to show that $M_w(\delta, G) \leq \alpha_w(G)$. We prove this by induction on the size of the graph. For the base case, consider the graph with 1 vertex and no edges. In this case, it is trivial to see that $M_w(\delta, G) \leq \alpha_w(G)$. *Induction Hypothesis:* For any graph G' with $|V(G')| < n$, for a given set of weights w and $\delta \geq 1$, we have $M_w(\delta, G') \leq \alpha_w(G')$.

Let G be a graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and let x^* be the maximizer of $w^\top x$ where $x \in \text{SOL}_\delta(G)$. Let S be a w -weighted maximum independent set of G .

Case 1: $\sigma(x^*) = V(G)$

Here, we have $x_i^* > 0 \forall i \in V(G)$. Thus, the LCP conditions (4),(5) and (6) dictate

$$C_i(x^*) = x_i^* + \delta \sum_{j=1}^n a_{ij} x_j^* = 1 \forall i \in V(G).$$

Taking the w -weighted sum over $i \in S$, we get

$$\sum_{i \in S} w_i x_i^* = \sum_{i \in S} w_i - \delta \sum_{i \in S} w_i \sum_{j=1}^n a_{ij} x_j^*. \quad (21)$$

Now,

$$\begin{aligned} M_w(\delta, G) &= \sum_{i \in V(G)} w_i x_i^* \\ &= \sum_{i \in S} w_i x_i^* + \sum_{i \in V(G) \setminus S} w_i x_i^* \\ &= \sum_{i \in S} w_i - \delta \sum_{i \in S} w_i \sum_{j=1}^n a_{ij} x_j^* + \sum_{i \in V(G) \setminus S} w_i x_i^* \end{aligned} \quad (22)$$

$$= \alpha_w(G) - \sum_{j \in V(G) \setminus S} x_j^* (\delta \sum_{i \in S} w_i a_{ij} - w_j), \quad (23)$$



where Equation ((22)) follows by using Equation ((21)) and Equation ((23)) follows because $a_{ij} = 0$ when $i, j \in S$. We claim that $\sum_{i \in S} w_i a_{ij} - w_j \geq 0 \forall j \in V(G) \setminus S$. Suppose not, then we construct an independent set with higher weight than that of S . Let $l \in V(G) \setminus S$ be such that $w_l - \sum_{i \in S} w_i a_{il} > 0$. Consider $\tilde{S} = (S \setminus N_G(l)) \cup l$. It is easy to see that $\tilde{S}$ is an independent set and the difference between weights of $\tilde{S}$ and S is $w_l - \sum_{i \in S} w_i a_{il} > 0$. This contradicts the fact that S is a w -weighted maximum independent set. Thus, $\sum_{i \in S} w_i a_{ij} - w_j \geq 0 \forall j \in V(G) \setminus S$. Since $\delta \geq 1$, this implies $\delta \sum_{i \in S} w_i a_{ij} - w_j \geq 0 \forall j \in V(G) \setminus S$ which gives $M_w(\delta, G) \leq \alpha_w(G)$.

Case 2: $\sigma(x^*) \subset V(G)$, a strict subset.

From Lemma 1, we know that since $x^* \in \text{SOL}_\delta(G)$, we have $x_{\sigma(x^*)}^* \in \text{SOL}_\delta(G_{\sigma(x^*)})$. For brevity let $y^* := x_{\sigma(x^*)}^*$. Now,

$$\begin{aligned} M_w(\delta, G) &= \sum_{i \in V(G)} w_i x_i^* = \sum_{i \in V(G_{\sigma(x^*)})} w_i y_i^* \leq M_w(\delta, G_{\sigma(x^*)}) \\ &\leq \alpha_w(G_{\sigma(x^*)}) \leq \alpha_w(G), \end{aligned}$$

where $M_w(\delta, G_{\sigma(x^*)}) \leq \alpha_w(G_{\sigma(x^*)})$ follows from the induction hypothesis since $|V(G_{\sigma(x^*)})| < n$. Hence, by the principle of mathematical induction, we have that for any graph G , with given weights w and $\delta \geq 1$, we have $M_w(\delta, G) \leq \alpha_w(G)$.

References

1. B. Balasundaram and S. Butenko. On a polynomial fractional formulation for independence number of a graph. **Journal of Global Optimization**, 35(3):405–421, July 2006. <https://doi.org/10.1007/s10898-005-5185-6>. URL <https://doi.org/10.1007/s10898-005-5185-6>
2. A. Boral, M. Cygan, T. Kociumaka, and M. Pilipczuk. A fast branching algorithm for cluster vertex deletion. **Theory of Computing Systems**, 58(2):357–376, Apr. 2015. <https://doi.org/10.1007/s00224-015-9631-7>. URL <https://doi.org/10.1007/s00224-015-9631-7>
3. Y. Bramoullé and R. Kranton. Public goods in networks. **Journal of Economic Theory**, 135(1):478–494, 2007.
4. Y. Bramoullé, R. Kranton, and M. D’Amours. Strategic interaction and networks. **American Economic Review**, 104(3):898–930, March 2014. <https://doi.org/10.1257/aer.104.3.898>. URL <http://www.aeaweb.org/articles?id=10.1257/aer.104.3.898>
5. R. W. Cottle, J.-S. Pang, and R. E. Stone. **The Linear Complementarity Problem**. Academic Press, Inc., Boston, MA, 1992.
6. Z. Ertem, E. Lykhovyd, Y. Wang, and S. Butenko. The maximum independent union of cliques problem: complexity and exact approaches. **Journal of Global Optimization**, 76(3):545–562, Aug. 2018. <https://doi.org/10.1007/s10898-018-0694-2>. URL <https://doi.org/10.1007/s10898-018-0694-2>
7. G. Hopkins and W. Staton. Graphs with unique maximum independent sets. **Discrete Mathematics**, 57(3):245–251, 1985.
8. S. Hosseini and S. Butenko. Polyhedral properties of the induced cluster subgraphs, 2019.
9. J. Hu, J. E. Mitchell, and J.-S. Pang. An LPCC approach to nonconvex quadratic programs. **Math. Program.**, 133(1-2):243–277, 2012.
10. J. Hu, J. E. Mitchell, J. S. Pang, and B. Yu. On linear programs with linear complementarity constraints. **J. Global Optimization**, 53(1):29–51, 2012.
11. A. A. Kulkarni and U. V. Shanbhag. On the variational equilibrium as a refinement of the generalized Nash equilibrium. **Automatica**, 48(1):45–55, 2012.
12. M. Locatelli, I. M. Bomze, and M. Pelillo. The combinatorics of pivoting for the maximum weight clique. **Operations Research Letters**, 32(6):523–529, 2004.
13. A. Massaro, M. Pelillo, and I. M. Bomze. A complementary pivoting approach to the maximum weight clique problem. **SIAM Journal on Optimization**, 12(4):928–948, 2002.
14. T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turán. **Canad. J. Math.**, 17(4):533–540, 1965.
15. K. G. Murty and F.-T. Yu. **Linear complementarity, linear and nonlinear programming**. Citeseer, 1988.
16. P. Pandit and A. A. Kulkarni. A linear complementarity based characterization of the weighted independence number and the independent domination number in graphs. **Discrete Applied Mathematics**, 244: 155–169, 2018.
17. P. Pandit and A. A. Kulkarni. Refinement of the equilibrium of public goods games over networks: Efficiency and effort of specialized equilibria. (in press, available online) **Journal of Mathematical Economics**, 2018.
18. W. Siemes, J. Topp, and L. Volkmann. On unique independent sets in graphs. **Discrete Mathematics**, 131(1-3):279–285, aug 1994. [https://doi.org/10.1016/0012-365x\(94\)90389-1](https://doi.org/10.1016/0012-365x(94)90389-1). URL [https://doi.org/10.1016/0012-365x\(94\)90389-1](https://doi.org/10.1016/0012-365x(94)90389-1)

