Indian J Pure Appl Math (2022) 53:1090–1109
https://doi.org/10.1007/s13226-022-00221-0

ORIGINAL RESEARCH

Sums of Fourier coefficients of holomorphic cusp forms over integers without large prime factors

Dan Wang

Received: 17 August 2021 / Accepted: 13 January 2022 / Published online: 14 February 2022

Abstract Our goal in this paper is to prove an asymptotic estimate for sums of Fourier coefficients of holomorphic cusp forms over integers without large prime factors by using a Chebyshev-type functional equation.

Keywords Asymptotic results on arithmetic functions, Fourier coefficients of automorphic forms

Mathematics Subject Classification 11N37 · 11F30

1 Introduction and main results

Integers whose prime factors are small have attracted a lot of mathematicians to study in many areas of number theory. Let $P(n)$ denote the largest prime divisor of $n > 1$, with $P(1) = 1$. Positive integers n with $P(n) \leqslant y$ for a fixed number y are particularly interesting. The well known counting function of integers not exceeding x with prime factors at most y is

$$\Psi(x, y) := \sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} 1.$$

The distribution of $\Psi(x, y)$ has been extensively studied, see, e.g. papers of Norton [16], Moree [15], Hildebrand and Tenenbaum [8]. All these works originate from the following asymptotic distribution result due to Dickman [5]. He proved that

$$\Psi(x, y) \sim x\varrho(u), x \to \infty, u := \frac{\log x}{\log y},$$

where $\varrho(u)$ satisfies

$$u\varrho'(u) = -\varrho(u-1), u > 1,$$

and subjects to the initial conditions

$$\varrho(u) = 0, u < 0,$$
$$\varrho(u) = 1, 0 \leqslant u \leqslant 1.$$

Communicated by B. Sury.

D. Wang (✉)
School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, Shandong, China
E-mail: wangdan-10@hotmail.com

The function $\varrho(u)$ is now known as Dickman's function.

De Bruijn [3] obtained that, for any fixed $\epsilon > 0$,

$$\Psi(x, y) = x\varrho(u)\left\{1 + O\left(\frac{\log(u+1)}{\log y}\right)\right\}, \tag{1.1}$$

uniformly for

$$1 \leqslant u := \frac{\log x}{\log y} \leqslant (\log x)^{3/8-\epsilon}, \quad \text{as} \quad x \to \infty.$$

Later, Hildebrand [7] improved the range of u by using a Chebyshev-type functional equation for $\Psi(x, y)$. He showed that (1.1) holds uniformly for

$$x \geqslant 3,\ 1 \leqslant u := \frac{\log x}{\log y} \leqslant \log x/(\log\log x)^{5/3+\epsilon}.$$

Alladi [1], Hildebrand and Tenenbaum [9] respectively studied an interesting distribution which is that

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \mu(n).$$

Ivić [10,11] studied the squarefree numbers with restricted prime factors, i.e,

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \mu^2(n).$$

Xuan [22,23] studied the average order of divisor functions over integers without large prime factors, i.e,

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \tau^k(n).$$

Based on the above research results, we are interested in the sums of Fourier coefficients of holomorphic cusp forms over integers without large prime factors, and want to know the distribution law of Fourier coefficients of holomorphic cusp forms in sparse sequences. Studying the oscillatory of the Fourier coefficients of holomorphic cusp forms or Hecke Maass forms in number theory is an important problem. It has attracted a lot of mathematicians to study, such as, Walfisz, Rankin, Hafner, Ivić. We firstly introduce some information about the holomorphic cusp forms.

Let H_k denote the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2, \mathbb{Z})$. For $f(z) \in H_k$, it has a Fourier expansion at the cusp ∞ given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz),$$

where $\lambda_f(n)$ is real, $\lambda_f(1) = 1$, and $\lambda_f(n)$ satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geqslant 1$ and $n \geqslant 1$ are integers, obviously $\lambda_f^2(p) = 1 + \lambda_f(p^2)$. For the holomorphic cusp forms, Deligne [4] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leqslant d(n).$$

Our goal of this paper is to prove an asymptotic estimate about sums of Fourier coefficients of holomorphic cusp forms, which the sums are over integers without large prime factors. i.e,

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \lambda_f^2(n).$$

Before we give the asymptotic estimate for sums of Fourier coefficients of holomorphic cusp forms over integers without large prime factors, we need to prove the asymptotic estimate for the sums, i.e,

$$\sum_{\substack{n\leqslant x\\P(n)\leqslant y}}\frac{\lambda_f^2(n)}{n}.$$

Our main results are the following.

Theorem 1 *Suppose $f(z)$ are the normalized primitive holomorphic forms for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$. Let $P(n)$ denote the largest prime divisor of $n>1$, with $P(1)=1$, then for all sufficiently large y, we have*

$$\sum_{\substack{n\leqslant x\\P(n)\leqslant y}}\frac{\lambda_f^2(n)}{n}=V(y)\left\{j(u)+O_f\left(\frac{\log(u+1)}{\log y}\right)\right\},$$

uniformly for

$$1\leqslant u:=\frac{\log x}{\log y}\leqslant\exp\left(\frac{1}{a}\log y\right),$$

where a is a suitable constant, $uj'(u)=j(u)-j(u-1), u>1$; $j(u)=e^{-\gamma}u, 0\leqslant u<1$; $j(u)=0, u<0$, and

$$V(y)=\prod_{p\leqslant y}\left(1+\sum_{k=1}^{\infty}\frac{\lambda_f^2(p^k)}{p^k}\right).$$

Theorem 2 *Suppose $f(z)$ are the normalized primitive holomorphic forms for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$. Let $P(n)$ denote the largest prime divisor of $n>1$, with $P(1)=1$, then for all sufficiently large y, we have*

$$\sum_{\substack{n\leqslant x\\P(n)\leqslant y}}\lambda_f^2(n)=\frac{x}{\log x}V(y)\left\{\varrho(u)+O_f\left(\frac{\log(u+1)}{u\log y}\right)\right\},$$

uniformly for

$$1\leqslant u:=\frac{\log x}{\log y}\leqslant\exp\left(\frac{1}{a}\log y\right),$$

where a is a suitable constant, ϱ is Dickman's function, and

$$V(y)=\prod_{p\leqslant y}\left(1+\sum_{k=1}^{\infty}\frac{\lambda_f^2(p^k)}{p^k}\right).$$

2 The lemmas

Lemma 2.1 *Suppose $f(z)$ are the normalized primitive holomorphic forms for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$, then*

$$\sum_{n\leqslant x}\lambda_f^2(n)=C_f x+O_f(x^{3/5}).\tag{2.1}$$

By partial summation, we have

$$\sum_{n\leqslant x}\frac{\lambda_f^2(n)}{n}=C_f\log x+O(1),\tag{2.2}$$

where C_f is a positive constant depending on f, i.e,

$$C_f = \lim_{s\to 1+0} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_f^2(p^k)}{p^{ks}}\right)\left(1 - \frac{1}{p^s}\right).$$

Proof The result (2.1) obtained independently by Rankin [19] and Selberg [20]. Next we mainly compute C_f. By the multiplicative property of $\lambda_f(n)$ and partial summation, for $\Re s > 1$, we have

$$\begin{aligned}\prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_f^2(p^k)}{p^{ks}}\right) &= \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)}{n^s} = (s-1)\int_1^{\infty} \frac{h(x)}{x^s}dx \\ &= (s-1)\int_1^{\infty} \frac{C_f \log x + O(1)}{x^s}dx \\ &= \frac{C_f}{s-1} + O(1),\end{aligned} \tag{2.3}$$

hence,

$$C_f = \lim_{s\to 1+0} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_f^2(p^k)}{p^{ks}}\right)(s-1).$$

On the other hand,

$$\lim_{s\to 1+0}(s-1)\zeta(s) = 1,$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \Re s > 1.$$

Hence

$$C_f = \lim_{s\to 1+0} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_f^2(p^k)}{p^{ks}}\right)\left(1 - \frac{1}{p^s}\right).$$

□

Lemma 2.2 *(de la Vallée Poussin) Let $\Lambda(n)$ denote the Mangoldt function, as $x \to \infty$, there exists an absolute positive constant $c > 0$ such that*

$$\psi(x) = \sum_{n\leqslant x} \Lambda(n) = x + O(xe^{-c\sqrt{\log x}}).$$

Proof See Ivić [13]. □

Lemma 2.3 *Suppose $L(s, F)$ is the L-function associated with a Hecke holomorphic form F for $SL(3, \mathbb{Z})$, and let $A_F(n, 1)$ be the n-th coefficient of the Dirichlet series for it. For $N \geqslant 2$ and any $\alpha \in R$, there exists an effective constant $c > 0$ such that*

$$\sum_{n\leqslant x} \Lambda(n)A_F(n, 1)e(n\alpha) \ll xe^{-c\sqrt{\log x}},$$

where $\Lambda(n)$ is the von Mangoldt function, and the implied constant depends only on F.

Proof Theorem 1.1 of [6] has given the detailed proof for the Hecke Maass form. The Hecke holomorphic form also has the same result, we shall not give a duplicate proof. □

Lemma 2.4 *Let $\lambda_f(n)$ be the n-th Fourier coefficients of the normalized primitive holomorphic forms f for the full modular group $SL(2,\mathbb{Z})$, $\Lambda(n)$ is the Mangoldt function, then we have*

$$\sum_{n\leqslant x}\lambda_f^2(n)\Lambda(n)=x+O(xe^{-c\sqrt{\log x}}).$$

Proof See Perelli [17]. □

Lemma 2.5 *Suppose $f(z)$ are the normalized primitive holomorphic forms for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$, then*

$$\sum_{y<p\leqslant z}\frac{\lambda_f^2(p)}{p^s}-\frac{1}{p^s}=O\left(\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}}\right),\quad z>y, \tag{2.4}$$

uniformly in $\Re s\geqslant 1$.

Proof Let

$$R(t):=\sum_{y<p\leqslant t}\frac{\lambda_f^2(p)-1}{p}\log p,$$

then $R(t)=0$ when $t\leqslant y$. By partial summation and Lemma 2.3, we have

$$R(t):=\sum_{y<p\leqslant t}\frac{\lambda_f^2(p)-1}{p}\log p=O\left(\frac{\sqrt{\log y}}{e^{c\sqrt{\log y}}}\right).$$

Using partial summation, we have

$$\begin{aligned}
&\sum_{y<p\leqslant z}\frac{\lambda_f^2(p)}{p^s}-\frac{1}{p^s}\\
&\quad=\int_y^z\frac{dR(t)}{t^{s-1}\log t}=\frac{R(z)}{z^{s-1}\log z}+\int_y^z\frac{R(t)}{t^s\log t}\left(s-1+\frac{1}{\log t}\right)dt\\
&\quad\ll\frac{\sqrt{\log y}}{z^{s-1}e^{c\sqrt{\log y}}\log z}+\int_y^z\frac{\sqrt{\log y}(s-1)}{e^{c\sqrt{\log y}}\log t}\frac{dt}{t^s}+\int_y^z\frac{\sqrt{\log y}dt}{t^se^{c\sqrt{\log y}}(\log t)^2}\\
&\quad\leqslant\frac{1}{z^{s-1}e^{c\sqrt{\log y}}\sqrt{\log y}}+\frac{1}{y^{s-1}e^{c\sqrt{\log y}}\sqrt{\log y}}+\frac{2}{e^{c\sqrt{\log y}}\sqrt{\log y}}\ll\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}},
\end{aligned}\tag{2.5}$$

uniformly for $\Re s\geqslant 1$. □

Lemma 2.6 *For $x>1$, we have*

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\leqslant\frac{e^{-\gamma}}{\log x}\left(1+\frac{1}{\log^2 x}\right)$$

and

$$\prod_{p\leqslant x}\frac{p}{p-1}\leqslant e^{\gamma}\log x\left(1+\frac{1}{\log^2 x}\right).$$

where $\gamma=0.57721566...$ is the Euler-Mascheroni constant.

Proof The result was proved by Rosser and Schoenfeld [18]. □

In order to state our results, we introduce a function $j(u)$.

Lemma 2.7 *Let $j(u)$ be strictly increasing and positive for all $u > 0$. $j(u)$ converges to 1 as $u \to \infty$, and also satisfies the following properties:*

$$uj'(u) = j(u) - j(u-1), u > 1;\ j(u) = e^{-\gamma}u, 0 \leqslant u < 1;\ j(u) = 0, u < 0.$$
$$(uj(u))' = 2j(u) - j(u-1),\ j'(u) = \varrho(u), u > 1.$$
$$j(u) = \frac{1}{u}\int_{u-1}^{u} j(t)dt + \frac{1}{u}\int_{0}^{u} j(t)dt = \frac{1}{u}\int_{u-1}^{u} j(t)dt + \frac{1}{u}\int_{1}^{u} j(t)dt + \frac{1}{2e^{\gamma}u}, u \geqslant 1.$$

Proof See Ankeny and Onishi [2]. □

3 Proof of Theorem 1 for small u

Define

$$h(x,y) := \sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \frac{\lambda_f^2(n)}{n},$$

using the Chebyshev-type functional equation of $h(x,y)$, we have

$$h(x,y)\log x = G(x,y) + \sum_{\substack{mp^k\leqslant x\\ P(mp)\leqslant y\\ p\nmid m}} \frac{\lambda_f^2(m)}{m}\frac{\lambda_f^2(p^k)}{p^k}\log p^k, \tag{3.1}$$

where

$$G(x,y) := \int_1^x \frac{h(t,y)}{t}dt = \sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \frac{\lambda_f^2(n)}{n}\log\left(\frac{x}{n}\right).$$

Firstly, we give an asymptotic formula for $h(x,y)\log x$.

Proposition 1 *Suppose $f(z)$ are the normalized primitive holomorphic forms for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$. Let $P(n)$ denote the largest prime divisor of $n > 1$, with $P(1) = 1$, then we have*

$$h(x,y)\log x = G(x,y) + \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \frac{\lambda_f^2(m)}{m} \sum_{p\leqslant \min(x/m,y)} \frac{\lambda_f^2(p)}{p}\log p + O(h(x,y)), 1 < y \leqslant x.$$

Proof We firstly consider the contribution of terms corresponding to $k = 1$ from the sum on the right of (3.1),

$$\begin{aligned}
&\sum_{\substack{mp\leqslant x\\ P(mp)\leqslant y\\ p\nmid m}} \frac{\lambda_f^2(m)}{m}\frac{\lambda_f^2(p)}{p}\log p\\
&= \sum_{\substack{mp\leqslant x\\ P(mp)\leqslant y}} \frac{\lambda_f^2(m)}{m}\frac{\lambda_f^2(p)}{p}\log p - \sum_{\substack{mp\leqslant x\\ P(mp)\leqslant y\\ p\mid m}} \frac{\lambda_f^2(m)}{m}\frac{\lambda_f^2(p)}{p}\log p \qquad (3.2)\\
&= \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \frac{\lambda_f^2(m)}{m} \sum_{p\leqslant \min(x/m,y)} \frac{\lambda_f^2(p)}{p}\log p - \sum_{\substack{lp^{k+1}\leqslant x\\ P(lp)\leqslant y\\ p\nmid l, k\geqslant 1}} \frac{\lambda_f^2(l)}{l}\frac{\lambda_f^2(p^k)}{p^k}\frac{\lambda_f^2(p)}{p}\log p,
\end{aligned}$$

the double sum on the right of (3.2) is at most

$$\begin{aligned}\sum_{\substack{lp^{k+1}\leqslant x\\ P(lp)\leqslant y\\ p\nmid l,k\geqslant 1}} \frac{\lambda_f^2(l)}{l}\frac{\lambda_f^2(p^k)}{p^k}\frac{\lambda_f^2(p)}{p}\log p &\leqslant \sum_{\substack{p^{k+1}\leqslant x\\ p\leqslant y,k\geqslant 1}} h\left(\frac{x}{p^{k+1}},y\right)\frac{\lambda_f^2(p^k)}{p^k}\frac{\lambda_f^2(p)}{p}\log p\\ &\leqslant h(x,y)\sum_{\substack{p^{k+1}\leqslant x\\ p\leqslant y}}\frac{\lambda_f^2(p^k)}{p^k}\frac{\lambda_f^2(p)}{p}\log p.\end{aligned} \tag{3.3}$$

By $|\lambda_f(n)|\leqslant d(n)\ll n^{\epsilon}$, we can easily get

$$\sum_{\substack{p^{k+1}\leqslant x\\ p\leqslant y}}\frac{\lambda_f^2(p^k)}{p^k}\frac{\lambda_f^2(p)}{p}\log p \ll \sum_{p^{k+1}\leqslant x}\frac{1}{p^{(k+1)(1-\epsilon)}}\ll 1.$$

Therefore, the terms corresponding to $k=1$ from the sum on the right of (3.1) contribute

$$\sum_{\substack{mp\leqslant x\\ P(mp)\leqslant y\\ p\nmid m}}\frac{\lambda_f^2(m)}{m}\frac{\lambda_f^2(p)}{p}\log p = \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}}\frac{\lambda_f^2(m)}{m}\sum_{p\leqslant \min(x/m,y)}\frac{\lambda_f^2(p)}{p}\log p + O\left(h(x,y)\right). \tag{3.4}$$

For the remaining terms $k\geqslant 2$, using $|\lambda_f(n)|\leqslant d(n)\ll n^{\epsilon}$, we can get the contribution

$$\sum_{\substack{mp^k\leqslant x\\ P(mp)\leqslant y\\ p\nmid m,k\geqslant 2}}\frac{\lambda_f^2(m)}{m}\frac{\lambda_f^2(p^k)}{p^k}\log p^k \leqslant \sum_{p,k\geqslant 2} h\left(\frac{x}{p^k},y\right)\frac{\lambda_f^2(p^k)}{p^k}\log p^k \ll h(x,y). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$h(x,y)\log x = G(x,y)+\sum_{\substack{m\leqslant x\\ P(m)\leqslant y}}\frac{\lambda_f^2(m)}{m}\sum_{p\leqslant\min(x/m,y)}\frac{\lambda_f^2(p)}{p}\log p + O(h(x,y)).$$

□

Remark if $1\leqslant x\leqslant y$, we take $y=x$ in the above lemma to obtain

$$h(x)\log x-\int_1^x\frac{h(t)}{t}dt-\sum_{m\leqslant x}\frac{\lambda_f^2(m)}{m}\sum_{p\leqslant x/m}\frac{\lambda_f^2(p)}{p}\log p\ll h(x),$$

where

$$h(x):=\sum_{n\leqslant x}\frac{\lambda_f^2(n)}{n}.$$

Define

$$V(y):=\prod_{p\leqslant y}\left(1+\sum_{k=1}^{\infty}\frac{\lambda_f^2(p^k)}{p^k}\right),$$

Obviously, we have

$$h(x,y)\leqslant\sum_{P(n)\leqslant y}\frac{\lambda_f^2(n)}{n}=V(y).$$

Let

$$H_p(s) := \sum_{k=2} \frac{\lambda_f^2(p^k)}{p^{ks}}, H_p := H_p(1),$$

then

$$V(y) = \prod_{p \leqslant y} \left(1 + \frac{\lambda_f^2(p)}{p} + H_p\right).$$

Secondly, we give an asymptotic formula for $V(y)$.

Proposition 2 *For $V(y)$, we have the following result.*

$$V(y) = C_f e^{\gamma} \log y \left\{1 + O_f\left(\frac{1}{\log^2 y}\right)\right\}, \tag{3.6}$$

for all sufficiently large y, $\gamma = 0.57721566...$ is the Euler-Mascheroni constant.

Proof We firstly consider the product

$$\begin{aligned} &\prod_{y<p\leqslant z} \left(1 + \frac{\lambda_f^2(p)}{p^s} + H_p(s)\right)\left(1 - \frac{1}{p^s}\right) \\ &= \exp\left\{\sum_{y<p\leqslant z} \log\left(1 + \frac{\lambda_f^2(p)}{p^s} + H_p(s)\right) + \log\left(1 - \frac{1}{p^s}\right)\right\}. \end{aligned} \tag{3.7}$$

By Taylor expansion of $\log(1 + x)$, we have

$$\begin{aligned} \log\left(1 + \frac{\lambda_f^2(p)}{p^s} + H_p(s)\right) &= \frac{\lambda_f^2(p)}{p^s} + H_p(s) + O\left(\frac{\lambda_f^4(p)}{p^{2s}} + H_p^2(s)\right) \\ &= \frac{\lambda_f^2(p)}{p^s} + O\left(\frac{\lambda_f^4(p)}{p^{2s}} + H_p\right), s \geqslant 1. \end{aligned} \tag{3.8}$$

By $x - \frac{1}{2}x^2 \leqslant \log(1 + x) \leqslant x$, we obtain that for $s \geqslant 1$,

$$0 \geqslant \log\left(1 - \frac{1}{p^s}\right) + \frac{1}{p^s} = -\sum_{k=2}^{\infty} \frac{1}{kp^{ks}} \geqslant -\frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{p^k} = -\frac{1}{2p(p-1)} \geqslant -\frac{1}{p^2}. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned} &\prod_{y<p\leqslant z} \left(1 + \frac{\lambda_f^2(p)}{p^s} + H_p(s)\right)\left(1 - \frac{1}{p^s}\right) \\ &= \exp\left\{\sum_{y<p\leqslant z} \frac{\lambda_f^2(p) - 1}{p^s} + O\left(\sum_{p>y}\left(H_p + \frac{\lambda_f^4(p)}{p^{2s}} + \frac{1}{p^2}\right)\right)\right\}. \end{aligned} \tag{3.10}$$

For the first O sums on the right of (3.10), by $|\lambda_f(n)| \leqslant d(n) \ll n^{\epsilon}$, we have

$$\sum_{p>y} H_p = \sum_{p>y}\sum_{k=2} \frac{\lambda_f^2(p^k)}{p^k} \ll \sum_{p>y}\sum_{k=2} \frac{1}{p^{k(1-\epsilon)}} \ll \frac{1}{y^{1-\epsilon}}. \tag{3.11}$$

For the second O sums on the right of (3.10), by $|\lambda_f(n)| \leqslant d(n) \ll n^{\epsilon}$, and $s \geqslant 1$, we have

$$\sum_{p>y} \frac{\lambda_f^4(p)}{p^{2s}} \ll \sum_{p>y} \frac{1}{p^2} \ll \frac{1}{y}. \tag{3.12}$$

Combining (3.10), (3.11), (3.12) and Lemma 2.5, we obtain

$$\begin{aligned} \prod_{y<p\leqslant z} &\left(1+\frac{\lambda_f^2(p)}{p^s}+H_p(s)\right)\left(1-\frac{1}{p^s}\right) = \exp\left\{O\left(\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}}\right)\right\} \\ &= 1+O\left(\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}}\right). \end{aligned} \tag{3.13}$$

By the definition of $V(y)$, C_f and Lemma 2.6, we get

$$\begin{aligned} C_f &= \prod_{p\leqslant y}\left(1+\frac{\lambda_f^2(p)}{p}+H_p\right)\left(1-\frac{1}{p}\right)\left\{1+O\left(\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}}\right)\right\} \\ &= V(y)\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\left\{1+O\left(\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}}\right)\right\} \\ &= V(y)\frac{e^{-\gamma}}{\log y}\left\{1+O\left(\frac{1}{\log^2 y}\right)\right\}\left\{1+O\left(\frac{1}{e^{c\sqrt{\log y}}\sqrt{\log y}}\right)\right\}. \end{aligned} \tag{3.14}$$

Therefore, we deduce that

$$V(y) = C_f e^{\gamma}\log y\left\{1+O_f\left(\frac{1}{\log^2 y}\right)\right\}, \tag{3.15}$$

for all sufficiently large y. □

Now we are ready to prove Theorem 1. For different ranges of $u := \frac{\log x}{\log y}$, we use different methods to study.

3.1. By Lemma 2.1 and Proposition 2, it follows that for $u \leqslant 1$,

$$\begin{aligned} h(x,y) = h(x) &= C_f\log x\left\{1+O\left(\frac{1}{\log x}\right)\right\} = C_f u\log y\left\{1+O\left(\frac{1}{\log x}\right)\right\} \\ &= V(y)e^{-\gamma}u\left\{1+O\left(\frac{1}{\log^2 y}\right)\right\}\left\{1+O\left(\frac{1}{\log x}\right)\right\}, \end{aligned} \tag{3.16}$$

Hence,

$$h(x,y) = V(y)\left\{e^{-\gamma}u+O\left(\frac{1}{\log y}\right)\right\}, 0<u\leqslant 1. \tag{3.17}$$

3.2. Assume $u > 1$, by Lemma 2.5 and partial summation, the double sums on the right of Proposition 2 is equal to

$$\begin{aligned} &\sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \frac{\lambda_f^2(m)}{m} \sum_{p\leqslant \min(x/m,y)} \frac{\lambda_f^2(p)}{p}\log p \\ &= \sum_{\substack{m\leqslant x/y\\ P(m)\leqslant y}} \frac{\lambda_f^2(m)}{m}\sum_{p\leqslant y}\frac{\lambda_f^2(p)}{p}\log p + \sum_{\substack{x/y<m\leqslant x\\ P(m)\leqslant y}}\frac{\lambda_f^2(m)}{m}\sum_{p\leqslant x/m}\frac{\lambda_f^2(p)}{p}\log p \\ &= G(x,y)-G(x/y,y)+O\left(h(x,y)\frac{\sqrt{\log x}}{e^{c\sqrt{\log x}}}\right). \end{aligned} \tag{3.18}$$

By Proposition 1, we obtain

$$h(x, y) \log x = 2G(x, y) - G(x/y, y) + O\left(V(y) + V(y)\frac{\sqrt{\log x}}{e^{c\sqrt{\log x}}}\right). \tag{3.19}$$

By the definition of $G(x, y)$, we have

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{G(x, y)}{(\log x)^2}\right) =& \frac{h(x, y)\log x - 2G(x, y)}{x(\log x)^3} \\ =& \frac{-G(x/y, y)}{x(\log x)^3} + O\left(\frac{V(y)}{x(\log x)^3} + \frac{V(y)}{x(\log x)^{5/2}e^{c\sqrt{\log x}}}\right).\end{aligned} \tag{3.20}$$

Next we integrate this relation with respect to x from η to κ, we obtain the approximate integral equation

$$\frac{G(\kappa, y)}{(\log \kappa)^2} = \frac{G(\eta, y)}{(\log \omega)^2} - \int_\eta^\kappa \frac{G(x/y, y)}{x(\log x)^3}dx + O\left(\frac{V(y)}{(\log \eta)^2} + \frac{V(y)}{(\log \eta)^{3/2}e^{c\sqrt{\log \eta}}}\right), \tag{3.21}$$

where $y \leqslant \eta \leqslant \kappa$.

By the definition of $G(x, y)$ and (3.17), it is reasonable to estimate $G(\kappa, y)$ as follow:

$$\int_1^\kappa V(y) j\left(\frac{\log t}{\log y}\right)\frac{dt}{t} = V(y)\log y \int_0^{\frac{\log \kappa}{\log y}} j(v)dv. \tag{3.22}$$

Define

$$J(u) = \int_0^u j(v)dv,$$

by Lemma 2.7, we have

$$J(u) = \frac{u^2}{2e^\gamma}, 0 \leqslant u < 1;\quad uj(u) = uJ'(u) = 2J(u) - J(u-1), u > 1.$$

Particularly,

$$\frac{J(\tau)}{\tau^2} = \frac{J(\omega)}{\omega^2} - \int_\omega^\tau \frac{J(v-1)}{v^3}dv. \tag{3.23}$$

Suppose

$$G(\kappa, y) = J(\tau)V(y)\log y + E(\kappa, y), \tau = \frac{\log \kappa}{\log y}. \tag{3.24}$$

Combining (3.21), (3.24) and (3.25), we obtain

$$\frac{E(\kappa, y)}{(\log \kappa)^2} = \frac{E(\eta, y)}{(\log \eta)^2} - \int_\eta^\kappa \frac{E(x/y, y)}{x(\log x)^3}dx + O\left(\frac{V(y)}{(\log \eta)^2} + \frac{V(y)}{(\log \eta)^{3/2}e^{c\sqrt{\log \eta}}}\right), \tag{3.25}$$

where $y \leqslant \eta \leqslant \kappa$.

Next, by means of an inductive argument, we mainly study the remainders $E(., y)$. Suppose $\kappa \leqslant y$, i.e. $\tau = \frac{\log \kappa}{\log y} \leqslant 1$, then by (3.17), we have

$$G(\kappa, y) = V(y)\log y\left\{J(\tau) + O\left(\frac{1}{\log y}\right)\right\},$$

which agree with (3.24) for $\tau \leqslant 1$, where

$$|E(\kappa, y)| \leqslant A_1 V(y), \tau \leqslant 1, \tag{3.26}$$

the constant A_1 is at least as large as the constant in (3.17).

Suppose $y < \kappa \leqslant y^2$, i.e. $1 < \tau \leqslant 2$, in the integral we have $x/y \leqslant \kappa/y \leqslant y$. Let B_1 denote the O-constant in (3.21) and apply (3.21) with $\eta = y$, by (3.22), we have

$$\begin{aligned}\frac{|E(\kappa, y)|}{(\log \kappa)^2} &\leqslant A_1 V(y) \times \left\{ \frac{1}{(\log y)^2} + \int_y^{y^2} \frac{dx}{x(\log x)^3} + \frac{B_1}{A_1} \left(\frac{1}{(\log y)^2} + \frac{1}{(\log y)^{3/2} e^{c\sqrt{\log y}}} \right) \right\} \\ &\leqslant \frac{A_1 V(y)}{(\log y)^2} \left(1 + \frac{3}{8} + \frac{B_1}{A_1} \left(1 + \frac{(\log y)^{1/2}}{e^{c\sqrt{\log y}}} \right) \right) \leqslant \frac{2A_1 V(y)}{(\log y)^2}, \; y < \kappa \leqslant y^2, \end{aligned} \tag{3.27}$$

provided that $A_1 \geqslant 2B_1$.

By recursive method, suppose $\nu = 1, 2, \cdots$, there exists a constant C^* independent of ν such that

$$\left| \frac{E(\kappa, y)}{(\log \kappa)^2} \right| \leqslant C^* \eta_0(\nu) \frac{V(y)}{(\log y)^2}, \quad y^\nu < \kappa \leqslant y^{\nu+1}, \quad \eta_0(\nu) = \nu \log(\nu + 1). \tag{3.28}$$

We can prove that, for all $\nu \geqslant 1$,

$$|E(\kappa, y)| \leqslant C^* \tau^3 \log(\tau + 1) V(y), \; \tau > 1.$$

By (3.24), we then obtain

$$G(\kappa, y) = V(y) \log y \left\{ J(\tau) + O\left(\frac{\tau^3 \log(\tau + 1)}{\log y} \right) \right\}, \tau > 1.$$

Combining with (3.19), once with $\kappa = x$ and once with $\kappa = x/y$, we deduce

$$\begin{aligned} &h(x, y) \log x \\ &= V(y) \log y \left\{ 2J(u) - J(u-1)) + O\left(\frac{u^3 \log(u+1) + 1}{\log y} + \frac{1}{\sqrt{\log y} e^{c\sqrt{\log y}}} \right) \right\}, u > 1. \end{aligned} \tag{3.29}$$

Dividing by $\log x = u \log y$, we get

$$\begin{aligned} h(x, y) =& V(y) \left\{ j(u) + O\left(\frac{u^2 \log(u+1)}{\log y} + \frac{1}{\log x} + \frac{\sqrt{\log y}}{e^{c\sqrt{\log y}} \log x} \right) \right\} \\ =& \; V(y) \left\{ j(u) + O\left(\frac{u^2 \log(u+1)}{\log y} \right) \right\}, u > 1. \end{aligned} \tag{3.30}$$

4 Proof of Theorem 1 for large *u*

By Proposition 1, we have

$$h(x, y) \log x = \int_1^x h(t, y) \frac{dt}{t} + \sum_{p \leqslant y} h\left(\frac{x}{p}, y \right) \frac{\lambda_f^2(p)}{p} \log p + O(h(x, y)). \tag{4.1}$$

By Lemma 2.1, there exists a constant K and x_0 such that

$$\left| h(x) - C_f \log x \right| \leqslant K, \quad x \geqslant x_0. \tag{4.2}$$

For large u, we need two Propositions.

Proposition 3 *Let $\lambda_f(n)$ be the n-th Fourier coefficients of the normalized primitive holomorphic forms f for the full modular group $SL(2, \mathbb{Z})$, then we have*

$$\int_1^y h(t, y) \frac{dt}{t} = \frac{1}{2e^\gamma} V(y) \log y \left\{ 1 + O\left(\frac{1}{\log y} \right) \right\},$$

for $y \geqslant y_1$, y_1 is the least value of y satisfying

$$\log y \geqslant \log^2 x_0,$$

where x_0 satisfies (4.2).

Proof For $1 \leqslant t \leqslant x_0 \leqslant y$, we have $h(t, y) = h(t) \leqslant V(x_0)$. By Lemma 2.1,

$$\int_1^y h(t)\frac{dt}{t} = \int_1^{x_0} h(t)\frac{dt}{t} + \int_{x_0}^y (h(t) - C_f \log t)\frac{dt}{t} + \frac{C_f}{2}(\log^2 y - \log^2 x_0),$$

hence,

$$\left|\int_1^y h(t)\frac{dt}{t} - \frac{C_f}{2}\log^2 y\right| \leqslant V(x_0)\log x_0 + K\int_{x_0}^y \frac{dt}{t} + \frac{C_f}{2}\log^2 x_0. \tag{4.3}$$

By Proposition 2 and $\log y \geqslant \log^2 x_0$, we have

$$\frac{V(x_0)\log x_0}{V(y)\log y} \ll \left(\frac{\log x_0}{\log y}\right)^2 \leqslant \frac{1}{\log y}.$$

The integral on the right of (4.3) is obviously $O(\log y)$, and the third expression on the right of (4.3) is the same order by $\log y \geqslant \log^2 x_0$. Finally by Proposition 2, we get

$$\frac{C_f}{2}\log^2 y = \frac{1}{2e^\gamma}V(y)\log y\left\{1 + O\left(\frac{1}{\log y}\right)\right\}.$$

This proves the Proposition 3. □

Proposition 4 *Let θ be a fixed number such that $1/2 \leqslant \theta \leqslant 1$, then*

$$\sum_{p\leqslant y^\theta} j\left(u - \frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p = \log y\int_0^\theta j(u-v)dv + O(1).$$

Proof Let $g(x)$ denote the sum $\sum_{p\leqslant x}\frac{\lambda_f^2(p)}{p}\log p$. By Lemma 2.4 and partial summation, we have

$$g(x) - \log x = r(x),\ |r(x)| \leqslant A_2,\ x \geqslant 2,$$

for a suitable constant A_2. By $j(u) < 1$ and $|r(x)| \leqslant A_2$, we get

$$\begin{aligned}
&\sum_{p\leqslant y^\theta} j\left(u - \frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p\\
&= \int_1^{y^\theta} j\left(u - \frac{\log t}{\log y}\right)d\{g(t) - \log t\} + \int_1^{y^\theta} j\left(u - \frac{\log t}{\log y}\right)\frac{dt}{t}\\
&= \int_1^{y^\theta} j\left(u - \frac{\log t}{\log y}\right)\frac{dt}{t} + \int_1^{y^\theta} j\left(u - \frac{\log t}{\log y}\right)dr(t)\\
&= \log y\int_0^\theta j(u-v)dv + j(u-\theta)r(y^\theta) + \frac{1}{\log y}\int_1^{y^\theta} r(t)j'\left(u - \frac{\log t}{\log y}\right)\frac{dt}{t}\\
&= \log y\int_0^\theta j(u-v)dv + O(1).
\end{aligned}$$

□

We have studied u in some bounded range, suppose $y \geqslant y_0$ and $1 \leqslant u \leqslant 3$, then

$$h(x, y) = V(y)\{j(u) + N(y, u)\}, \tag{4.4}$$

where

$$N(y, u) = \frac{u^2\log(u+1)}{\log y} \ll \frac{1}{\log y}.$$

If $u > 3$, then we assume

$$h(x,y) = V(y)j(u)\left\{1+\frac{N(y,u)}{j(u)}\right\} := V(y)j(u)\{1+\Theta(y,u)\}, \tag{4.5}$$

where $\Theta(y,u) \leqslant 2N(y,u)$.

Let

$$\Theta^*(y,u) \ll \sup_{1\leqslant u'\leqslant u} |\Theta(y,u')|,$$

then we have

$$\Theta^*(y,u) \ll \frac{1}{\log y}, 1 < u \leqslant 3.$$

For $u \geqslant 3$ and all sufficiently large y, we will prove that

$$\Theta^*(y,u) \ll \frac{\log u}{\log y}. \tag{4.6}$$

We first consider the integral in (4.1),

$$\int_1^x h_0(t,y)\frac{dt}{t} = \int_1^y h(t)\frac{dt}{t} + \int_y^x h(t,y)\frac{dt}{t}. \tag{4.7}$$

By Proposition 3, we have

$$\int_1^y h(t)\frac{dt}{t} = \frac{1}{2e^\gamma}V(y)\log y\left\{1+O\left(\frac{1}{\log y}\right)\right\}. \tag{4.8}$$

By (4.4), the portion of the second integral of (4.7) that corresponds to $y \leqslant t \leqslant y^3$ contributes,

$$\begin{aligned} V(y)\int_y^{y^3} j\left(\frac{\log t}{\log y}\right)\frac{dt}{t} + O\left(V(y)\frac{\log y^3/y}{\log y}\right) \\ = \ V(y)\log y\int_1^3 j(v)dv + O(V(y)). \end{aligned} \tag{4.9}$$

By (4.5), the remaining portion of (4.7) contributes,

$$\begin{aligned} V(y)\int_{y^3}^x j\left(\frac{\log t}{\log y}\right)\left\{1+\Theta\left(y,\frac{\log t}{\log y}\right)\right\}\frac{dt}{t} \\ = \ V(y)\log y\int_3^u j(v)\{1+\Theta(y,v)\}dv. \end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we have

$$\begin{aligned} \int_1^x h(t,y)\frac{dt}{t} =& \frac{1}{2e^\gamma}V(y)\log y + V(y)\log y\left\{\int_1^u j(v)dv + \int_3^u j(v)\Theta(y,v)dv\right\} \\ &+ \ O(V(y)), \quad y \geqslant y_0, \quad u \geqslant 3. \end{aligned} \tag{4.11}$$

Next we turn to the second expression on the right of (4.1). By (4.5) and Proposition 4, for $u > 3$, we have

$$\begin{aligned} &\sum_{p\leqslant y} h\left(\frac{x}{p},y\right)\frac{\lambda_f^2(p)}{p}\log p \\ &= \ V(y)\sum_{p\leqslant y} j\left(u-\frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p\left\{1+\Theta\left(y,u-\frac{\log p}{\log y}\right)\right\} \\ &= \ V(y)\log y\int_0^1 j(u-t)dt + O(V(y)) \\ &\quad + \ V(y)\sum_{p\leqslant y} j\left(u-\frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p\,\Theta\left(y,u-\frac{\log p}{\log y}\right). \end{aligned} \tag{4.12}$$

Combining (4.1), (4.11) and (4.12), we get

$$
\begin{aligned}
1+\Theta(y,u) =& \frac{1}{2e^{\gamma}uj(u)}+\frac{1}{uj(u)}\int_1^u j(v)dv+\frac{1}{uj(u)}\int_0^1 j(u-v)dv \\
&+ \frac{1}{uj(u)\log y}\sum_{p\leqslant y} j\left(u-\frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p\Theta\left(y,u-\frac{\log p}{\log y}\right) \\
&+ \frac{1}{uj(u)}\int_3^u j(v)\Theta(y,v)dv+O\left(\frac{1}{uj(u)\log y}\right), u>3.
\end{aligned}
\tag{4.13}
$$

By Lemma 2.7, we have

$$
\begin{aligned}
\Theta(y,u) =& \frac{1}{uj(u)\log y}\sum_{p\leqslant y} j\left(u-\frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p\Theta\left(y,u-\frac{\log p}{\log y}\right) \\
&+ \frac{1}{uj(u)}\int_3^u j(v)\Theta(y,v)dv+O\left(\frac{1}{uj(u)\log y}\right), u>3.
\end{aligned}
\tag{4.14}
$$

By Proposition 4, the terms in the sum on the right of (4.14) that correspond to $p \leqslant y^{1/2}$ contribute at most

$$
\Theta^*(y,u)\sum_{p\leqslant y^{1/2}} j\left(u-\frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p=\Theta^*(y,u)\left(\log y\int_0^{1/2} j(u-v)dv+O(1)\right), \tag{4.15}
$$

and the remaining terms contribute at most

$$
\begin{aligned}
&\Theta^*(y,u-1/2)\sum_{y^{1/2}<p\leqslant y} j\left(u-\frac{\log p}{\log y}\right)\frac{\lambda_f^2(p)}{p}\log p \\
=& \Theta^*(y,u-1/2)\left(\log y\int_{1/2}^1 j(u-v)dv+O(1)\right).
\end{aligned}
\tag{4.16}
$$

The integral on the right of (4.14) is at most

$$
\Theta^*(y,u-1)\int_1^{u-1} j(v)dv+\Theta^*(y,u)\int_{u-1}^u j(v)dv. \tag{4.17}
$$

Combining (4.14), (4.15), (4.16) and (4.17),

$$
\begin{aligned}
\Theta(y,u) \leqslant& \Theta^*(y,u)\left\{\alpha(u)+\frac{1}{uj(u)}\int_{u-1}^u j(v)dv\right\} \\
&+ \Theta^*(y,u-1/2)\left\{\beta(u)+\frac{1}{uj(u)}\int_1^{u-1} j(v)dv\right\}+O\left(\frac{1+\Theta^*(y,u)}{uj(u)\log y}\right),
\end{aligned}
\tag{4.18}
$$

where

$$
\alpha(u)=\frac{1}{uj(u)}\int_0^{1/2} j(u-v)dv,\quad \beta(u)=\frac{1}{uj(u)}\int_{1/2}^1 j(u-v)dv.
$$

By Lemma 2.7, we have

$$
\alpha(u),\beta(u)\leqslant 1/2u,\quad u>1.
$$

Define

$$
\alpha_1(u)=\alpha(u)+\frac{1}{uj(u)}\int_{u-1}^u j(v)dv,
$$

by Lemma 2.7, we have

$$\beta(u)+\frac{1}{uj(u)}\int_1^{u-1} j(v)dv = 1-\alpha_1(u)-\frac{1}{2e^{\gamma}uj(u)} \leqslant 1-\alpha_1(u),$$

hence,

$$|\Theta(y,u)| \leqslant \Theta^*(y,u)\alpha_1(u)+\Theta^*(y,u-1/2)(1-\alpha_1(u))+O\left(\frac{1+\Theta^*(y,u)}{uj(u)\log y}\right), \quad u\geqslant 3.$$

Next we prove that

$$|\Theta(y,u)| \leqslant \frac{1}{2}\left(\Theta^*(y,u)+\Theta^*(y,u-1/2)\right)+O\left(\frac{1+\Theta^*(y,u)}{uj(u)\log y}\right) \tag{4.19}$$

uniformly for $u>3$ and if y is sufficiently large.

For the monotonicity of Θ^*, and

$$\alpha_1(u)=\alpha(u)+\frac{1}{uj(u)}\int_{u-1}^{u} j(v)dv \leqslant \frac{1}{2u}+\frac{1}{u} \leqslant 1/2, \ u\geqslant 3,$$

we deduce

$$\begin{aligned}&\frac{1}{2}\left(\Theta^*(y,u)+\Theta^*(y,u-1/2)\right)-\left(\Theta^*(y,u)\alpha_1(u)+\Theta^*(y,u-1/2)(1-\alpha_1(u))\right)\\ &=\left(\frac{1}{2}-\alpha_1(u)\right)(\Theta^*(y,u)-\Theta^*(y,u-1/2))\geqslant 0.\end{aligned}$$

In order to prove that (4.6) holds for $u\geqslant 3$, Suppose $u-1/2\leqslant u'\leqslant u$. Let A_3 denote the O-constant in (4.19), by the monotonicity of Θ^*,

$$\begin{aligned}|\Theta(y,u')| &\leqslant \frac{1}{2}\left(\Theta^*(y,u')+\Theta^*(y,u'-1/2)\right)+A_3\left(\frac{1+\Theta^*(y,u')}{u'j(u')\log y}\right)\\ &\leqslant \frac{1}{2}\left(\Theta^*(y,u)+\Theta^*(y,u-1/2)\right)+\frac{2A_3}{3}\left(\frac{1+\Theta^*(y,u)}{uj(u)\log y}\right).\end{aligned} \tag{4.20}$$

For $3\leqslant u'\leqslant u-1/2$,

$$|\Theta(y,u')| \leqslant \Theta^*(y,u-1/2) \leqslant \frac{1}{2}\left(\Theta^*(y,u)+\Theta^*(y,u-1/2)\right).$$

By the definition of $\Theta^*(y,u)$, and taking the supremum on the right of (4.19), we have uniformly for $u\geqslant 3$,

$$|\Theta^*(y,u)| \leqslant \frac{1}{2}\left(\Theta^*(y,u)+\Theta^*(y,u-1/2)\right)+\frac{2A}{3}\left(\frac{1+\Theta^*(y,u)}{uj(u)\log y}\right). \tag{4.21}$$

After rearranging terms, we get the inequality

$$\Theta^*(y,u) \leqslant \Theta^*(y,u-1/2)+\frac{4A_3}{3}\left(\frac{1+\Theta^*(y,u)}{uj(u)\log y}\right). \tag{4.22}$$

By the iterative method, we obtain

$$\Theta^*(y,u) \leqslant \Theta^*(y,u_0)+\frac{4A_3}{3}\left(\frac{(1+\Theta^*(y,u))\log u}{\log y}\right), \tag{4.23}$$

where $\frac{5}{2}\leqslant u_0\leqslant 3$.

By $\Theta^*(y,u)\ll \frac{1}{\log y}$, $1<u\leqslant 3$, we have

$$\Theta^*(y,u_0) \leqslant \frac{A_4}{\log y}, \tag{4.24}$$

where A_4 is an appropriate constant, and thus

$$\Theta^*(y,u) \leqslant A^*\left((1+\Theta^*(y,u))\frac{\log u}{\log y}\right), \quad u\geqslant 3, \tag{4.25}$$

where $A^*=\max(4A_3/3, A_4)$.

5 Proof of Theorem 2

Define

$$H(x,y) := \sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \lambda_f^2(n).$$

Next we mainly study the relation between $H(x, y)$ and $h(x, y)$.

Proposition 5 *Let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$. Let $P(n)$ denote the largest prime divisor of $n > 1$, with $P(1) = 1$, then we have*

$$\sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \lambda_f^2(n) \leqslant B_2 x \frac{h(x,y)}{\log ex}, \quad x \geqslant 1,$$

where B_2 is a suitable constant.

Proof We have the equation

$$H(x,y)\log x = \sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \lambda_f^2(n) \log \frac{x}{n} + \sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \lambda_f^2(n) \log n = T + S. \tag{5.1}$$

By $\log x \leqslant x - 1$ for all $x > 0$, we have

$$0 \leqslant T \leqslant x h(x,y) - H(x,y).$$

By the multiplicative property of $\lambda_f(n)$, the obvious inequality $M(x, y) \leqslant x m(x, y)$, Merten's prime number estimate, and

$$\sum_{\substack{p,k \geqslant 2 \\ p \leqslant y}} \frac{\lambda_f^2(p^k) \log p^k}{p^k} \leqslant b,$$

we have

$$\begin{aligned} S &= \sum_{\substack{mp^k \leqslant x \\ p \nmid m \\ P(mp) \leqslant y}} \lambda_f^2(m) \lambda_f^2(p^k) \log p^k \\ &\leqslant \sum_{\substack{mp \leqslant x \\ P(mp) \leqslant y}} \lambda_f^2(m) \lambda_f^2(p) \log p + \sum_{\substack{mp^k \leqslant x \\ k \geqslant 2 \\ P(mp) \leqslant y}} \lambda_f^2(m) \lambda_f^2(p^k) \log p^k \\ &\leqslant \sum_{\substack{m \leqslant x \\ P(m) \leqslant y}} \lambda_f^2(m) \sum_{p \leqslant x/m} \lambda_f^2(p) \log p + \sum_{\substack{p,k \geqslant 2 \\ p \leqslant y}} H\left(\frac{x}{p^k}, y\right) \lambda_f^2(p^k) \log p^k \\ &\leqslant axh(x,y) + \sum_{\substack{p,k \geqslant 2 \\ p \leqslant y}} \frac{x}{p^k} h\left(\frac{x}{p^k}, y\right) \lambda_f^2(p^k) \log p^k \leqslant axh(x,y) + bxh(x,y). \end{aligned} \tag{5.2}$$

Therefore,

$$H(x,y) \leqslant B_2 x \frac{h(x,y)}{\log ex}, \quad x \geqslant 1, B_2 = 1 + a + b.$$

□

Proposition 6 *Let $\lambda_f(n)$ be the Fourier coefficients of $f(z)$. Let $P(n)$ denote the largest prime divisor of $n > 1$, with $P(1) = 1$, then we have*

$$H(x, y)\log x = \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \lambda_f^2(m) \sum_{p\leqslant \min(x/m, y)} \lambda_f^2(p)\log p + O(xh(x, y)\log^{-1} x), \quad x, y \geqslant 2.$$

In particular, when $y \geqslant x \geqslant 2$, we have

$$H(x)\log x = \sum_{m\leqslant x} \lambda_f^2(m) \sum_{p\leqslant x/m} \lambda_f^2(p)\log p + O(xh(x, y)\log^{-1} x), \quad x, y \geqslant 2.$$

Proof We begin from the identity (5.1), suppose $2 \leqslant y \leqslant x$, we have

$$S = \sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \lambda_f^2(n)\log n = \sum_{\substack{mp^k\leqslant x\\ p\nmid m\\ P(mp)\leqslant y}} \lambda_f^2(m)\lambda_f^2(p^k)\log p^k,$$

the main contribution to the sums S comes from the terms corresponding to $k = 1$,

$$\begin{aligned} \sum_{\substack{mp\leqslant x\\ p\nmid m\\ P(mp)\leqslant y}} \lambda_f^2(m)\lambda_f^2(p)\log p &= \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \lambda_f^2(m) \sum_{\substack{p\leqslant \min(x/m, y)\\ p\nmid m}} \lambda_f^2(p)\log p \\ &= \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \lambda_f^2(m) \sum_{p\leqslant \min(x/m, y)} \lambda_f^2(p)\log p - \sum_{\substack{mp\leqslant x\\ p\mid m\\ P(mp)\leqslant y}} \lambda_f^2(m)\lambda_f^2(p)\log p. \end{aligned} \tag{5.3}$$

This gives

$$\begin{aligned} &\Big|H(x, y)\log x - \sum_{\substack{m\leqslant x\\ P(m)\leqslant y}} \lambda_f^2(m) \sum_{p\leqslant \min(x/m, y)} \lambda_f^2(p)\log p\Big| \\ &\leqslant T + \sum_{\substack{mp\leqslant x\\ p\mid m\\ P(mp)\leqslant y}} \lambda_f^2(m)\lambda_f^2(p)\log p + \sum_{\substack{mp^k\leqslant x\\ p,k\geqslant 2\\ P(mp)\leqslant y}} \lambda_f^2(m)\lambda_f^2(p^k)\log p^k \\ &= T + \sum_{\substack{lp^{k+1}\leqslant x\\ p\nmid l\\ P(lp)\leqslant y}} \lambda_f^2(l)\lambda_f^2(p^k)\lambda_f^2(p)\log p + \sum_{\substack{mp^k\leqslant x\\ p,k\geqslant 2\\ P(mp)\leqslant y}} \lambda_f^2(m)\lambda_f^2(p^k)\log p^k \\ &\leqslant T + \sum_{\substack{p^{k+1}\leqslant x\\ p\leqslant y}} H\left(\frac{x}{p^{k+1}}, y\right)\lambda_f^2(p^k)\lambda_f^2(p)\log p + \sum_{\substack{p,k\geqslant 2\\ p\leqslant y}} H\left(\frac{x}{p^k}, y\right)\lambda_f^2(p^k)\log p^k. \end{aligned} \tag{5.4}$$

Next we estimate T, by Proposition 5, then

$$\begin{aligned} T &= \sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} \lambda_f^2(n)\log\frac{x}{n} = \int_1^x H(t, y)\frac{dt}{t} \ll \int_1^x \frac{h(t, y)}{\log(et)} \\ &\leqslant h(x, y)\int_1^x \frac{dt}{\log(et)} \ll x\frac{h(x, y)}{\log x}. \end{aligned} \tag{5.5}$$

For the first sum of (5.4), we consider the cases $k = 1$ and $k \geqslant 2$. When $k = 1$, by Proposition 5, and $|\lambda_f(n)| \leqslant d(n)$,

$$\begin{aligned}\sum_{\substack{p^2\leqslant x\\ p\leqslant y}} H\left(\frac{x}{p^2}, y\right)\lambda_f^2(p)\lambda_f^2(p)\log p &\leqslant xB_2\sum_{p\leqslant\sqrt{x}}\frac{h(x/p^2, y)}{\log(ex/p^2)}\frac{\lambda_f^2(p)\lambda_f^2(p)\log p}{p^2}\\ &\ll \frac{xh(x, y)}{\log(ex^{1/3})}\sum_{p\leqslant x^{1/3}}\frac{\lambda_f^2(p)\lambda_f^2(p)\log p}{p^2} + xh(x, y)\sum_{p>x^{1/3}}\frac{\lambda_f^2(p)\lambda_f^2(p)\log p}{p^2}\\ &\ll \frac{xh(x, y)}{\log x}.\end{aligned} \tag{5.6}$$

When $k \geqslant 2$, by Proposition 5,

$$\begin{aligned}&\sum_{\substack{p^{k+1}\leqslant x\\ p,k\geqslant 2\\ p\leqslant y}} H\left(\frac{x}{p^{k+1}}, y\right)\lambda_f^2(p^k)\lambda_f^2(p)\log p\\ &\quad\leqslant \frac{xh(x, y)}{\log(ex^{1/2})}\sum_{\substack{p^{k+1}\leqslant x\\ p,k\geqslant 2}}\frac{\lambda_f^2(p^k)\lambda_f^2(p)\log p}{p^{k+1}} + xh(x, y)\sum_{\substack{\sqrt{x}\leqslant p^{k+1}\leqslant x\\ p,k\geqslant 2}}\frac{\lambda_f^2(p^k)\lambda_f^2(p)\log p}{p^{k+1}\log(ex/p^{k+1})}\\ &\ll \frac{xh(x, y)}{\log x}.\end{aligned} \tag{5.7}$$

For the second sum of (5.4), using the same method, we also have

$$\sum_{\substack{p,k\geqslant 2\\ p\leqslant y}} H\left(\frac{x}{p^k}, y\right)\lambda_f^2(p^k)\log p^k \ll \frac{xh(x, y)}{\log x}. \tag{5.8}$$

□

Now we begin to deduce Theorem 2, firstly, suppose $u \geqslant 2$. By Proposition 6, we have

$$\begin{aligned}H(x, y)\log x = &\sum_{\substack{m\leqslant x/y\\ P(m)\leqslant y}}\lambda_f^2(m)\sum_{p\leqslant y}\lambda_f^2(p)\log p\\ &+\sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}}\lambda_f^2(m)\sum_{p\leqslant x/m}\lambda_f^2(p)\log p + O(xh(x, y)\log^{-1} x).\end{aligned} \tag{5.9}$$

Consider the first sum on the right of (5.9). By Lemma 2.1, we have

$$\begin{aligned}\sum_{\substack{m\leqslant x/y\\ P(m)\leqslant y}}\lambda_f^2(m)\sum_{p\leqslant y}\lambda_f^2(p)\log p &= \sum_{\substack{m\leqslant x/y\\ P(m)\leqslant y}}\lambda_f^2(m)\{y + O(ye^{-c\sqrt{\log y}})\}\\ &= yH\left(\frac{x}{y}, y\right)\{1 + O(e^{-c\sqrt{\log y}})\}.\end{aligned} \tag{5.10}$$

By Proposition 5, and $h(x, y) \leqslant V(y)$, we have

$$yH\left(\frac{x}{y}, y\right) \ll x\frac{h(x/y, y)}{\log y} \ll x\frac{V(y)}{\log y}, \quad u \geqslant 2.$$

Consider the second sum on the right of (5.9). By Lemma 2.1, we have

$$\begin{aligned}
&\sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}} \lambda_f^2(m) \sum_{p\leqslant x/m} \lambda_f^2(p)\log p \\
&= \sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}} \lambda_f^2(m) \left\{\frac{x}{m} + O\left(\frac{x}{m} e^{-c\sqrt{\log \frac{x}{m}}}\right)\right\} \\
&= x\{h(x/2, y) - h(x/y, y)\} + O\left(\sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}} \lambda_f^2(m)\frac{x}{m} e^{-c\sqrt{\log \frac{x}{m}}}\right).
\end{aligned} \tag{5.11}$$

To estimate the error term of (5.11), by Proposition 5, we have

$$\begin{aligned}
&\sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}} \lambda_f^2(m) \int_{3/2}^{x/m} \frac{dt}{e^{c\sqrt{\log t}}} \\
&\ll \int_2^y H\left(\frac{x}{t}, y\right)\frac{dt}{e^{c\sqrt{\log t}}} \leqslant x\int_2^y \frac{m(x/t, y)}{t\log(x/t)}\frac{dt}{e^{c\sqrt{\log t}}} \\
&\ll x\frac{V(y)}{\log(x/y)}\int_2^y \frac{dt}{te^{c\sqrt{\log t}}} \ll x\frac{V(y)}{\log y}.
\end{aligned} \tag{5.12}$$

The main term on the right side of (5.11), by Theorem 1, we have

$$\begin{aligned}
&V(y)x\left(j\left(u - \frac{\log 2}{\log y}\right) - j(u-1)\right) + O\left(V(y)x\frac{\log(u+1)}{\log y}\right) \\
&= V(y)x(j(u) - j(u-1)) + O\left(V(y)x\left(j(u) - j\left(u - \frac{\log 2}{\log y}\right) + \frac{\log(u+1)}{\log y}\right)\right) \\
&= V(y)x(j(u) - j(u-1)) + O\left(V(y)x\frac{\log(u+1)}{\log y}\right),
\end{aligned} \tag{5.13}$$

here we use the bound

$$j(u) - j\left(u - \frac{\log 2}{\log y}\right) = \int_{u-\frac{\log 2}{\log y}}^{u} j'(t)dt \ll \frac{1}{\log y}.$$

Thus,

$$H(x, y)\log x = V(y)x(j(u) - j(u-1)) + O\left(V(y)x\frac{\log(u+1)}{\log y}\right), \quad u \geqslant 2,$$

after division $\log x$, by Lemma 2.7, we have

$$H(x, y) = \frac{V(y)x}{\log y}\left\{j'(u) + O\left(\frac{\log(u+1)}{u\log y}\right)\right\}, \quad u \geqslant 2.$$

$$\begin{aligned}
H(x, y)\log x &= \sum_{m\leqslant x/y} \lambda_f^2(m) \sum_{p\leqslant y} \lambda_f^2(p)\log p + \sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}} \lambda_f^2(m) \sum_{p\leqslant x/m} \lambda_f^2(p)\log p \\
&\quad + O(xh(x, y)\log^{-1} x) \\
&= yH\left(\frac{x}{y}\right)\left\{1 + O(e^{-c\sqrt{\log y}})\right\} + x\{h(x/2, y) - h(x/y, y)\} \\
&\quad + O\left(x\frac{h(x, y)}{\log x}\right) + O\left(\sum_{\substack{x/y<m\leqslant x/2\\ P(m)\leqslant y}} \lambda_f^2(m)\frac{x}{m} e^{-c\sqrt{\log \frac{x}{m}}}\right).
\end{aligned} \tag{5.14}$$

By Lemma 2.1 and Proposition 2, suppose $1 \leqslant u < 2$, i.e $1 < x/y \leqslant y$, we have

$$h\left(\frac{x}{y}, y\right) = h\left(\frac{x}{y}\right) = C_f \log\left(\frac{x}{y}\right) + O(1) = V(y)\left\{j(u-1) + O\left(\frac{1}{\log y}\right)\right\},$$

here, we use $0 \leqslant u - 1 < 1$ and $j(u) = e^{-\gamma} u$, $u < 1$. Similarly, we have

$$H(x, y)\log x = xV(y)\left\{uj'(u) + \frac{\log(u+1)}{\log y} + \frac{1}{\log y}\right\} + yH\left(\frac{x}{y}\right)\left\{1 + O(e^{-c\sqrt{\log y}})\right\}, \tag{5.15}$$

By lemma 2.1, we obtain

$$yH\left(\frac{x}{y}\right) \ll x.$$

Thus, (5.15) reduces to

$$H(x, y)\log x = xV(y)\left\{uj'(u) + \frac{\log(u+1)}{\log y} + \frac{1}{\log y}\right\}.$$

After division by $\log x$ and $1 \leqslant u < 2$, we get

$$H(x, y) = \frac{xV(y)}{\log y}\left\{j'(u) + \frac{\log(u+1)}{u \log y}\right\}.$$

Acknowledgements The author is very grateful to the reviewers for many valuable suggestions and comments. The author is supported by NSF(No.ZR2021QA060) of Shandong Province.

References

1. K. Alladi, Asymptotic estimates of sums involving the Moebius function. II, *Trans. Amer. Math. Soc.*, **272** (1982), 87–105.
2. N. C. Ankeny and H. Onishi, The general sieve, *Acta Arith.*, **10** (1964), 31–62.
3. N. G. de Bruijn, On the number of positive integers $\leqslant x$ and free of prime factors $>y$, *Indag. Math.*, **13** (1951), 50–60.
4. P. Deligne, *La conjecture de Weil,*. Inst. Hautes Etudes Sci. Pul. Math.,**43** (1974), 29–39.
5. K. Dickman, On the frequency of numbers containing primes of a certain relative magnitude, *Ark. Mat. Aston. Fys.*, **22** (1930), 1–14.
6. F. Hou, G. S. Lü and Y. J. Jiang, On exponential sums involvinf coefficents of L-functions for $SL(3, \mathbb{Z})$ over primes. *Quart. J. Math.*, **67** (2016), 285–301.
7. A. Hildebrand, On the numbers of positive integers $\leqslant x$ and free of prime factors $>y$, *J. Number Theory.*, **22** (1986), 289–307.
8. A. Hildebrand and G. Tenenbaum, Integers without large prime factors, *J. Theor. Nombrés Bordeaux.*, **5** (1933), 411–484.
9. A. Hildebrand and G. Tenenbaum, On integers free of large prime factors, *Tans. Amer. Math. Soc.*, **296** (1986), 265–290.
10. A. Ivić, On squarefree numbers with restricted prime factors, *Studia. Math. Hungar.*, **20** (1985), 189–192.
11. A. Ivić and G. Tenenbaum, Local densities over integers free of large prime factors, *Quart. J. Math. Oxford Ser.*, **37(2)** (1986), 401–417.
12. H. Iwanice and E. Kowalski, Analytic number theory.
13. A. Ivić, The Riemann Zeta-function theory and applications.
14. Karachuba, Basic analytic number theory.
15. P. Moree, Psixyology and Diophantine equations, *thesis, Leiden University*, (1994).
16. K. K. Norton, Numbers with small prime factors, and least kth power nonresidue, *Mem. Amer. Math. Soc.*, **106** (1971).
17. A. Perelli, On the prime number theorem for the coefficients of certain modular forms, *Banach Center Publications.*, **17** (1985).
18. J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.*, **6** (1962), 64–94.
19. R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. I. The zeros of the function $\sum_{n=1}^{\infty} \tau(n)/n^s$ on the line $Res = 13/2$. II. The order of the Fourier coefficients of integral modular forms, *Proc. Cambridge Philos. Soc.*, **35** (1939), 351–372.
20. A. Selberg, Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, *Arch. Math. Naturvid.*, **43** (1940), 47–50.
21. J. M. Song, Sums of nonnegative multiplicative functions over integers without large prime factors I, *Acta Arith.*, **2(4)** (2001), 329–351.
22. T. Z. Xuan, The average order of $d_k(n)$ over integers free of large prime factors, *Acta Arith.*, **55** (1990), 249–260.
23. T. Z. Xuan, The average order of divisor functions over integers without large prime factors, *Chinese Ann. Math. Ser. A.*, **12** (1991), 28–33.