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ORIGINAL RESEARCH

Weight distributions of some irreducible cyclic codes of length n

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Abstract Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, where p_i's are distinct odd primes, be a positive integer and F_l be a finite field of prime order l where $gcd(l, n) = 1$. Here, we compute the weight distributions of all irreducible cyclic codes of length n over F_l for the case when multiplicative order of l modulo $p_i^{\alpha_i}$; $O_{p_i^{\alpha_i}}(l) := 2p_i^{\alpha_i - 1}$ for each $\alpha_i \geq 1$. It is also shown that computation of weight distributions of irreducible cyclic codes of length $p_1 p_2 ... p_r$ over F_l is sufficient to compute the weight distributions of all irreducible cyclic codes of length n.

Keywords Irreducible cyclic codes · cyclotomic cosets · weight distribution

Mathematics Subject Classification 94B15

1 Introduction

Let n be a positive integer and let F_l be a finite field of order l; $gcd(n, l) = 1$. Then $R_n = \frac{F_l[x]}{<x^n - 1>}$ is a semisimple ring. Every cyclic code over F_l is an ideal of R_n and, in particular, an irreducible cyclic code C is a minimal ideal in R_n (Chapter 7 [5]). If elements of C are such that for $g(x) \in C$, the $x^i g(x)$ (modulo $x^n - 1$) is also in C, then C is called a cyclic code. Since elements of R_n are polynomials over F_l of degree at most $n - 1$, the mapping $b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} \rightarrow (b_0, b_1, b_2, \cdots, b_{n-1})$ is an isomorphism from $R_n \rightarrow F_l^n$, where F_l^n is a vector space of all n-tuples over F_l, therefore, we can identify $b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$ by $(b_0, b_1, b_2, \cdots, b_{n-1})$ in F_l^n. The weight of an n-tuple $\tau := (b_0, b_1, b_2, \cdots, b_{n-1})$ is number of non-zero coordinates in it. Let $A_t(C)$ denotes the number of codewords of weight t in a cyclic code C, then the collection $\{A_0(C), A_1(C), ..., A_n(C)\}$ is weight distribution of C.

Since the error detecting and error correcting capacity of a code is directly related to its weight distribution, many authors have studied the weight distributions of cyclic codes. Authors in [1,4,7,8,10–12] have computed the weight distributions of some irreducible cyclic codes of prime power length. In [9], Vega computed weight distribution for any irreducible cyclic code of length p^m. He used MacWilliams identity to compute these weight

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distributions. In [2], Fang studied some specific codes of composite length $2^{2^r} - 1$ having two zeros whose dual codes have three weights. In [3] and [6], the authors computed the weight distributions of some cyclic codes of composite length p^n and $2p^n$, and $2^n p^m$ respectively over F_q, where q is a primitive root modulo p^n and $O_{2^n}(q) = 1$. Here, in this paper, we compute the weight distributions of some irreducible cyclic codes of composite length $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, where $p_i's$ are distinct odd primes.

Throughout the paper, $p_1, p_2, ..., p_r, l$ are distinct odd primes and $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$. Further, for each $\alpha_i \geq 1$, the multiplicative order of l modulo $p_i^{\alpha_i}$ i.e. $O_{p_i^{\alpha_i}}(l) = 2p_i^{\alpha_i - 1}$. In Section 2, we have discussed some results on the cyclotomic cosets modulo n. In [9], Vega used MacWilliams identity to compute weight enumerator polynomial $A(Z)'$ of a code C which is a direct sum of p^d equivalent codes each having same weight enumerator polynomial $A(Z)$. In Theorem 1 [9], he showed that $A(Z)' = (A(Z))^{p^d}$. Since, for cases discussed in Theorem 1 [9], the $A(Z)$ is at most a trinomial, therefore, it becomes easy to compute $(A(Z))^{p^d}$. If $A(Z)$ has $k+1$ terms, then $A(Z)'$ has $C(p^d + k, k)$ terms, where $C(p^d + k, k)$ stands for number of combinations of k things out of $p^d + k$ distinct things . Therefore, a Weight set corresponding to codewords of weight t is defined to compute the weight distributions of cyclic codes of length $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ over F_l with the help of weight distributions of cyclic codes of length $p_1 p_2 ... p_r$ over F_l easily. In Section 3, we have computed the weight distributions of all irreducible cyclic codes of length $m = p_1 p_2 ... p_r$. In Section 4, it is shown that for computation of weight distributions of all cyclic codes of length $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, the weight distributions computed in Section 3 are sufficient. In the end, weight distributions of all irreducible cyclic codes of length 15 and 225 are computed over F_{29}.

2 Preliminaries and some notations

Throughout the paper $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, where $p_i's$ are distinct odd primes, l is an odd prime with $gcd(l, n) = 1$, F_l is a finite field of order l. For each $\alpha_i \geq 1$, the multiplicative order of l modulo $p_i^{\alpha_i}$; $O_{p_i^{\alpha_i}}(l) = 2p_i^{\alpha_i - 1}$, $1 \leq i \leq r$. The l-cyclotomic coset modulo n containing s is denoted by $C_s^{n(l)}$ and, the irreducible cyclic code corresponding to $C_s^{n(l)}$ by M_s^n. For a code C, $A_t(C)$ denotes the number of codewords of weight t in C and, in particular $A_t(M_s^n)$ is number of codewords of weight t in M_s^n. Throughout the paper, δ is a fixed primitive nth root of unity in some extension of F_l. Since every code has a minimal and a generating polynomial, therefore,

Definition 2.1 Minimal polynomial and generating polynomial: The polynomial $h_s^n(x)$ denotes the minimal polynomial of M_s^n. It is given by $h_s^n(x) = \prod_{s \in C_s^{n(l)}} (x - \delta^s)$. The generating polynomial of M_s^n is denoted by $g_s^n(x)$. It is given by $g_s^n(x) = \frac{x^n - 1}{h_s^n(x)}$.

In [9], Vega used MacWilliams identity to compute weight distribution of a code C which is a direct sum of p^d equivalent codes each having same weight enumerator polynomial $A(Z)$. In this case weight enumerator polynomial $A(Z)'$ of C is $A(Z)' = (A(Z))^{p^d}$. Since, for cases discussed in Theorem 1 [9], $A(Z)$ is at most a trinomial, therefore, it becomes easy to compute $(A(Z))^{p^d}$. If $A(Z)$ has $k+1$ terms, then $A(Z)'$ has $C(p^d + k, k)$ terms. Therefore, we define a set and call it the Weight Set. With the help of this set it is easy to compute the weight distribution of a code which is a direct sum of equivalent codes (see Example 4.6).

Definition 2.2 Weight Set corresponding to weight t: Let $\bar{D}$ be a cyclic code such that $\bar{D} = D_1 \oplus D_2 \oplus ... \oplus D_r$ i.e., $\bar{D}$ is a direct sum of r equivalent cyclic codes. For $1 \leq i \leq r$, by [7], each D_i has same weight distribution. Let $\{A_{j_1}, A_{j_2}, ... A_{j_k}\}$ be the weight distribution of each D_i. Further let $c \in \bar{D}$ has weight t. Define

$X_t = \{(i_1, i_2, ..., i_k) | j_1 i_1 + j_2 i_2 + ... + j_k i_k = t$, where $i_1, i_2 ..., i_k$ are non-negative integers$\}$. Then, we call X_t the weight set for $\bar{D}$ corresponding to weight t, since the number of codewords of weight t in $\bar{D}$ i.e., $A_t(\bar{D}) =$

$$\sum C(r, i_1 + ... + i_k) C(i_1 + ... + i_k, i_1) C(i_2 + ... + i_k, i_2) .... C(i_{k-1} + i_k, i_{k-1}) A_{j_1}^{i_1} A_{j_2}^{i_2} ... A_{j_k}^{i_k}$$
$$= \sum \frac{r!}{(r - (i_1 + i_2 + ... + i_k))! (i_1)! (i_2)! ... (i_k)!} A_{j_1}^{i_1} A_{j_2}^{i_2} ... A_{j_k}^{i_k}$$

where the sum runs over each element of X_t, $C(r, i)$ denotes the number of combinations of r distinct things taking i at a time and $(i)!$ stands for i factorial.

By our choice on multiplicative order of l modulo p^α; p prime, the $O_p(l) = 2$, $O_{p^2}(l) = 2p$, $O_{p^3}(l) = 2p^2$, ..., therefore, we have following result:

Lemma 2.3 *Let $p_1, p_2, ..., p_r$ be distinct primes. If $0 \leq \gamma_i < \alpha_i$ for each i, $1 \leq i \leq r$ and $O_{p_i^{\alpha_i-\gamma_i}}(l) = 2p_i^{\alpha_i-\gamma_i-1}$, then*

$$O_{p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}...p_r^{\alpha_r-\gamma_r}}(l) = 2p_1^{\alpha_1-\gamma_1-1}p_2^{\alpha_2-\gamma_2-1}...p_r^{\alpha_r-\gamma_r-1}.$$

Proof Since $O_{p_i^{\alpha_i-\gamma_i}}(l) = 2p_i^{\alpha_i-\gamma_i-1}$, therefore,

$$l^{2p_i^{\alpha_i-\gamma_i-1}} \cong 1(mod\ p_i^{\alpha_i-\gamma_i}); 1 \leq i \leq r$$

Simultaneously,

$$l^{lcm(2p_1^{\alpha_1-\gamma_1-1},2p_2^{\alpha_2-\gamma_2-1},...,2p_r^{\alpha_r-\gamma_r-1})} \cong 1(mod\ p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}...p_r^{\alpha_r-\gamma_r}).$$

Equivalently, $l^{2p_1^{\alpha_1-\gamma_1-1}p_2^{\alpha_2-\gamma_2-1}...p_r^{\alpha_r-\gamma_r-1}} \cong 1(mod\ p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2}...p_r^{\alpha_r-\gamma_r})$ since $lcm(2p_1^{\alpha_1-\gamma_1-1}, 2p_2^{\alpha_2-\gamma_2-1}, ..., 2p_r^{\alpha_r-\gamma_r-1}) = 2p_1^{\alpha_1-\gamma_1-1}p_2^{\alpha_2-\gamma_2-1}...p_r^{\alpha_r-\gamma_r-1}$. This proves the Lemma. ◇

The set $C_s^{n(l)} = \{s, sl, sl^2, ..., sl^{u-1}\}$, where u is smallest positive integer such that $sl^u \cong s(mod\ n)$ is called l-cyclotomic coset modulo n. As a consequence of Lemma 2.3, we have following two theorems:

Theorem 2.4 *For each i, let $0 \leq \gamma_i < \alpha_i$. If $gcd(s, n) = p_1^{\gamma_1}p_2^{\gamma_2}...p_r^{\gamma_r}$, where $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$, then order of $C_s^{n(l)}$ i.e., $O(C_s^{n(l)}) = 2p_1^{\alpha_1-\gamma_1-1}...p_r^{\alpha_r-\gamma_r-1}$.*

Theorem 2.5 *Let $\{1, 2, ...r\} = \{i_1, i_2, ..., i_r\}$. If*

$$gcd(s, n) = p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_k}^{\alpha_{i_k}}p_{i_{k+1}}^{\gamma_{i_{k+1}}}p_{i_{k+2}}^{\gamma_{i_{k+2}}}...p_{i_r}^{\gamma_{i_r}},$$

where, $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$ and $0 \leq \gamma_{i_u} < \alpha_{i_u}$, then

$$O(C_s^{n(l)}) = 2p_{i_{k+1}}^{\alpha_{i_{k+1}}-\gamma_{i_{k+1}}-1}p_{i_{k+2}}^{\alpha_{i_{k+2}}-\gamma_{i_{k+2}}-1}...p_{i_r}^{\alpha_{i_r}-\gamma_{i_r}-1}.$$

3 Weight distributions of irreducible cyclic codes of length $p_1p_2...p_r$ over F_l

In this section we compute the weight distributions of all two dimensional irreducible cyclic codes of length $p_1p_2...p_r$, where $p_i's$ are distinct primes.

Lemma 3.1 *Let $m = p_1p_2...p_r$ and l be a prime with $gcd(l, m) = 1$. If λ is a primitive mth root of unity and, for each i, $1 \leq i \leq r$, the $O_{p_i}(l) = 2$, then, for every positive integer s; $s < m$ and $gcd(s, m) = 1$, the polynomial $\frac{x^m-1}{(x-\lambda^s)(x-\lambda^{sl})} := x^{m-2} + \beta_1x^{m-3} + \beta_2x^{m-4} + ... + \beta_{m-2}$ is a complete polynomial and $\begin{vmatrix} \beta_i & \beta_{i+1} \\ \beta_j & \beta_{j+1} \end{vmatrix} \neq 0$.*

Proof By Synthetic Devision, let

$$\frac{x^m - 1}{(x - \lambda^s)(x - \lambda^{ls})} = x^{m-2} + \beta_1x^{m-3} + \beta_2x^{m-4} + ... + \beta_{m-2}.$$

Then $\beta_i = \frac{\lambda^{si}(\lambda^{(l-1)s(i+1)}-1)}{\lambda^{s(l-1)}-1}$. We will now show that each β_i is nonzero. As λ is a primitive mth root of unity, therefore, β_i is zero only when $m|(l-1)s(i+1)$. The $O_{p_i}(l) = 2$ ensures that $gcd(m, l-1) = 1$. Moreover, $gcd(s, m) = 1$ and $(i+1) < m$, therefore, $m \nmid (l-1)s(i+1)$. Consequently, each β_i is nonzero. As $\beta_i = \frac{\lambda^{si}(\lambda^{(l-1)s(i+1)}-1)}{\lambda^{s(l-1)}-1}$, therefore, it is easy to see that $\begin{vmatrix} \beta_i & \beta_{i+1} \\ \beta_j & \beta_{j+1} \end{vmatrix} \neq 0$. ◇

If $gcd(s, , m) = 1$, then by Theorem 1 [7], M_s^m and M_1^m are equivalent codes, there in the following theorem we compute weight distribution of M_1^m.

Theorem 3.2 *Let $m = p_1 p_2 ... p_r$, l be a prime with $gcd(l, m) = 1$ and, for each i, $1 \leq i \leq r$, $O_{p_i}(l) = 2$. Then $A_0(M_1^m) = 1$, $A_{m-1}(M_1^m) = m(l-1)$, $A_m(M_1^m) = (l-1)(l-m+1)$ and $A_k(M_1^m) = 0$ otherwise.*

Proof As $O_{p_i}(l) = 2$, therefore, $O_m(l) = 2$. Then $M_1^m :=< x^{m-2} + \beta_1 x^{m-3} + \beta_2 x^{m-4} + ... + \beta_{m-2} >$ is a two dimensional cyclic code of length m over F_l, where β_i's are same as defined in Lemma 3.1. Clearly, the generating matrix G(say) of M_1^m is

$$G = \begin{bmatrix} \beta_{m-2} & \beta_{m-3} & \beta_{m-4} & ... & \beta_1 & 1 & 0 \\ 0 & \beta_{m-2} & \beta_{m-3} & ... & \beta_2 & \beta_1 & 1 \end{bmatrix}$$

By Lemma 3.1,

$$\begin{vmatrix} \beta_i & \beta_{i+1} \\ \beta_j & \beta_{j+1} \end{vmatrix} \neq 0,$$

therefore, all columns of G are linearly independent. Since M_1^m is a two dimensional code, every codeword in M_1^m is of the form $\begin{bmatrix} a & b \end{bmatrix} G$, where $a, b \in F_l$. We consider $\begin{bmatrix} a & b \end{bmatrix} G$ as an m-tuple over F_l and denote it by σ. The equation $\beta_i x + \beta_{i+1} y = 0$ always has $l-1$ solutions over F_l (if $x = a$, $y = b$ is a solution of $\beta_i x + \beta_{i+1} y = 0$, then $x = \eta a$, $y = \eta b$; $\eta \in F_l$ is also a solution of $\beta_i x + \beta_{i+1} y = 0$). Consequently, there are $l-1$ choices of the form (a, b) over F_l which makes exactly one coordinate of σ zero. Equivalently, we get a codeword of weight $m-1$ in M_1^m. The G has m columns, therefore, there are $m(l-1)$ codewords of weight $m-1$ in M_1^m. The number of choices of nonzero tuples of the form (a, b); $a, b \in F_l$ is $l^2 - 1$ out of which only $m(l-1)$ tuples produce a single zero in σ, therefore, $l^2 - 1 - m(l-1) = (l-1)(l+1-m)$ codewords in M_1^m have weight m. Moreover, there is exactly one codeword of weight zero. Hence we have $A_0(M_1^m) = 1$, $A_m(M_1^m) = (l-1)(l+1-m)$, $A_{m-1}(M_1^m) = m(l-1)$ and $A_k(M_1^m) = 0$ otherwise. ◇

If $gcd(s, m) = P$, then by Theorem 1 [7], M_s^m and M_P^m are equivalent codes, therefore, in the following theorem we compute weight distribution of M_P^m.

Theorem 3.3 *Let $m = p_1 p_2 ... p_r$ and P is divisor of m. If $O_{p_i}(l) = 2$; $1 \leq i \leq r$, then $A_{m-P}(M_P^m) = \frac{m}{P}(l-1)$, $A_m(M_P^m) = (l-1)(l+1-\frac{m}{P})$, $A_0(M_P^m) = 1$ and $A_k(M_P^m) = 0$ otherwise.*

Proof As $gcd(s, m) = P < m$, $O_{p_i}(l) = 2$, therefore, the minimal polynomial $h_P^m(x)$ corresponding to the cyclotomic coset $C_P^{m(l)}$ is $(x - \lambda^P)(x - \lambda^{lP})$, where λ is a fixed primitive mth root of unity. Then, the generating polynomial of the code M_P^m is

$$\begin{aligned} g_P^m(x) &= \frac{x^m - 1}{h_P^m(x)} \\ &= \frac{(x^{\frac{m}{P}} - 1)(1 + x^{\frac{m}{P}} + ... + x^{\frac{m(P-1)}{P}})}{(x - \lambda^P)(x - \lambda^{lP})}. \end{aligned}$$

Let C be a code of length $\frac{m}{P}$ generated by $\frac{(x^{\frac{m}{P}} - 1)}{(x-\lambda^P)(x-\lambda^{lP})}$. Then, by Theorem 3.2, $A_{\frac{m}{P}-1}(C) = \frac{m}{P}(l-1)$, $A_{\frac{m}{P}}(C) = (l-1)(l+1-\frac{m}{P})$, $A_0(C) = 1$ and $A_k(C) = 0$ otherwise. Clearly M_P^m is a repetition code of C repeated P times, the result follows. ◇

Since $h_0^m(x) = (x-1)$, therefore, the $g_0^m(x) = (1 + x + x^2 + ...x^{m-1})$. Then we have following simple corollary:

Corollary 3.4 *The $A_0(M_0^m) = 1$, $A_m(M_0^m) = l-1$, $A_k(M_0^m) = 0$ otherwise.*

4 Weight distributions of irreducible cyclic codes of length $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, where $p_i's$ are distinct odd primes

In this section, we observe that the weight distributions of two dimensional irreducible cyclic codes of length $p_1 p_2 ... p_r$ over F_l are sufficient to compute the weight distributions of all irreducible cyclic codes of length $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$. First we prove a lemma.

Lemma 4.1 *Let C be an irreducible cyclic code of dimension $d \geq 1$ over F_l having length m. Further, let $g(x)$ be the generating polynomial of C. If $C^* =< g(x^t), x^t g(x^t), x^{2t} g(x^t), ..., x^{t(d-1)} g(x^t) >$ is a cyclic code of length mt over F_l, then C and C^* have same weight distribution.*

Proof Since C is a d dimensional irreducible cyclic code of length m, it is a d dimensional vector space over F_l with basis $\{g(x), xg(x), ..., x^{d-1}g(x)\}$. Then, for $c(x) \in C$,

$$c(x) = \alpha_0 g(x) + \alpha_1 x g(x) + ... + \alpha_{d-1} x^{d-1} g(x); \alpha_i \in F_l$$

Equivalently,

$$c(x^t) = \alpha_0 g(x^t) + \alpha_1 x^t g(x^t) + ... + \alpha_{d-1} x^{t(d-1)} g(x^t); \alpha_i \in F_l.$$

From above equation, we get that $c(x^t)$ is a linear combination of

$$g(x^t), x^t g(x^t), x^{2t} g(x^t), ..., x^{t(d-1)} g(x^t)$$

over F_l. Therefore, $c(x^t) \in C^*$. As C and C^* both have same dimension and there is a one to one correspondence between codewords of C and C^*, C and C^* have same weight distribution. ◇

As a result of above lemma we have following observations:

Observations 4.2 (i) Clearly $C^* =< g(x^t), x^t g(x^t), x^{2t} g(x^t), ..., x^{t(d-1)} g(x^t) >$ is a d dimensional code over F_l.

(ii) For each positive integer i, the code

$$C_i =< x^{i-1} g(x^t), x^{t+i-1} g(x^t), x^{2t+i-1} g(x^t), ..., x^{t(d-1)+i-1} g(x^t) >$$

and

$$C^* =< g(x^t), x^t g(x^t), x^{2t} g(x^t), ..., x^{t(d-1)} g(x^t) >$$

have same weight distribution.

(iii) Codes $C_1, C_2, ..., C_t$, where

$$C_i =< x^{i-1} g(x^t), x^{t+i-1} g(x^t), ..., x^{t(d-1)+i-1} g(x^t) >$$

are linearly independent also.

By Lemma 4.1 and Observations 4.2, we have following theorem:

Theorem 4.3 *Let C be a cyclic code of dimension $d \geq 1$ over F_l having length m. Further, let $g(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + ... + \beta_{m-d} x^{m-d}$ be the generating polynomial of C. If C^{**} is a td dimensional code of length mt generated by $f(x) = \beta_0 + \beta_1 x^t + \beta_2 x^{2t} + ... + \beta_{m-d} x^{(m-d)t}$, then $C^{**} = C_1 \oplus C_2 \oplus ... \oplus C_t$, where $C_1, C_2, ..., C_t$ are same as defined in Observations 4.2(ii).*

Proof Clearly $f(x) = g(x^t)$. Therefore,

$$\begin{aligned}
&\{g(x^t), xg(x^t), x^2 g(x^t), ..., x^{t-1} g(x^t),\\
&x^t g(x^t), x^{t+1} g(x^t), x^{t+2} g(x^t), ..., x^{2t-1} g(x^t),\\
&x^{2t} g(x^t), x^{2t+1} g(x^t), x^{2t+2} g(x^t), ..., x^{3t-1} g(x^t),\\
&.....,\\
&x^{(d-1)t} g(x^t), x^{(d-1)t+1} g(x^t), x^{(d-1)t+2} g(x^t), ..., x^{dt-1} g(x^t)\}
\end{aligned}$$

is a basis of C^{**} over F_l. We regroup above basis of C^{**} as:

$$\begin{aligned}
&\{g(x^t), x^t g(x^t), x^{2t} g(x^t), ..., x^{(d-1)t} g(x^t),\\
&xg(x^t), x^{t+1} g(x^t), x^{2t+1} g(x^t), ..., x^{(d-1)t+1} g(x^t),\\
&x^2 g(x^t), x^{t+2} g(x^t), x^{2t+2} g(x^t), ..., x^{(d-1)t+2} g(x^t),\\
&.....\\
&x^{t-1} g(x^t), x^{2t-1} g(x^t), x^{3t-1} g(x^t), ..., x^{dt-1} g(x^t)\}.
\end{aligned}$$

Then, by Observation 4.2(iii), $C^{**} = C_1 \oplus C_2 \oplus ... \oplus C_t$. ◇

Next we compute weight distributions of all irreducible cyclic codes of length $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ over F_l.

Theorem 4.4 *Let s be a positive integer with $gcd(s, n) = d = p_1^{\gamma_1} p_2^{\gamma_2} ... p_r^{\gamma_r}$, where, $0 \leq \gamma_i < \alpha_i$. For a nonzero positive integer v, if $X_v = \{(a, b)|ap_1p_2...p_r + b(p_1p_2...p_r - 1) = v\}$ is nonempty, then $A_{vp_1^{\gamma_1} p_2^{\gamma_2} ... p_r^{\gamma_r}}(M_s^n) = \sum C(p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}, (a+b))C(a+b, a)(l-1)^{a+b}(l+1-p_1p_2...p_r)^a(p_1p_2...p_r)^b$, where the sum runs over each element of X_v.*

Proof Since $gcd(s, n) = d = p_1^{\gamma_1} p_2^{\gamma_2} ... p_r^{\gamma_r}$, by Theorem 2.4,

$$O(C_s^{n(l)}) = O(C_d^{n(l)}) = 2p_1^{\alpha_1-\gamma_1-1} p_2^{\alpha_2-\gamma_2-1} ... p_r^{\alpha_r-\gamma_r-1},$$

therefore, M_s^n and M_d^n are equivalent codes. Hence we compute the weight distribution of M_d^n. Clearly M_d^n is a $2p_1^{\alpha_1-\gamma_1-1} p_2^{\alpha_2-\gamma_2-1} ... p_r^{\alpha_r-\gamma_r-1}$ dimensional code having

$$(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} - \delta^{p_1^{\alpha_1-1} ... p_r^{\alpha_r-1}})(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} - \delta^{lp_1^{\alpha_1-1} ... p_r^{\alpha_r-1}}),$$

where δ is a primitive nth root of unity, as its minimal polynomial.

Then, generating polynomial $g_d^n(x)$ of M_d^n is

$$\begin{aligned} g_d^n(x) &= \frac{x^n - 1}{h_d^n(x)} \\ &= \frac{(x^{p_1^{\alpha_1-\gamma_1} ... p_r^{\alpha_r-\gamma_r}} - 1)(1 + x^{p_1^{\alpha_1-\gamma_1} ... p_r^{\alpha_r-\gamma_r}} + ... + x^{(p_1^{\alpha_1-\gamma_1} .... p_r^{\alpha_r-\gamma_r})(p_1^{\gamma_1} ... p_r^{\gamma_r}-1)})}{(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} - \delta^{p_1^{\alpha_1-1} ... p_r^{\alpha_r-1}})(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} - \delta^{lp_1^{\alpha_1-1} ... p_r^{\alpha_r-1}})}. \end{aligned}$$

Let C be a cyclic code generated by

$$g(x) = \frac{(x^{p_1^{\alpha_1-\gamma_1} p_2^{\alpha_2-\gamma_2} ... p_r^{\alpha_r-\gamma_r}} - 1)}{(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} - \delta^{p_1^{\alpha_1-1} ... p_r^{\alpha_r-1}})(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} - \delta^{lp_1^{\alpha_1-1} ... p_r^{\alpha_r-1}})}.$$

Putting

$$x^{p_1^{\alpha_1-\gamma_1-1} p_2^{\alpha_2-\gamma_2-1} ... p_r^{\alpha_r-\gamma_r-1}} = y$$

and

$$\delta^{p_1^{\alpha_1-1} ... p_r^{\alpha_r-1}} = \lambda$$

in $g(x)$, we get a polynomial $f(y) = \frac{y^{p_1p_2...p_r}-1}{(y-\lambda)(y-\lambda^l)}$. By Lemma 3.1, $f(y) = \beta_0 + \beta_1 y + ... + \beta_{p_1p_2...p_r-2} y^{p_1p_2...p_r-2}$.

Then

$$\begin{aligned} g(x) &= f(x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}}) \\ &= \beta_0 + \beta_1 x^{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}} + ... + \beta_{p_1p_2...p_r-2} x^{(p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1})(p_1p_2...p_r-2))}. \end{aligned}$$

Therefore, by Theorem 4.3,

$$C = C_1 \oplus C_2 \oplus ... \oplus C_{p_1^{\alpha_1-\gamma_1-1} ... p_r^{\alpha_r-\gamma_r-1}}. \quad (1)$$

By Lemma 4.1, for each i, C_i is equivalent to a two dimensional code of length $p_1p_2...p_r$. Therefore, by Theorem 3.2, $A_{p_1p_2...p_r}(C_i) = (l-1)(l - p_1p_2...p_r + 1)$ and $A_{p_1p_2...p_r-1}(C_i) = (l-1)p_1p_2...p_r$. Consequently, each block of (1) contributes $p_1p_2...p_r$ or $p_1p_2...p_r - 1$ to the weight of a codeword in C. For a nonzero positive integer v, if $X_v = \{(a, b)|ap_1p_2...p_r + b(p_1p_2...p_r - 1) = v\}$ is nonempty, then C has codewords of weight v and, by Definition 2.2,
$A_v(C) = \sum C(p_1^{\alpha_1-\gamma_1-1} p_2^{\alpha_2-\gamma_2-1} ... p_r^{\alpha_r-\gamma_r-1}, (a+b))C(a+b, a)(l-1)^{a+b}(l+1-p_1p_2...p_r)^a(p_1p_2...p_r)^b$, where the sum runs over each element of X_v. Since M_d^n is a repetition code of C repeated $p_1^{\gamma_1} p_2^{\gamma_2} ... p_r^{\gamma_r}$ times, for each codeword of weight v in C, there is a codeword of weight $vp_1^{\gamma_1} p_2^{\gamma_2} ... p_r^{\gamma_r}$ in M_s^n. Therefore, $A_{vp_1^{\gamma_1} p_2^{\gamma_2} ... p_r^{\gamma_r}}(M_s^n) = A_v(C)$. ◇

In the following theorem we have $\{1, 2, 3, ..., r\} = \{i_1, i_2, i_3, ..., i_r\}$ i.e., $\{i_1, i_2, i_3, ..., i_r\}$ is an index set of $\{1, 2, 3, ..., r\}$.

Theorem 4.5 *Let s be a positive integer with*

$$gcd(s, n) = d = p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} ... p_{i_k}^{\alpha_{i_k}} p_{i_{k+1}}^{\gamma_{i_{k+1}}} p_{i_{k+2}}^{\gamma_{i_{k+2}}} ... p_{i_r}^{\gamma_{i_r}}$$

where, $0 \leq \gamma_{i_u} < \alpha_{i_u}$; $k+1 \leq u \leq r$. For a nonzero v, if $X_v = \{(a, b) | a p_{i_{k+1}} p_{i_{k+2}} ... p_{i_r} + b(p_{i_{k+1}} p_{i_{k+2}} ... p_{i_r} - 1) = v\}$ is nonempty, then

$$A_{v p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} ... p_{i_k}^{\alpha_{i_k}} p_{i_{k+1}}^{\gamma_{i_{k+1}}} p_{i_{k+2}}^{\gamma_{i_{k+2}}} ... p_{i_r}^{\gamma_{i_r}}}(M_s^n)$$

$$= \sum C(p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1}, (a + b)) C(a + b, a)(l - 1)^{a+b} (l + 1 - p_{i_{k+1}} p_{i_{k+2}} ... p_{i_r})^a (p_{i_{k+1}} p_{i_{k+2}} ... p_{i_r})^b,$$

where the sum runs over each element of X_v.

Proof Since $gcd(s, n) = d = p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} ... p_{i_k}^{\alpha_{i_k}} p_{i_{k+1}}^{\gamma_{i_{k+1}}} p_{i_{k+2}}^{\gamma_{i_{k+2}}} ... p_{i_r}^{\gamma_{i_r}}$, where, $0 \leq \gamma_{i_u} < \alpha_{i_u}$, by Theorem 2.5,

$$O(C_s^{n(l)}) = O(C_d^{n(l)}) = 2 p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1},$$

therefore, M_s^n and M_d^n are equivalent codes. Hence we will compute the weight distribution of M_d^n. As $O(C_d^{n(l)}) = 2 p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1}$, therefore, M_d^n is a $2 p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1}$ dimensional code having

$$h_d^n(x) = (x^{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1}} - \delta^{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} ... p_{i_k}^{\alpha_{i_k}} p_{i_{k+1}}^{\alpha_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - 1}})$$
$$(x^{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1}} - \delta^{l p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} ... p_{i_k}^{\alpha_{i_k}} p_{i_{k+1}}^{\alpha_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - 1}})$$

as its minimal polynomial. Then, generating polynomial $g_d^n(x)$ of M_d^n is

$$g_d^n(x) = \frac{x^n - 1}{h_d^n(x)} = (\frac{x^{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r}}} - 1}{h_d^n(x)})$$
$$(1 + x^{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r}}} + ... + x^{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r}} (p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} ... p_{i_k}^{\alpha_{i_k}} p_{i_{k+1}}^{\gamma_{i_{k+1}}} p_{i_{k+2}}^{\gamma_{i_{k+2}}} ... p_{i_r}^{\gamma_{i_r}} - 1)}.$$

Let C be the cyclic code generated by

$$\left(\frac{x^{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r}}} - 1}{h_d^n(x)} \right).$$

Then, By Lemma 3.1 and Theorem 4.3,

$$C = C_1 \oplus C_2 \oplus ... \oplus C_{p_{i_{k+1}}^{\alpha_{i_{k+1}} - \gamma_{i_{k+1}} - 1} p_{i_{k+2}}^{\alpha_{i_{k+2}} - \gamma_{i_{k+2}} - 1} ... p_{i_r}^{\alpha_{i_r} - \gamma_{i_r} - 1}}. \qquad (2)$$

By Lemma 4.1, for each i, C_i is equivalent to a two dimensional code of length $p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}$. Therefore, by Theorem 3.3,

$$A_{p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}}(C_i) = (l-1)(l+1-p_{i_{k+1}}p_{i_{k+2}}...p_{i_r})$$

and

$$A_{p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}-1}(C_i) = (l-1)p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}.$$

Consequently, each block of (2) contributes $p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}$ or $p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}-1$ to the weight of a codeword in C. For a nonzero v, if $X_v = \{(a,b)|ap_{i_{k+1}}p_{i_{k+2}}...p_{i_r} + b(p_{i_{k+1}}p_{i_{k+2}}...p_{i_r}-1) = v\}$ is nonempty, then C has codewords of weight v, and by Definition 2.2

$A_v(C) = \sum C(p_{i_{k+1}}^{\alpha_{i_{k+1}}-\gamma_{i_{k+1}}-1}p_{i_{k+2}}^{\alpha_{i_{k+2}}-\gamma_{i_{k+2}}-1}p_{i_r}^{\alpha_{i_r}-\gamma_{i_r}-1},(a+b))C(a+b,a)(l-1)^{a+b}(l+1-p_{i_{k+1}}p_{i_{k+2}}...p_{i_r})^a(p_{i_{k+1}}p_{i_{k+2}}...p_{i_r})^b$, where the sum runs over each element of X_v. Since M_d^n is a repetition code of C repeated

$$p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_k}^{\alpha_{i_k}}p_{i_{k+1}}^{\gamma_{i_{k+1}}}p_{i_{k+2}}^{\gamma_{i_{k+2}}}...p_{i_r}^{\gamma_{i_r}}$$

times, for each codeword of weight v in C, there is a codeword of weight

$$vp_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_k}^{\alpha_{i_k}}p_{i_{k+1}}^{\gamma_{i_{k+1}}}p_{i_{k+2}}^{\gamma_{i_{k+2}}}...p_{i_r}^{\gamma_{i_r}}$$

in (M_s^n). Therefore, $A_{vp_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_k}^{\alpha_{i_k}}p_{i_{k+1}}^{\gamma_{i_{k+1}}}p_{i_{k+2}}^{\gamma_{i_{k+2}}}...p_{i_r}^{\gamma_{i_r}}}(M_s^n) = A_v(C)$ ◇

Example 4.6 Here we compute the weight distributions of all irreducible cyclic codes of length 225 over F_{29}. By Lemma 3.1 and Theorem 3.2, we have following table for weight distributions of all irreducible cyclic codes of length 15 over F_{29}.

Table 1 .

t	$A_t(M_s^{15})$; $gcd(s,15)=1$
0	1
$1 \le t \le 13$	0
14	420
15	420

By Theorem 3.3, we have

Table 2 .

t	$A_t(M_s^{15})$; $gcd(s,15)=3$
0	1
$1 \le t \le 11$	0
12	140
$13 \le t \le 14$	0
15	700

Table 3 .

t	$A_t(M_s^{15})$; $gcd(s,15)=5$
0	1
$1 \le t \le 9$	0
10	84
$11 \le t \le 14$	0
15	756

We now have following tables for weight distributions of all irreducible cyclic codes of length 225 over F_{29}. Table 4, Table 5, Table 6 and Table 7 are consequences of Theorem 4.4 while Table 8 to Table 11 are due to Theorem 4.5

Table 4 .

t	$A_t(M_s^{225})$; $gcd(s, 225) = 1$
0	1
$14 \leq t \leq 15$	C(15,1)(420)
$28 \leq t \leq 30$	$C(15, 2)C(2, t-28)(420)^2$
$42 \leq t \leq 45$	$C(15, 3)C(3, t-42)(420)^3$
$56 \leq t \leq 60$	$C(15, 4)C(4, t-56)(420)^4$
$70 \leq t \leq 75$	$C(15, 5)C(5, t-70)(420)^5$
$84 \leq t \leq 90$	$C(15, 6)C(6, t-84)(420)^6$
$98 \leq t \leq 105$	$C(15, 7)C(7, t-98)(420)^7$
$112 \leq t \leq 120$	$C(15, 8)C(8, t-112)(420)^8$
$126 \leq t \leq 135$	$C(15, 9)C(9, t-126)(420)^9$
$140 \leq t \leq 150$	$C(15, 10)C(10, t-140)(420)^{10}$
$154 \leq t \leq 165$	$C(15, 11)C(11, t-154)(420)^{11}$
$168 \leq t \leq 180$	$C(15, 12)C(12, t-168)(420)^{12}$
$182 \leq t \leq 195$	$C(15, 13)C(13, t-172)(420)^{13}$
$196 \leq t \leq 209$	$C(15, 14)C(14, t-196)(420)^{14}$
210	$C(15, 14)C(14, 14)(420)^{14} + C(15, 15)C(15, 15)(420)^{15}$
$211 \leq t \leq 225$	$C(15, 15)C(15, t-210)(420)^{15}$

Table 5 .

t	$A_{3t}(M_s^{225})$; $gcd(s, 225) = 3$
0	1
$14 \leq t \leq 15$	C(5,1)(420)
$28 \leq t \leq 30$	$C(5, 2)C(2, t-28)(420)^2$
$42 \leq t \leq 45$	$C(5, 3)C(3, t-42)(420)^3$
$56 \leq t \leq 60$	$C(5, 4)C(4, t-56)(420)^4$
$70 \leq t \leq 75$	$C(5, 5)C(5, t-70)(420)^5$

Table 6 .

t	$A_{5t}(M_s^{225})$; $gcd(s, 225) = 5$
0	1
$14 \leq t \leq 15$	C(3,1)(420)
$28 \leq t \leq 30$	$C(3, 2)C(2, t-28)(420)^2$
$42 \leq t \leq 45$	$C(3, 3)C(3, t-42)(420)^3$

Table 7 .

t	$A_{15t}(M_s^{225})$; $gcd(s, 225) = 15$
0	1
14	420
15	420

Table 8 .

t	$A_{9t}(M_s^{225})$; $gcd(s, 225) = 9$
0	1
$4 \leq t \leq 5$	$C(5,1)C(1,t-4)(140)^{1-t+4}(700)^{t-4}$
$8 \leq t \leq 10$	$C(5,2)C(2,t-8)(140)^{2-t+8}(700)^{t-8}$
$12 \leq t \leq 15$	$C(5,3)C(3,t-12)(140)^{3-t+12}(700)^{t-12}$
$16 \leq t \leq 19$	$C(5,4)C(4,t-16)(140)^{4-t+16}(700)^{t-16}$
20	$C(5,4)C(4,4)(700)^4 + C(5,5)C(5,5)(140)^5$
$21 \leq t \leq 25$	$C(5,5)C(5,t-20)(140)^{5-t+20}(700)^{t-20}$

Table 9 .

t	$A_{45t}(M_s^{225})$; $gcd(s, 225) = 45$
0	1
4	140
5	700

Table 10 .

t	$A_{25t}(M_s^{225})$; $gcd(s, 225) = 25$
0	1
$2 \leq t \leq 3$	$C(3,1)C(1,t-2)(84)^{1-t+2}(756)^{t-2}$
$4 \leq t \leq 5$	$C(3,2)C(2,t-4)(84)^{2-t+4}(756)^{t-4}$
6	$C(3,2)C(2,0)(756)^2 + C(3,3)C(3,3)(84)^3$
$7 \leq t \leq 9$	$C(3,3)C(3,t-6)(84)^{3-t+6}(756)^{t-6}$

Table 11 .

t	$A_{75t}(M_s^{225})$; $gcd(s, 225) = 75$
0	1
2	84
3	756

References

1. C. Ding, The weight distribution of some irreducible cyclic codes, *IEEE Trans. Inform.Theory*, **55** (2009), 955-960.
2. T. Fang, On cyclic codes of length $2^{2^r} - 1$ with two zeros whose dual codes have three weights, *Des. Codes Crytogr.*, **62** (2012), 253-258.
3. P. Kumar, M. Sangwan and S.K. Arora, The weight distributions of some irreducible cyclic codes of length p^n and $2p^n$, *Advance Math. Commun.*, **9** (2015), 277-289.
4. Y. Liu and H. Yan, A class of five-weight cyclic codes and their weight distribution, *Des. Codes Cryptogr.*, **79** (2016), 353-366.
5. F.J. MacWilliams and N.J.A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam (1977).
6. M. Sangwan and P. Kumar, The weight distributions of irreducible cyclic codes of length $2^n p^m$, *Asian. Eur. J. Math.*, **11** (2018), 1850085 (11 pages).
7. A. Sharma and G.K. Bakshi, The weight distributions of some irreducible cyclic codes, *Finite Fields Appl.*, **18** (2012), 144-159.
8. A. Sharma, G.K. Bakshi and M. Raka, The weight distributions of irreducible cyclic codes of length 2^m, *Finite Fields Appl.*, **13** (2007), 1086-1095.
9. G. Vega, The weight distribution for any irreducible cyclic code of length p^m, *Appli. algeb. Engineering commun. comput.*, **29(4)** (2018), 363-370.
10. M. Xiong, The weight distribution of a class of cyclic codes, *Finite Fields Appl.*, **18** (2012), 1086-1095.
11. S. Yang, X. Kong and X. Shi, Complete weight enumerators of a class of linear codes over finite fields, *Advance Math. Commun.*, **15** (2021), 99-112.
12. X. Zhu, Q. Yue, and L. Hu, Weight distributions of cyclic codes of length l^m, *Finite Fields Appl.*, **31** (2015), 241-257.