



Centralizers of the Riordan Group

Tian-Xiao He · Yuanziyi Zhang

Received: 5 July 2021 / Accepted: 5 January 2022 / Published online: 20 January 2022
© The Indian National Science Academy 2022

Abstract In this paper, we discuss centralizers in the Riordan group. We will see that Faà di Bruno's formula is an application of the Fundamental Theorem of Riordan arrays. Then the composition group of formal power series in $\mathcal{F}_1$ is studied to construct the centralizers of Bell type and Lagrange type Riordan arrays. Our tools are the A -sequences of Riordan arrays and Faà di Bruno's formula. Some combinatorial explanation and discussion about related algebraic topics are also given.

Keywords Centralizer · Riordan group · Formal power series · Generalized Riordan array · Bell polynomial, A -sequence · Faà di Bruno's formula · Fundamental theorem of Riordan arrays · Bell type Riordan arrays · Lagrange type Riordan arrays · Generating function

Mathematics Subject Classification 15A30 · 08A05 · 05A05 · 05A15 · 05E15 · 05E18

1 Introduction

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro, Getu, Woan and Woodson [21]).

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{K}[[z]]$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$); the *order* of $f(z) \in \mathcal{F}$, $f(z) = \sum_{k=0}^{\infty} f_k z^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r . Let $d(z) \in \mathcal{F}_0$ and $h(z) \in \mathcal{F}_1$; the pair $(d(z), h(z))$ defines the (*proper*) *Riordan array* $D = (d_{n,k})_{n,k \in \mathbb{N}} = (d(z), h(z))$ having

$$d_{n,k} = [z^n]d(z)h(z)^k \quad (1)$$

or, in other words, having $d(z)h(z)^k$ as the generating function whose coefficients make-up the entries of column k .

From the *fundamental theorem of Riordan arrays* (see [21]),

$$(d(z), h(z))f(z) = d(z)f(h(z)),$$

Communicated by Rahul Roy.

T-X He ()

Department of Mathematics, Illinois Wesleyan University, Bloomington, IL 61702, USA
E-mail: the@iwu.edu

Y. Zhang

Mathematics Department, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

it is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(z), h_1(z))(d_2(z), h_2(z)) = (d_1(z)d_2(h_1(z)), h_2(h_1(z))). \quad (2)$$

The Riordan array $I = (1, z)$ acts as an identity for this product, that is, $(1, t)(d(z), h(z)) = (d(z), h(z))(1, z) = (d(z), h(z))$. Let $(d(z), h(z))$ be a Riordan array. when its inverse is

$$(d(z), h(z))^{-1} = \left(\frac{1}{d(\bar{h}(z))}, \bar{h}(z) \right) \quad (3)$$

where $\bar{h}(z)$ is the compositional inverse of $h(z)$, i.e., $(h \circ \bar{h})(z) = (\bar{h} \circ h)(z) = z$. In this way, the set $\mathcal{R}$ of proper Riordan arrays forms a group (see [21]).

Here is a list of six important subgroups of the Riordan group (see [21]), where $d \in \mathcal{F}_0$ and $h \in \mathcal{F}_1$.

- the *Appell subgroup* $\mathcal{A} = \{(d(z), z)\}$.
- the *Lagrange (associated) subgroup* $\mathcal{L} = \{(1, h(z))\}$.
- the *Bell subgroup* $\mathcal{B} = \{(d(z), zd(z))\}$.
- the *hitting-time subgroup* $\mathcal{H} = \{(zh'(z)/h(z), h(z))\}$.
- the *derivative subgroup* $\mathcal{D} = \{(h'(z), h(z))\}$.
- the *checkerboard subgroup* $\mathcal{C} = \{(d(z), h(z))\}$, where d is an even function and h is an odd function.

An infinite lower triangular array $[d_{n,k}]_{n,k \in \mathbb{N}} = (d(z), h(z))$ is a Riordan array if and only if a unique sequence $A = (a_0 \neq 0, a_1, a_2, \dots)$ exists such that for every $n, k \in \mathbb{N}$ (see Merlini, Rogers, Sprugnoli, and Verri [16])

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \dots + a_n d_{n,n}. \quad (4)$$

This is equivalent to

$$h(z) = zA(h(z)) \quad \text{or} \quad z = \bar{h}(z)A(z). \quad (5)$$

Here, $A(z)$ is the generating function of the A -sequence. There exists a unique sequence $Z = (z_0, z_1, z_2, \dots)$ such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \dots + z_n d_{n,n}, \quad (6)$$

or equivalently,

$$d(z) = \frac{1}{1 - zZ(h(z))}. \quad (7)$$

Definition 1.1 The centralizer of a subset S of group G is defined to be

$$C_G(S) = \{g \in G : gs = sg \text{ for all } s \in S\}$$

It is worth noting that S may be a singleton set. Thus, the centralizer of a subset S of the group G is the set of all elements in G that fix each element of S under conjugation, i.e. commute with each element in S .

Example 1.1 Let $S = \{(1 + z^2, z)\}$. For any $(d, h) \in \mathcal{R}$ in $C_{\mathcal{R}}(S)$, we have

$$(d, h)(1 + z^2, z) = (1 + z^2, z)(d, h),$$

i.e.,

$$(d(1 + h^2), h) = ((1 + z^2)d, h),$$

which implies $h^2 = z^2$, or equivalently, $h = \pm z$. Thus, $C_{\mathcal{R}}(S) = \{(d, h) \in \mathcal{R} : h = \pm z\}$, the double Appell subgroup $A^{\pm} = (d(z), \pm z)$ of the Riordan group $\mathcal{R}$. To guarantee $C_{\mathcal{R}}(S)$ contains only elements of $\mathcal{A}$, the Appell subgroup of $\mathcal{R}$, we need an additional condition, which will be presented in Theorem 2.7.



A *generalized Riordan array* with respect to the sequence $(c_n)_{n \in \mathbb{N}}$ ($c_n \neq 0$) is a pair $(g(z), f(z))$ of formal power series, where $g(z) = \sum_{k=0}^{\infty} g_k z^k / c_k$ and $f(z) = \sum_{k=1}^{\infty} f_k z^k / c_k$ with $f_1 \neq 0$. The Riordan array $(g(z), f(z))$ defines an infinite, lower triangular array $(d_{n,k})_{0 \leq k \leq n < \infty}$ according to the rule:

$$d_{n,k} = \left[\frac{z^n}{c_n} \right] g(z) \frac{(f(z))^k}{c_k}, \quad (8)$$

where the functions $g(z)(f(z))^k / c_k$ are column generating functions of the Riordan array.

Let $D = (g(z), f(z)) = (d_{n,k})_{n,k \in \mathbb{N}}$ be a Riordan array with respect to $(c_n)_{n \in \mathbb{N}}$ and let $h(z) = \sum_{k=0}^{\infty} h_k z^k / c_k$ be the generating function of the sequence $(h_n)_{n \in \mathbb{N}}$. Then we have the fundamental theorem of Riordan arrays (FTRA)

$$\sum_{k=0}^n d_{n,k} h_k = \left[\frac{z^n}{c_n} \right] g(z) h(f(z)), \quad (9)$$

or equivalently, $(g(z), f(z)) * h(z) = g(z)h(f(z))$.

With every formal power series $f(z) = \sum_{k=1}^{\infty} f_k z^k / c_k$, we associate the following infinite lower *iteration matrix* with respect to $(c_n)_{n \in \mathbb{N}}$ (see p. 145 of Comtet [3]):

$$B(f(z)) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B_{1,1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & B_{2,1} & B_{2,2} & 0 & 0 & 0 & \cdots \\ 0 & B_{3,1} & B_{3,2} & B_{3,3} & 0 & 0 & \cdots \\ 0 & B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} & 0 & \cdots \\ 0 & B_{5,1} & B_{5,2} & B_{5,3} & B_{5,4} & B_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $B_{n,k} = B_{n,k}(f_1, f_2, \dots)$ is the Bell polynomial with respect to $(c_n)_{n \in \mathbb{N}}$, defined as follows:

$$\frac{1}{c_k} (f(z))^k = \sum_{n=k}^{\infty} B_{n,k} \frac{z^n}{c_n}. \quad (10)$$

The most obvious choices for (c_n) are (1) , $(n!)$, and $(n!n!)$ or $(n!)$.

Therefore, $B_{n,k} = [z^n / c_n] (f(z))^k / c_k$, which implies that the iteration matrix $B(f(z))$ is the Riordan array $(1, f(z))$. Now, the following important property of the iteration matrix (see Theorem A on p. 145 of Comtet [3], Roman [18], and Roman and Rota [19])

$$B(f(g(z))) = B(g(z)) * B(f(z))$$

is trivial in the context of the theory of Riordan arrays, i.e.,

$$(1, f(g(z))) = (1, g(z)) * (1, f(z)).$$

In the case where $c_n = n!$ we obtain that the following well-known Faà di Bruno's formula is an application of the FTRA:

$$\sum_{k=1}^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) f_k = \left[\frac{t^n}{n!} \right] f(g(t)).$$

More precisely, Let f and g be two formal (Taylor) series:

$$f = \sum_{k \geq 0} f_k \frac{z^k}{k!}, \quad g = \sum_{k \geq 0} g_k \frac{z^k}{k!},$$

where $g_0 = 0$, and let $h = f \circ g$ with formal Taylor expansion

$$h = f \circ g = \sum_{n \geq 0} h_n \frac{z^n}{n!}.$$



Then the coefficients h_n are given by the following expression (see Theorem A on p. 137 of Comtet [3]):

$$h_0 = f_0, \quad h_n = \sum_{k=1}^n f_k B_{n,k}(g_1, g_2, \dots, g_{n-k+1}),$$

where the $B_{n,k}$ are the exponential Bell polynomials. $B_{n,k}$ can be presented as

$$B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) = \sum \frac{n!}{c_1! c_2! \dots (1!)^{c_1} (2!)^{c_2} \dots} x_1^{c_1} x_2^{c_2} \dots,$$

where the summation takes place over all integers $c_1, c_2, c_3, \dots \geq 0$ such that there hold (the partition of n with k parts)

$$c_1 + 2c_2 + 3c_3 + \dots = n, \quad c_1 + c_2 + c_3 + \dots = k.$$

Example 1.2 Consider the problem of finding the n -th derivative of $F(x) = x^{\alpha x}$, where $x > 0$ and $\alpha \neq 0$ is a real number (cf. Comtet [3]). By using the Stirling numbers of the first kind, we expand $f(t) = F(x+t)$ as

$$\begin{aligned} f(t) &= (x+t)^{\alpha(x+t)} = x^{\alpha(x+t)} \left(1 + \frac{t}{x}\right)^{\alpha(x+t)} = x^{\alpha x} x^{\alpha t} \left(1 + \frac{t}{x}\right)^{\alpha x \left(1 + \frac{t}{x}\right)} \\ &= F(x) \exp(\alpha t \log x) \exp\left(\alpha x \left(1 + \frac{t}{x}\right) \log\left(1 + \frac{t}{x}\right)\right). \end{aligned}$$

Notice that there is a well-known expansion on P. 139 of Comtet [3] shown as

$$\frac{1}{k!} ((1+z) \log(1+z))^k = \sum_{n \geq k} b(n, k) \frac{z^n}{n!},$$

where $b(0, 0) = 1$ and

$$b(n, k) = \sum_{k \leq \ell \leq n} \binom{\ell}{k} k^{\ell-k} s(n, \ell)$$

with the Stirling numbers $s(n, \ell)$ of the first kind. The numbers $b(n, k)$, called Lehmer-Comtet numbers of first kind, follow the recurrence relation

$$b(n+1, k) = (k-n)b(n, k) + b(n, k-1) + nb(n-1, k-1),$$

and the first few values are listed below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & -1 & 6 & 1 & 0 & 0 & \dots \\ 0 & -6 & 0 & 5 & 10 & 1 & 0 & \dots \\ 0 & 24 & 4 & -15 & 25 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

More properties of Lehmer-Comtet numbers of first kind can be found in OEIS A008296, P. 139 of [3], and Gould [7]. However, the combinatorial explanation seems not to be known. By using Lehmer-Comtet numbers of first kind, we obtain consequently

$$f(t) = F(x) \sum_{j \geq 0, k \leq m} b(m, k) \frac{(\alpha t \log x)^j (t/x)^m}{j! m!} (\alpha x)^k,$$



which implies

$$f_n = \frac{d^n(x^{\alpha x})}{dx^n} = \alpha^n x^{\alpha x} \sum_{j=0}^n \left[\binom{n}{j} u^j \sum_{k=0}^{n-j} b(n-j, n-k-j) v^k \right],$$

where $u = \log x$ and $v = (\alpha x)^{-1}$. For instance, by using the above table of $b(n, k)$ we get

$$\begin{aligned} f_4 &= \frac{d^4(x^{\alpha x})}{dx^4} \\ &= \alpha^4 x^{\alpha x} \sum_{j=0}^4 \left[\binom{4}{j} u^j \sum_{k=0}^{4-j} b(4-j, 4-k-j) v^k \right] \\ &= \alpha^4 x^{\alpha x} \left[\sum_{k=0}^4 b(4, 4-k) v^k + 4u \sum_{k=0}^3 b(3, 3-k) v^k \right. \\ &\quad \left. + 6u^2 \sum_{k=0}^2 b(2, 2-k) v^k + 3u^3 \sum_{k=0}^1 b(1, 1-k) v^k + u^4 \right] \\ &= \alpha^4 x^{\alpha x} [1 + 6v - v^2 + 2v^3 + 4u(1 + 3v - v^2) + 6u^2(1 + v) + 4u^3 + u^4]. \end{aligned}$$

Additionally, for any $f \in \mathcal{F}_1$, the inverse array $(1, \bar{f}) = B(\bar{f})$ is also an iteration matrix. Thus, the set of iteration matrices with respect to $(c_n)_{n \in \mathbb{N}}$, denoted by $\mathcal{B}$, is a nonempty subset of the Riordan group $\mathcal{R}$ with respect to $(c_n)_{n \in \mathbb{N}}$, closed under multiplication and taking inverses in $\mathcal{R}$. These indicate that $\mathcal{B}$ is a subgroup of $\mathcal{R}$ and we call it the *associated subgroup*.

If $f(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$ and $g(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ and $f(g(x)) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n$, then the coefficient d_n (which would be the n th derivative of $f \circ g$ evaluated at 0 if we were dealing with convergent series rather than formal power series) is given by

$$d_n = \sum_{\pi = \{B_1, \dots, B_k\}} a_{|B_1|} \cdots a_{|B_k|} b_k$$

where π runs through the set of all partitions of the set $\{1, \dots, n\}$ and $B_1, \dots, B_k$ are the blocks of the partition π , and $|B_j|$ is the number of members of the j th block, for $j = 1, \dots, k$. This version of the formula is equivalent to Faà di Bruno's formula for exponential formal power series shown before, i.e.,

$$f(g(x)) = b_0 + \sum_{n=1}^{\infty} \frac{\sum_{k=1}^n b_k B_{n,k}(a_1, \dots, a_{n-k+1})}{n!} x^n,$$

where $B_{n,k}(a_1, \dots, a_{n-k+1})$ are Bell polynomials. Faà di Bruno's formula is useful in finding centralizers, commutative compositions, Riordan involutions and pseudo-involutions. The centralizers related to the Riordan pseudo-involution multiplication and palindromes are shown in a recent paper by Shapiro and one of the authors [10].

In the next section, we will use Faà di Bruno's formula and sequence characterization of Riordan arrays to construct centralizers of Bell type Riordan arrays and Riordan arrays of the form $(1/(1-kz), f)$, $f \in \mathcal{F}_1$. Some combinatorial explanation are also given. In Section 3, we will study the method of the construction of centralizers of the Riordan arrays related to the composition group of formal power series in $\mathcal{F}_1$ and its application for Lagrange type Riordan arrays and Bell type Riordan arrays. Section 4 diverts to some related topics such as reversers, conjugates, commutators, twisted subgroup, and pseudo-Riordan involutions.



2 Centralizers of the Riordan group

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are formal power series and $b_0 = 0$. Then the composition $f \circ g$ is again a formal power series,

$$f(g(x)) = \sum_{n=0}^{\infty} c_n x^n, \quad (11)$$

where $c_0 = a_0$ and the other coefficients c_n for $n \geq 1$ can be expressed as a sum over compositions of n or as an equivalent sum over partitions of n :

$$c_n = \sum_{\mathbf{i} \in \mathcal{C}_n} a_k b_{i_1} b_{i_2} \cdots b_{i_k},$$

where

$$\mathcal{C}_n = \{(i_1, i_2, \dots, i_k) : 1 \leq k \leq n, i_1 + i_2 + \cdots + i_k = n\}$$

is the set of compositions of n with k denoting the number of parts, or

$$c_n = \sum_{k=1}^n a_k \sum_{\pi \in \mathcal{P}_{n,k}} \binom{k}{\pi_1, \pi_2, \dots, \pi_n} b_1^{\pi_1} b_2^{\pi_2} \cdots b_n^{\pi_n}, \quad (12)$$

where

$$\mathcal{P}_{n,k} = \{(\pi_1, \pi_2, \dots, \pi_n) : \pi_1 + \pi_2 + \cdots + \pi_n = k, \pi_1 \cdot 1 + \pi_2 \cdot 2 + \cdots + \pi_n \cdot n = n\}$$

is the set of partitions of n into k parts, in frequency-of-parts form.

The first form is obtained by picking out the coefficient of x^n in $(b_1 x + b_2 x^2 + \cdots)^k$ "by inspection", and the second form is then obtained by collecting like terms, or alternatively, by applying the multinomial theorem.

The product of two Riordan arrays is a Riordan array. Sprugnoli and one of the authors [11] study the problem of how do the A -sequence and Z -sequence of the product depend on the analogous sequences of the two factors. More precisely, consider two proper Riordan arrays $D_1 = (d_1, h_1)$ and $D_2 = (d_2, h_2)$ and their product,

$$D_3 = D_1 D_2 = (d_1 d_2(h_1), h_2(h_1)).$$

Theorem 2.1 [11] Let $A_i(z)$, $i = 1, 2$, and 3 , be the generating functions of A -sequences of D_i , $i = 1, 2$, and 3 , respectively. Then

$$A_3(z) = A_2(z) A_1\left(\frac{z}{A_2(z)}\right).$$

Let $h \in \mathcal{F}_1$. We define

$$A_h(z) = \frac{z}{\bar{h}(z)} \quad (13)$$

where $\bar{h}$ is the compositional inverse of h . Thus, from Theorem 2.1, we have the following result.

Theorem 2.2 Let A_h be defined by (13), and let $h, f \in \mathcal{F}_1$. Then

$$A_{h \circ f}(z) = A_f(z) A_h\left(\frac{z}{A_f(z)}\right).$$



Corollary 2.3 Let A_h be defined by (13), and let $h, f \in \mathcal{F}_1$. Then $h \circ f = f \circ h$ if and only if

$$A_f(z)A_h\left(\frac{z}{A_f(z)}\right) = A_h(z)A_f\left(\frac{z}{A_h(z)}\right). \quad (14)$$

Particularly, $h \circ f = f \circ h = z$ if and only if

$$A_f(z)A_h\left(\frac{z}{A_f(z)}\right) = A_h(z)A_f\left(\frac{z}{A_h(z)}\right) = 1. \quad (15)$$

Proposition 2.4 Let B be the Bell subgroup of the Riordan group, and let $(h/z, h) \in B$. Then the centralizer of $(h/z, h)$ in the subgroup B is

$$C_B((h/z, h)) = \{(f/z, f) \in B : h \circ f = f \circ h\}.$$

Proof Let $(f/z, f) \in B$ with $h \circ f = f \circ h$. Then

$$(h/z, h)(f/z, f) = ((f \circ h)/z, f \circ h) = ((h \circ f)/z, h \circ f) = (f/z, f)(h/z, h),$$

completing the proof. $\square$

Theorem 2.5 Let $S = \{(1/(1-z), z/(1-z))\}$, and let B be the Bell subgroup of the Riordan group. Then the centralizer

$$C_B(S) = \{(1/(1-rz), z/(1-rz)) : r \in \mathbb{R}\}. \quad (16)$$

Proof From Proposition 2.4, to obtain (14), it is sufficient to prove the set of all elements in $\mathcal{F}_1$ that commute with $h = z/(1-z)$ is

$$E = \{z/(1-rz) : r \in \mathbb{R}\}. \quad (17)$$

Let $f \in \mathcal{F}_1$, and let A_f and A_h be defined by (13). Since $\bar{h} = z/(1+z)$, $A_h(z) = (1+z)$. From Corollary 2.3, $h \circ f = f \circ h$ if and only if (14) holds, i.e.,

$$A_f(z)\left(1 + \frac{z}{A_f(z)}\right) = (1+z)A_f\left(\frac{z}{1+z}\right),$$

or equivalently,

$$A_f(z) + z = (1+z)A_f\left(\frac{z}{1+z}\right).$$

Substituting $z = z/(1-z)$ into above equation yields

$$(1-z)A_f\left(\frac{z}{1-z}\right) + z = A_f(z). \quad (18)$$

Denote $A_f(z) = \sum_{k \geq 0} a_k z^k$ and substitute it into (18):

$$(1-z)(a_0 + a_1(z + z^2 + \cdots) + a_2(z + z^2 + \cdots)^2 + \cdots) = a_0 + a_1 z + a_2 z^2 + \cdots.$$

By comparing the coefficients of powers z^0, z^1 , and z^2 on both sides of the resulting equation, we obtain $a_0 = a_0$,

$$-a_0 + a_1 + 1 = a_1 \quad \text{and} \quad a_1 - a_1 + a_2 = a_2,$$

which implies $a_0 = 1$ and a_1 is free. To determine a_n for $n \geq 2$, we use the Lagrange Inversion Formula

$$[z^n]h(g(z)) = \frac{1}{n}[z^{-1}]\frac{h'(z)}{g^n}$$



for $n \geq 3$ to $A_f(z/(1-z))$ in (18), where $g \in \mathcal{F}_1$ and $\bar{g}$ is the compositional inversion of g . Then we can write the coefficients of z^n on the lefthand side of (18) as

$$\begin{aligned}
 & [z^n]A_f\left(\frac{z}{1-z}\right) - [z^{n-1}]A_f\left(\frac{z}{1-z}\right) \\
 &= \frac{1}{n}[z^{-1}]\frac{A'_f(Z)}{\left(\frac{z}{1+z}\right)^n} - \frac{1}{n-1}[z^{-1}]\frac{A'_f(Z)}{\left(\frac{z}{1+z}\right)^{n-1}} \\
 &= \frac{1}{n}[z^{n-1}](1+z)^n A'_f(z) - \frac{1}{n-1}[z^{n-2}](1+z)^{n-1} A'_f(Z) \\
 &= \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} (n-k) a_{n-k} - \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} (n-k-1) a_{n-k-1} \\
 &= \sum_{k=0}^{n-1} \binom{n-1}{k} a_{n-k} - \sum_{k=0}^{n-2} \binom{n-2}{k} a_{n-k-1} \\
 &= a_n + \sum_{k=1}^{n-1} \left(\binom{n-1}{k} - \binom{n-2}{k-1} \right) a_{n-k} \\
 &= a_n + \sum_{k=1}^{n-2} \binom{n-2}{k} a_{n-k}. \tag{19}
 \end{aligned}$$

Comparing the coefficients of z^n on both sides of (18), we have

$$a_n + \sum_{k=1}^{n-2} \binom{n-2}{k} a_{n-k} = a_n,$$

or equivalently,

$$\sum_{k=1}^{n-2} \binom{n-2}{k} a_{n-k} = 0.$$

Using the above equation and noting $a_2 = 0$ when $n = 3$, we may prove all $a_n = 0$ for $n \geq 2$ inductively. Thus $A_f(t) = 1 + a_1 t$ and $f(t) = t/(1 - rt)$ for $r = a_1 \in \mathbb{R}$. $\square$

We now give a combinatorial proof of Theorem 2.5 based on a combinatorial interpretation of expression of (12).

Theorem 2.6 Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be formal power series and $b_0 = 0$, and let $f \circ g$ be the composition of f and g shown in (11), $f(g(x)) = \sum_{n=0}^{\infty} c_n x^n$, where $c_0 = a_0$ and the other coefficients c_n for $n \geq 1$ are expressed as a sum over compositions of n or as an equivalent sum over partitions of n shown in (12). Then

$$c_n = \sum_{k=1}^n B_{n,k} a_k, \tag{20}$$

where $B_{n,k} = \sum b_{j_1} b_{j_2} \dots b_{j_k}$, where $|j| = j_1 + j_2 + \dots + j_k = n$ and $B_{n,k}$ could be interpreted as the combinations of ways to put n non-distinguished objects into k nonempty distinguished (labeled) boxes.

Proof As an example of $B_{n,k}$, $B_{2,2} = b_1^2$ can be interpreted as combinations of ways to put 2 objects into 2 boxes. Similarly, $B_{4,2} = 2b_1 b_3 + b_2^2$. We now give a combinatorial proof of Theorem 2.5. Let $h(t) = \frac{1}{1-t} = \sum_{k=1}^{\infty} a_k t^k$ where $a_k = 1$ for all $k \in \mathbb{N}$. We want to find elements f of $C_B(h)$, $f(t) = \sum_{k=1}^{\infty} b_k t^k$ where $b_1 = 1$ such that $h(f) = f(h)$. Let $h(f) = \sum_{k=1}^{\infty} c_k t^k$ and $f(h) = \sum_{k=1}^{\infty} \hat{c}_k t^k$. We first compute the first few terms.



For $k = 1$, we have

$$c_1 = B_{1,1}a_1 = b_1a_1.$$

Switch the terms accordingly, we obtain

$$\hat{c}_1 = a_1b_1.$$

Then $c_1 = \hat{c}_1$ naturally.

For $k = 2$, we have

$$\begin{aligned} c_2 &= B_{2,1}a_1 + B_{2,2}a_2 = b_2a_1 + b_1^2a_2, \\ \hat{c}_2 &= a_2b_1 + a_1^2b_2. \end{aligned}$$

Plugging in $a_k = 1$ and $b_1 = 1$, and then equate c_2 and $\hat{c}_2$ we have

$$b_2 + 1 = 1 + b_2.$$

Therefore, we conclude that b_2 is free.

For $k = 3$, we have

$$\begin{aligned} c_3 &= B_{3,1}a_1 + B_{3,2}a_2 + B_{3,3}a_3 \\ &= a_1b_3 + a_2(b_1b_2 + b_2b_1) + a_3b_1^3 \\ &= a_1b_3 + 2a_2b_1b_2 + a_3b_1^3, \\ \hat{c}_3 &= b_1a_3 + 2b_2a_1a_2 + b_3a_1^3. \end{aligned}$$

Equate c_3 and $\hat{c}_3$, the equation holds naturally.

For $k = 4$, we have

$$\begin{aligned} c_4 &= B_{4,1}a_1 + B_{4,2}a_2 + B_{4,3}a_3 + B_{4,4}a_4 \\ &= a_1b_4 + a_2(2b_1b_3 + b_2^2) + a_3(3b_1b_1b_2) + a_4b_1^4 \\ &= a_1b_4 + 2a_2b_1b_3 + a_2b_2^2 + 3a_3b_1b_1b_2 + a_4b_1^4, \\ \hat{c}_4 &= b_1a_4 + 2b_2a_1a_3 + b_2a_2^2 + 3b_3a_1a_1a_2 + b_4a_1^4. \end{aligned}$$

Plugging in $a_k = 1$ and $b_1 = 1$, and then equate c_4 and $\hat{c}_4$ we have

$$\begin{aligned} b_4 + 2b_3 + b_2^2 + 3b_2 + 1 \\ &= 1 + 2b_2 + b_2 + 3b_3 + b_4, \\ 2b_3 + b_2^2 &= 3b_3. \end{aligned}$$

Therefore, we conclude that $b_3 = b_2^2$.

For $k = 5$, we have

$$\begin{aligned} c_5 &= B_{5,1}a_1 + B_{5,2}a_2 + B_{5,3}a_3 + B_{5,4}a_4 + B_{5,5}a_5 \\ &= a_1b_5 + a_2(2b_1b_4 + 2b_2b_3) + a_3(3b_1^2b_3 + 3b_1b_2^2) + a_4(4b_1^3b_2) + a_5b_1^5, \\ \hat{c}_5 &= b_1a_5 + b_2(2a_1a_4 + 2a_2a_3) + b_3(3a_1^2a_3 + 3a_1a_2^2) + b_4(4a_1^3a_2) + b_5a_1^5. \end{aligned}$$

Equate c_5 and $\hat{c}_5$ we have

$$2b_4 + 2b_2b_3 + 3b_3 + 3b_2^2 = 6b_3 + 4b_4.$$

Therefore, we conclude that $b_4 = b_2^3$.

Base on the computational results above, we make the conjecture that $b_p = b_2^{p-1}$ for all $p \in \mathbb{N}$ and $p \geq 2$. We prove this result by induction.



For $p = 1$, the result holds naturally as demonstrated above. Suppose now the result holds up to $p = q$. Since $a_k = 1$ for all $k \in \mathbb{N}$, as a matter of simplicity, we put the coefficients $B_{n,k}$ as the entries of a lower triangular matrix below,

$$\begin{pmatrix} 1 & & & & \\ b_2 & 1 & & & \\ b_2^2 & 2b_2 & 1 & & \\ b_2^3 & 3b_2^2 & 3b_2 & 1 & \\ \vdots & & & & \ddots \\ b_2^q & qb_2^{q-1} & \dots & \dots & 1 \end{pmatrix}.$$

Observe that the matrix above is a Riordan array with A-sequence $a_0 = 1, a_1 = b_2$.

Now we want to show that the result holds for $p = q + 1$. It is sufficient to show that adding another row and column to the matrix above, we still obtain a Riordan array preserving the same A-sequence. First, we consider the combinatorial interpretation of $B_{q+1,k}$ for $k \geq 3$. If we have k boxes and $q + 1$ objects, each of the boxes contains at most $q - k + 2$ objects, i.e., the subscript of the terms that appear in $B_{q+1,k}$ is at most $q - k$. Since $q - k + 2 < q$, by induction, we conclude that the k th entries of the $q + 1$ th row preserve the same A-sequence for $k \geq 3$.

It can be seen that the matrix of the size $(q + 1) \times (q + 1)$ is uniquely determined. Now we need to check if the result holds for the two entries $B_{q+1,1}$ and $B_{q+1,2}$. Again, by the combinatorial interpretation, if we contract the objects of the entry $B_{q,1}$ into one, the entry $B_{q+1,1}$ can be interpreted as adding an object to the same box. Then it follows that $B_{q+1,1} = B_{q,1} \times b_2$. Similarly, the entry $B_{q+1,2}$ can be obtained by adding one object and one box to $B_{q,1}$ and adding one object into one existing box of $B_{q,2}$. It follows that $B_{q+1,2} = B_{q,1} + B_{q,2} \times b_2$.

Therefore, we conclude that the result indeed holds. Furthermore, if $b_p = b_2^{p-1}$, the corresponding power series is indeed $\frac{t}{1-b_2t}$. $\square$

We now return to the problem raised in Example 1.1.

Theorem 2.7 *Let S be a subset of Appell subgroup $\mathcal{A}$ of the Riordan group $\mathcal{R}$ defined by $S = \{(g, f) : (g, f) \in \mathcal{A}, g'(0) \neq 0\}$. Then $C_{\mathcal{R}}(S) = \mathcal{A}$. Particularly, if (g, f) does not satisfy the condition $g'(0) \neq 0$, then $C_{\mathcal{R}}$ contains elements that do not belong to $\mathcal{A}$. In general, $C_{\mathcal{R}}((g, t), g \in \mathcal{F}_0) = A^{\pm}$, the double Appell subgroup of $\mathcal{R}$.*

Proof Let $(g, f) \in S \subset \mathcal{A}$ with $g'(0) \neq 0$. Then $(g, f) = (g, t)$ and $g = \sum_{k \geq 0} g_k t^k$ with $g_0, g_1 \neq 0$. A Riordan array $(d, h) \in C_{\mathcal{R}}(S)$ with $h = \sum_{k \geq 0} h_k t^k$ if and only if

$$(d, h)(g, t) = (g, t)(d, h); \text{ i.e., } (d(g \circ h), h) = (gd, h).$$

Hence, $g(h) = g$. From (11), we have

$$g(h(t)) = \sum_{n \geq 0} c_n t^n, \quad (21)$$

where, by denoting $\mathcal{C}_n = \{(i_1, i_2, \dots, i_k) : 1 \leq k \leq n, i_1 + i_2 + \dots + i_k = n\}$,

$$\begin{aligned} c_n &= \sum_{\mathbf{i} \in \mathcal{C}_n} g_k h_{i_1} h_{i_2} \cdots h_{i_k} \\ &= g_1 h_n + g_n h_1^n + \sum_{\mathbf{i} \in \mathcal{C}'_n} g_k h_{i_1} h_{i_2} \cdots h_{i_k}, \end{aligned}$$

in which $\mathcal{C}'_n = \{(i_1, i_2, \dots, i_k) : 2 \leq k \leq n - 1, i_1 + i_2 + \dots + i_k = n\}$. It is clear that the summation in the last equation contains terms involving only h_k with $1 \leq k \leq n - 1$ because

$$i_1 + i_2 + \dots + i_k = n$$

and $k \geq 2$ imply $1 \leq i_j \leq n - 1$ for all $j = 1, 2, \dots, k$. In addition, the summation contains terms involving g_k with $2 \leq k \leq n - 1$, but not g_1 and g_n . For instance,



$$\begin{aligned}c_1 &= g_1 h_1, \\c_2 &= g_1 h_2 + g_2 h_1^2 \\c_3 &= g_1 h_3 + g_3 h_1^3 + 2g_2 h_1 h_2, \dots\end{aligned}$$

Comparing the coefficients of the same powers of $g(h) = g$ yields

$$g_n = c_n = g_1 h_n + g_n h_1^n + \sum_{\mathbf{i} \in C'_n} g_k h_{i_1} h_{i_2} \cdots h_{i_k}. \quad (22)$$

Therefore, for $n = 1$, we have

$$g_1 = g_1 h_1.$$

Since $g_1 \neq 0$, we obtain $h_1 = 1$. Considering $n = 2$, we have

$$g_2 = c_2 = g_1 h_2 + g_2 h_1^2 = g_1 h_2 + g_2,$$

which implies $h_2 = 0$. Similarly, we have $h_3 = 0$. Assume that $h_k = 0$ for $2 \leq k \leq n - 1$. Then from

$$\begin{aligned}g_n = c_n &= g_1 h_n + g_n h_1^n + \sum_{\mathbf{i} \in C'_n} g_k h_{i_1} h_{i_2} \cdots h_{i_k} \\&= g_1 h_n + g_n\end{aligned}$$

we obtain $h_n = 0$. Thus, $h = t$, which means $C_{\mathcal{R}}(S) = \mathcal{A}$. Combining the above and Example 1.1, we complete the proof. $\square$

Remark 2.1 In Theorem 2.7, the set S may be a singleton set. From Example 1.1, we may know that the condition $g'(0) \neq 0$ is necessary to insure that $C_{\mathcal{R}}(S) = \mathcal{A}$. In addition, this condition can not be replaced to be $g(z) \neq 1$ since $C_{\mathcal{R}}(1 + z^2, z) = \mathcal{A}^{\pm}$, which is shown in Example 1.1. For instance $(g, -z) \in C_{\mathcal{R}}(1 + z^2, z)$. As examples of nontrivial g with $g'(0) = 0$, we may see $g(z) = (1 - z + z^2)/(1 - z)$, $(1 - 2z - \sqrt{1 - 4z})/(2z)$, etc.

3 The subgroup of $\mathcal{F}_1$ and centralizers of Riordan group

Let $h(z) = z(1 + \sum_{i \geq 1} b_i z^i)$ and $g(z) = z(1 + \sum_{i \geq 1} a_i z^i)$. Then $(g \circ h)(z) = z(1 + \sum_{i \geq 1} d_i z^i)$. Successively carrying out multiplications and combining similar terms, we obtain the following expressions for the canonical coordinates d_i :

$$\begin{aligned}d_1 &= a_1 + b_1, \quad (4_1) \\d_2 &= a_2 + b_2 + 2a_1 b_1, \quad (4_2) \\d_3 &= a_3 + b_3 + 2a_1 b_2 + 3a_2 b_1 + a_1 b_1^2, \quad (4_3) \\&\dots\dots \\d_n &= a_n + b_n + \sum_{i=1}^{n-1} \alpha_i \phi_{n-i}(b_1, b_2, \dots, b_{n-i}). \quad (4_n)\end{aligned}$$

The expressions $\phi_r(b_1, b_2, \dots, b_r)$ are universal homogeneous polynomials of multidegree r with integer coefficients. It can be seen that the formulae (4_k) , $k = 1, 2, \dots, n$, define a group structure on the direct product $\mathbb{K}^n$. This group will be denoted by $\mathcal{F}_1^{(n)}(\mathbb{K})$. Clearly, $\mathcal{F}_1^{(1)}(\mathbb{K})$ is an Abelian group isomorphic to the additive group of the ring $\mathbb{K}$. The group $\mathcal{F}_1^{(2)}(\mathbb{K})$ is isomorphic to the additive group of $\mathbb{K} \oplus \mathbb{K}$, and the corresponding isomorphism depends non-linearly on the canonical coordinates (b_1, b_2) , etc. (see Babenko [2]).

Proposition 3.1 $(\mathcal{F}_1, \circ)$ is a group with the identity z . For any $f \in \mathcal{F}_1$, the inverse of f is its compositional inverse.

The properties of the substitution group of power series are discussed by Jennings [13], Johnson [14], and Babenko [2].

We denote the centralizers of a subset $S \in \mathcal{F}_1$ as $C_{\mathcal{F}_1}(S) = \{h \in \mathcal{F}_1 : h \circ f = f \circ h \text{ for all } f \in S\}$. We now use the centralizers of $\mathcal{F}_1$ to describe some centralizers of $\mathcal{R}$.



Proposition 3.2 *The centralizer of a Lagrange type Riordan array $(1, h)$ with $h \in \mathcal{F}_1$, $h'(0) = 1$, is*

$$C_{\mathcal{R}}(1, h) = \begin{cases} \mathcal{R} & \text{if } h = z \\ \{(1, f) : f \in C_{\mathcal{F}_1}(h)\} & \text{if } h \neq z. \end{cases} \quad (23)$$

Proof It is obvious that if $h = z$, then $C_{\mathcal{R}}(1, z) = \mathcal{R}$.

Suppose $h(z) = \sum_{n \geq 1} h_n z^n \neq z$, thus there exists a positive integer ℓ such that $h_\ell \neq 0$ and $h_2 = h_3 = \dots = h_{\ell-1} = 0$. Without loss of generality, we may assume $h_1 = 1$. From compositional formula (11) of $g \in \mathcal{F}_0$, where $g(z) = \sum_{n \geq 0} g_n z^n$ with $g_0 = 1$, and the above $h \in \mathcal{F}_1$, we have $(g \circ h)(z) = \sum_{n \geq 0} d_n z^n$, where $d_0 = g_0 = 1$ and the other coefficient c_n for $n \geq 1$ can be expressed as a sum over compositions of n or as an equivalent sum over partitions of n :

$$d_n = \sum_{\mathbf{i} \in \mathcal{D}_n} g_k h_{i_1} h_{i_2} \cdots h_{i_k},$$

where

$$\mathcal{D}_n = \{(i_1, i_2, \dots, i_k) : 1 \leq k \leq n, i_1 + i_2 + \dots + i_k = n\}$$

is the set of compositions of n with k denoting the number of parts. If $(g, f) \in C_{\mathcal{R}}(1, h)$, then $(g, f)(1, h) = (1, h)(g, f)$ implies

$$h(f) = f(h) \quad \text{and} \quad g = g(h), \quad (24)$$

where the first equation of (24) implies $f \in C_{\mathcal{F}_1}(h)$, while the second equation derives $[z^n]g = [z^n]g(h)$, i.e., $g_n = d_n$, which implies

$$g_n = \sum_{\mathbf{i} \in \mathcal{D}_n} g_k h_{i_1} h_{i_2} \cdots h_{i_k}. \quad (25)$$

For the h assumed above, i.e., $h(z) = \sum_{n \geq 1} h_n z^n \neq z$ with $h_\ell \neq 0$ and $h_2 = h_3 = \dots = h_{\ell-1} = 0$, we substitute $n = \ell$ and the conditions of $h_{i_1}, h_{i_2}, \dots$, and h_{i_k} into the (25) and get

$$g_\ell = g_1 h_\ell + 2g_2 h_1 h_2 + \dots + g_\ell h_1^\ell.$$

In the sum on the right-hand side of the above equation, every term except the first one and the last one contains $h_2, h_3, \dots$, and $h_{\ell-1}$. Thus the equation can be reduced to

$$g_\ell = g_1 h_\ell + g_\ell h_1^\ell = g_1 h_\ell + g_\ell,$$

which implies $g_1 = 0$ due to $h_\ell \neq 0$. For $n = \ell$, (25) gives

$$g_{\ell+1} = g_1 h_{\ell+1} + 2g_2 h_1 h_\ell + \dots + g_{\ell+1} h_1^{\ell+1} = 2g_2 h_\ell + g_{\ell+1}$$

because $g_1 = 0$, $h_1 = 1$, and $h_2 = h_3 = \dots = h_{\ell-1} = 0$. Thus from the assumption of $h_\ell \neq 0$, we obtain $g_2 = 0$. Similarly, by using the induction assumption of $g_2 = g_3 = \dots = g_m = 0$, we may have $g_{m+1} = 0$ from

$$\begin{aligned} g_{\ell+m} &= g_1 h_{\ell+k} + 2g_2 h_1 h_{\ell+m-1} + \dots \\ &\quad + g_{m+1}((m+1)h_1^m h_\ell + \text{terms contain } h_2, h_3, \dots, h_{\ell-1}) + \dots \\ &\quad + g_{\ell+m} h_1^{\ell+m}. \end{aligned}$$

More precisely, from the induction assumption and the conditions of $h_2 = h_3 = \dots = h_{\ell-1} = 0$, the above equation can be reduced to

$$g_{\ell+m} = (m+1)g_{m+1} h_\ell + g_{\ell+m},$$

which implies $g_{m+1} = 0$. Hence, we have proved that for $h(z) \neq z$, every element (g, f) of the centralizers of $(1, h)$ in Riordan group must has $g = 1$, completing the proof of the proposition. $\square$

Similarly, we have the following result.



Proposition 3.3 *The centralizer of a Bell type Riordan array $(h/z, h)$ is*

$$C_{\mathcal{R}}(h/z, h) = \begin{cases} \mathcal{R} & \text{if } h = z \\ \{(f/z, f) : f \in C_{\mathcal{F}_1}(h)\} & \text{if } h \neq z. \end{cases}$$

Corollary 3.4 *Let S , and T be subsets of the Appel and Bell subgroups of $\mathcal{R}$, which contain more than one element. Then we have $C_{\mathcal{R}}(S) = \{(1, f) : f \in C_{\mathcal{F}_1}(h)\}$ and $C_{\mathcal{R}}(T) = \{(f/z, f) : f \in C_{\mathcal{F}_1}(h)\}$, respectively.*

Remark 3.1 Propositions 4.8 and 3.3 are not trivial. Although the maps $\gamma_1 : \mathcal{L} \rightarrow \mathcal{F}_1$ and $\gamma_2 : \mathcal{B} \rightarrow \mathcal{F}_1$ defined by $\gamma_1(1, f) = f$ and $\gamma_2(f/z, f) = f$, respectively, for all $f \in \mathcal{F}_1$ are group isomorphisms, $\gamma_k, k = 1, 2$, do not transfer the centralizers from $\mathcal{F}_1$ to $\mathcal{R}$. This is because $\mathcal{F}_1$ contains only one type formal power series, $f \in \mathcal{F}_1$ with $f(0) = 0$ and $f'(0) \neq 0$, while Riordan array (g, f) contains two type formal power series including the above f and $g \in \mathcal{F}_0$ with $g(0) \neq 0$.

4 Related topics

In this section, first we recall the definition of a reverser of a group and study the collection of all reversers of a group as well as their applications to the centralizers of a group. The part on the reversers is a slight modification of a survey shown on pages 25-27 of O'Farrell and Short [17]. Secondly, we consider the set of all elements of a group that can be reversed by an element of a group, particularly, the set of all elements of the Riordan group $\mathcal{R}$ that can be reversed by $M(1, -z)$, i.e., the set of pseudo-involutions of $\mathcal{R}$. Shapiro and one of the authors [10] found that this set is a twisted subgroup of $\mathcal{R}$.

Definition 4.1 An element g of a group G is said to be *reversible* in G if there is another element h of G such that

$$hgh^{-1} = g^{-1}.$$

We say that h reverses g or that h is a *reverser* for g . We denote by $R_g(G)$, or just R_g , the set of reversers of g .

Denote by $R_G(h)$ the set of all elements in G that can be reversed by h , namely,

$$R_G(h) = \{g \in G : hgh^{-1} = g^{-1}\}. \quad (26)$$

We call $R_G(h)$ the reverse set in G reversed by h .

Definition 4.2 Two elements f and g of a group G are said to be *conjugate* in G if there is a third element h such that

$$f = h^{-1}gh,$$

which can be denoted by g^h . Then f is called the conjugate of g by h .

Definition 4.3 The commutator of two elements, g and h , of a group G , is the element

$$[g, h] = g^{-1}h^{-1}gh.$$

It is equal to the group's identity if and only if g and h commute. The subgroup of G generated by all commutators is called the derived group or the commutator subgroup of G . The commutator subgroup of the Riordan group $\mathcal{R}$ is discussed in Luzón, Morón, and Prieto-Martínez [15].

The relationship between the conjugate of g by h and the commutator of g and h is

$$g^h = g[g, h].$$

Particularly, if $g \in C_G(h)$, then $[g, h] = e$ and $g^h = g$.

Proposition 4.4 [17] *Let g be reversible in G , and suppose that $h \in R_g$. Then*

$$R_g = C_g h = h C_g.$$



Here is an explanation of the last proposition. Let $f \in C_g$. Since $h \in R_g$, then $fg = gf$ and $hgh^{-1} = g^{-1}$. We have

$$(hf)g(hf)^{-1} = hfgf^{-1}h^{-1} = hgf f^{-1}h^{-1} = hgh^{-1} = g^{-1}.$$

Thus $hC_g \subset R_g$. Conversely, let $k \in R_g$, then

$$kgk^{-1} = g^{-1}. \quad (27)$$

We have

$$(h^{-1}k)g(h^{-1}k)^{-1} = h^{-1}kgk^{-1}h = h^{-1}g^{-1}h = g.$$

Thus $h^{-1}k \in C_g$, which implies that $k \in hC_g$. Hence, $R_g \subset hC_g$. The same method could be applied to show $R_g = C_g h$.

The above proposition motivates the definition of the extended centralizer:

$$E_g(G) := \{h \in G : g^h = g \text{ or } g^h = g^{-1}\} = C_g \cup R_g.$$

This is also a subgroup of G .

Proposition 4.5 [17] *Let $g \in G$. Then*

- (i) *if g is not reversible, then $E_g = C_g$ and $R_g = \emptyset$.*
- (ii) *if g is an involution, then $E_g = C_g = R_g$.*

Proposition 4.6 [17] *Let g and h be elements of a group G . Then (i) $C_{g^{-1}} = C_g$; (ii) $R_{g^{-1}} = R_g$; (iii) $C_{hgh^{-1}} = hC_g h^{-1}$; (iv) $R_{hgh^{-1}} = hR_g h^{-1}$.*

The extended centralizer group E_g consists of C_g and a single coset R_g . If C_g is abelian, then we have a generalized dihedral group. Hence, there is a question arises: are there examples where C_g is non-abelian? The answer is "yes". Here are two examples. If G is non-abelian group with the identity e , then the centralizer $C_G(e) = G$ is not abelian. Another example is the dihedral group of order $4n$. This has an element, the rotation of 180° , in its center.

We now give the definition of a twisted subgroup.

Definition 4.7 The set of elements reversed by $h \in G$ forms a twisted subgroup $T \equiv T_h := \{g \in G : hgh^{-1} = g^{-1}\}$ of G defined by (see Foguel and Ungar [4,5]):

- (i) $e \in T$ (h reverses e)
- (ii) T is closed under taking inverse (if h reverses g , then h reverses g^{-1}).
- (iii) if $x, y \in T$, then $xyx \in T$.

The set $R(G)$ of reversible elements in G is a union of twisted subgroups. $R(G)$ contains the identity and is closed under taking inverse, but it is not necessarily closed under composition. However, if $hfh^{-1} = f^{-1}$ and $hgh^{-1} = g^{-1}$, i.e., $f, g \in R(G)$ and both are reversed by the same h , then

$$(hg)(fg)(hg)^{-1} = (fg)^{-1},$$

i.e., $hg \in R_{fg}$.

Considering the set of the pseudo-involutions of Riordan group $\mathcal{R}$, which means the set of all $D \in \mathcal{R}$ such that MD (and DM) is an involution, where $M = (1, -t)$. Then $MDM^{-1} = D^{-1}$. Shapiro [20] proved that the centralizer of M in $\mathcal{R}$ is $\mathcal{C}$, the checkerboard subgroup (cf. also Jean-Louis and Nkwanta [12]). Hence, from [10] and [20] we have the the first half and the second half, respectively, of the following result.

Proposition 4.8 *The reserves set in $\mathcal{R}$ reversed by M , $R_{\mathcal{R}}(M)$, is a collection of all elements of pseudo-involutions form a twisted subgroup, which is denoted by T_M . The centralizers of M in $\mathcal{R}$, $C_{\mathcal{R}}(M)$, is the checkerboard subgroup of $\mathcal{R}$.*



Proof For the sake of the readers' convenience, we give a brief proof. First, $I = (1, z) \in T_M$ because $MIM^{-1} = I$. Secondly, $D = (d, h)$ and $E = (g, f) \in R_{\mathcal{R}}(M)$ implies that $DE^{-1}D \in R_{\mathcal{R}}(M)$ due to

$$\begin{aligned} MDE^{-1}DM^{-1} &= (MDM^{-1})(ME^{-1}M^{-1})(MDM^{-1}) \\ &= D^{-1}ED^{-1} = (DE^{-1}D)^{-1}. \end{aligned}$$

Hence, $R_{\mathcal{R}}(M)$ is the twisted subgroup T_M .

If $D = (d, h) \in C_G(M)$, then $MD = DM$ implies that $(1, -t)(d, h) = (d, h)(1, -t)$, i.e.,

$$(d(-t), h(-t)) = (d, -h).$$

Hence, $d(-t) = d(t)$ and $h(-t) = -h(t)$, i.e., (d, h) is in the checkerboard subgroup of $\mathcal{R}$. $\square$

Let G be a group and let $T \subset G$ be a twisted subgroup, from Lemma 1.2 of Aschbacher [1] we have $\langle x \rangle \subset T$ for each $x \in T$. Furthermore, if T is a 2-divisible twisted subgroup, i.e., for each $x \in T$, there exists a unique element $x^{1/2} \in T$ such that $(x^{1/2})^2 = x$, from Corollary 3 of Glauberman [6], we have Lagrange's Theorem on twisted subgroup, i.e., for every $x \in T$, the order of x divides $|T|$. Since T_M is not finite, the Lagrange's Theorem is invalid on T_M .

Shapiro and one of the authors [10] presents a palindromic property of pseudo-involutions. More precisely, if A and B are both pseudo-involutions, then so is the triple product ABA . With this it follows that if are pseudo-involutions so is any palindromic word using these symbols.

Acknowledgements The authors wish to express their sincere gratitude and appreciation to Professor Louis Shapiro for his helpful comments and remarks that led to the current version of the manuscript.

References

1. M. Aschbacher, *Near subgroups of finite groups*, J. Group Theory. **1** (1998), 113–129.
2. I. K. Babenko, *Algebra, geometry and topology of the substitution group of formal power series (Russian)*, Uspekhi Mat. Nauk. **409** (2013), no. 1, 3–76 (2013); translation in Russian Math. Surveys. **68** (2013), no. 1, 1–68.
3. L. Comtet, *Advanced Combinatorics*, French, 1974.
4. T. Foguel and A. A. Ungar, *Involutory decomposition of groups into twisted subgroups and subgroups*, J. Group Theory. **3** (2000), no. 1, 27–46.
5. T. Foguel and A. A. Ungar, *Gyrogroups and the decomposition of groups into twisted subgroups and subgroups*, Pacific J. Math. **197** (2001), no. 1, 1–11.
6. G. Glauberman, *On loops of odd order I*, J. Algebra. **1** (1964), 374–396.
7. H. W. Gould, *A set of polynomials associated with the higher derivatives of $y = x^x$* , Rocky Mountain J. Math. **26** (1996), no. 2, 615–625.
8. R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989.
9. T.-X. He, *Parametric Catalan numbers and Catalan triangles*, Linear Algebra Appl. **438** (2013), no. 3, 1467–1484.
10. T.-X. He and L. W. Shapiro, *Palindromes and pseudo-involution multiplication*, Linear Algebra Appl. **593** (2020), 1–17.
11. T.-X. He and R. Sprugnoli, *Sequence characterization of Riordan arrays*, Discrete Math. **309** (2009), 3962–3974.
12. C. Jean-Louis and A. Nkwanta, *Some algebraic structure of the Riordan group*, Linear Algebra Appl. **438** (2013), no. 5, 2018–2035.
13. S. A. Jennings, *Substitution groups of formal power series*, Canadian J. Math. **6** (1954), 325–340.
14. Johnson, D. L.: The group of formal power series under substitution. J. Austral. Math. Soc. Ser. A. **45**, no. 3, 296–302 (1988).
15. A. M. A. Luzón, M. A. Morón, and L. F. Prieto-Martínez, *The group generated by Riordan involutions*, arXiv:1803.06872, 2018.
16. D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, *On some alternative characterizations of Riordan arrays*, Canadian J. Math. **49** (1997), 301–320.
17. A. G. O'Farrell and I. Short, *Reversibility in dynamics and group theory*, London Mathematical Society Lecture Note Series, 416, Cambridge University Press, Cambridge, 2015.
18. S. Roman, *The Umbral Calculus*, Academic Press, Inc. New York, 1984.
19. S. Roman and G.-C. Rota, *The umbral calculus*, Advances in Math. **27** (1978), no. 2, 95–188.
20. L. W. Shapiro, *Bijections and the Riordan group*, Theoretical Computer Science. **307** (2003), 403–413.
21. L. W. Shapiro, S. Getu, W. J. Woan, L. and Woodson, *The Riordan group*, Discrete Appl. Math. **34** (1991), 229–239.

