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ORIGINAL RESEARCH

On the complex twisted Laplacian on $\mathbb{C}^n$ and Poisson transform for the Heisenberg group

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Abstract We compute the point spectrum of the complex twisted Laplacian. We also refine a Theorem proved by Thangavelu.

Keywords Heisenberg group · Sub-Laplacian · Twisted Laplacian · global hypoellipticity · Poisson transform for the Heisenberg group

1 Introduction

The Heisenberg group $\mathbb{H}^n$ is simply $\mathbb{C}^n \times \mathbb{R}$ endowed with the group law

$$(z,t)(z',t') = (z+z', t+t' + \frac{1}{2}\,\mathrm{Im} < z, \overline{z'} >)$$

where $< z, \overline{z'} >= \sum_{j=1}^{n} z_j \overline{z'_j}$.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. Then a basis for $\mathfrak{h}$ is given by the following vector fields

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \quad and \quad T = \frac{\partial}{\partial t},$$

where $j = 1 \ldots, n$. The sub-Laplacian, denoted by $\mathcal{L}$, is defined by

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

this second order differential operator plays the role of Laplacian for $\mathbb{H}^n$, which is hypoelliptic, self-adjoint and non-negative.

A simple computation gives in terms of the coordinates (x, y, t)

$$\mathcal{L} = -\Delta_x - \Delta_y - \frac{1}{4}(|x|^2 + |y|^2)\partial_t^2 + N\partial_t, \tag{1.1}$$

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where Δ_x and Δ_y are the standard Laplacian on $\mathbb{R}^n$ and

$$N = \sum_{j=1}^{n}(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}).$$

Now if we replace ∂_t by $i\lambda$ in 1.1, we obtain the so-called twisted Laplacian operator L_λ given by

$$L_\lambda = -\Delta_x - \Delta_y + \frac{\lambda^2}{4}(|x|^2 + |y|^2) + i\lambda N.$$

The operator L_λ was introduced by R.S. Strichartz [9], and also known as the special Hermite operator, because it is intimately connected with the Hermite functions on $\mathbb{R}^n$.

In the literature there are a lot of works concerning the twisted Laplacian e.g. [[2,3,7,10,11,13]]. Nevertheless, to our knowledge there are no works dealing with the case when λ is complex not even when λ is purely imaginary.

Let us recall in [7] for $\lambda = 1$ Koch and Ricci found the optimal exponent $\rho(p)$ in the estimate

$$\|P_\beta u\|_{L^p(\mathbb{R}^d)} \leq \rho(p)\|u\|_{L^2(\mathbb{R}^d)} \tag{1.2}$$

for all $p \in [2, +\infty]$ where $\rho(p)$ is

$$\rho(p) = \begin{cases} \frac{1}{p} - \frac{1}{2} \text{ if} 2 \leq p \leq \frac{2(d+1)}{d-1} \\ \frac{d-2}{2} - \frac{d}{p} \text{ if} \frac{2(d+1)}{d-1} \leq \infty \end{cases}$$

and P_β is the spectral projection corresponding to an eigenvalue β^2 of L_1.

By the explicit formulas of the heat kernel and the Green function of L_{-1} (see [4]), it is shown in [13] by Wong that L_{-1} is globally hypoelliptic in the sense that

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), Lu \in \mathcal{S}(\mathbb{R}^{2n}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{2n})$$

where $\mathcal{S}(\mathbb{R}^{2n})$ is the Schwartz space and $\mathcal{S}'(\mathbb{R}^{2n})$ is the space of all tempered distributions.

In [10] Thangavelu has defined a Poisson transform and has used it to establish a necessary and sufficient condition for some eigenfunctions for $L_{-\lambda}$, $\lambda > 0$ to be represented by a Poisson integrable of square integrable functions on the unit sphere S^{2n-1}.

In this paper, we compute the eigenfunctions of $L_{-\lambda}$, Re $(\lambda) > 0$ with their corresponding eigenvalues explicitly and establish a necessary condition so that some eigenfunctions of $L_{-\lambda}$ are in $L^p(\mathbb{C}^n)$. We also give a refinement of Theorem [3.5, [10]].

2 Notations and preliminaries

To state our results, we fix notations and recall some definitions and results.

Fix $n \geq 1$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, by $|\alpha|$ we understand its length $\alpha_1 + \cdots + \alpha_n$.

By $< ., . >$ we denote the usual inner product in $L^2(\mathbb{C}^n)$ and, more generally, for any functions f and g on $\mathbb{C}^n$, $< f, g >$ stands for $\int_{\mathbb{C}^n} f(z)\overline{g(z)}dz$ whenever the integral makes sense.

For $\alpha, \beta \in \mathbb{N}^n$ and $\lambda \in \mathbb{R}^*$ let $\Phi_n(x)$ stand for the n-dimensional Hermite functions and let $\Phi_n^\lambda(x) = |\lambda|^{\frac{n}{4}}\Phi_n(|\lambda|^{\frac{1}{2}}x)$. We define the so called special Hermite functions:

$$\Phi_{\alpha\beta}^\lambda(z) = (2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}} < \pi_\lambda(z)\Phi_\alpha^\lambda, \Phi_\beta^\lambda >$$

and they form an orthonormal basis for $L^2(\mathbb{C}^n)$. Moreover,

$$L_\lambda \Phi_{\alpha\beta}^\lambda = (2|\alpha| + n)|\lambda|\Phi_{\alpha\beta}^\lambda.$$

They are also eigenfunctions of the operators $H_\lambda = -\Delta_z + \frac{\lambda^2}{4}|z|^2$ and $i\lambda N$. Indeed, we can show that

$$H_\lambda \Phi_{\alpha\beta}^\lambda = (|\alpha| + |\beta| + n)|\lambda|\Phi_{\alpha\beta}^\lambda$$

and

$$i\lambda N\Phi^{\lambda}_{\alpha\beta} = (|\alpha| - |\beta|)|\lambda|\Phi^{\lambda}_{\alpha\beta}.$$

Let $L^{\alpha}_{n}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}(-1)^{k}\frac{x^{k}}{k!}$, $\alpha > -1$ be Laguerre polynomials of type k. We define

$$\varphi^{n-1}_{k,\lambda}(z) = L^{n-1}_{k}\left(\frac{1}{2}|\lambda||z|^{2}\right)e^{-\frac{1}{4}|\lambda||z|^{2}}.$$

Now, let $\mathcal{H}^{p,q}$ denote the space of restrictions to $S^{2n-1} = \{z \in \mathbb{C}^{n}, |z| = 1\}$ of harmonic polynomials $h^{p,q}$ which are homogeneous of degree p in z and degree q in $\overline{z}$ with $\dim \mathcal{H}^{p,q} = d(p,q) = \frac{(n+p+q-1)(n+q-2)!(n+p-2)!}{p!q!(n-2)!(n-1)!}$ and if $n = 1$, we make the convention that $pq = 0$, so that $d(p,q) = 1$ (see [5] and [8]).

Basic properties: The spaces $\mathcal{H}^{p,q}$ are pairwise orthogonal in $L^{2}(S^{2n-1})$ and we have

$$L^{2}(S^{2n-1}) = \oplus_{p,q\geq 0}\mathcal{H}^{p,q}.$$

Notice that for each $f_{p,q} \in \mathcal{H}^{p,q}$ with $(p,q) \neq (0,0)$, we have

$$\int_{S^{2n-1}} f_{p,q}(w)d\sigma(w) = 0.$$

Now, we fix an orthonormal basis $h^{p,q}_{j}$, $1 \leq j \leq d(p,q)$ for $\mathcal{H}^{p,q}$ so that

$$\{h^{p,q}_{j}(\omega) : 1 \leq j \leq d(p,q);\ p,q \geq 0\}$$

forms an orthonormal basis for $L^{2}(S^{2n-1})$.

Therefore for a continuous function f on $\mathbb{C}^{n}$, writing $z = r\omega$, where $r > 0$ and $\omega \in S^{2n-1}$, we can expand the function f in terms of spherical harmonics as

$$f(r\omega) = \sum_{p,q\geq 0}\sum_{j=1}^{d(p,q)} a^{p,q}_{j}(r)h^{p,q}_{j}(\omega).$$

The $(p,q)^{th}$ spherical harmonic projection, $a^{p,q}(r,\omega)$ of the function f is then defined as

$$a^{p,q}(r,w) = \sum_{j=1}^{d(p,q)} a^{p,q}_{j}(r)h^{p,q}_{j}(\omega).$$

Hence

$$f(r\omega) = \sum_{p,q\geq 0} a^{p,q}(r,\omega)$$

which the series converges in $L^{2}(S^{2n-1})$.

Here we refer the reader to the monographs of Thangavelu [10,11] and Folland [5].

Special functions. The confluent hypergeometric [6, p. 204] or [12] is defined as

$${}_{1}F_{1}(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)}\frac{z^{k}}{k!}$$

For $a \in \mathbb{C}$, a not a non-positive integer, ${}_{1}F_{1}(a,b,z)$ has the following asymptotic behavior [[6], p. 209]

$${}_{1}F_{1}(a,b,z) = \Gamma(b)\Big(\frac{(-z)^{-a}}{\Gamma(b-a)} + \frac{e^{z}z^{a-b}}{\Gamma(a)}\Big)\Big(1 + O\Big(\frac{1}{|z|}\Big)\Big)\ as\ |z| \to \infty, -\pi < \arg(z) \leq \pi. \qquad (2.1)$$

where by Γ we denote the usual Euler Gamma function.

In particular, when $\operatorname{Re}(z) > 0$, we have

$${}_{1}F_{1}(a,b,z) \underset{|z|\to+\infty}{\sim} \frac{\Gamma(b)}{\Gamma(a)}e^{z}z^{a-b}. \qquad (2.2)$$

3 Eigenfunctions of $L_{\lambda+i\beta}$, $(\lambda,\beta)\in\mathbb{R}^\star * \mathbb{R}$

Given $\lambda\in\mathbb{C}^*$. We are interested in the eigenfunctions of $L_{-\lambda}$, for this purpose we will write the operator $L_{-\lambda}$ into the polar coordinates $z=r\omega(r\in[0,+\infty[,\omega\in S^{2n-1})$. We have the following polar coordinates decomposition,

$$L_{-\lambda}f(r\omega)=[-\frac{d^2}{dr^2}-\frac{2n-1}{r}\frac{d}{dr}-\frac{1}{r^2}\Delta_s+\frac{\lambda^2}{4}r^2+2\lambda E_\omega]f(r\omega) \tag{3.1}$$

where Δ_s is the spherical Laplacian on S^{2n-1} and where E_ω is the spherical part of the complex Euler operator $E=\sum_{j=1}^n z_j\frac{\partial}{\partial z_j}$ on S^{2n-1} given by

$$E=\frac{1}{2}(r\frac{d}{dr})+E_\omega.$$

Theorem 3.1 *Let $\mu\in\mathbb{C}$. Then $f:\mathbb{C}^n\to\mathbb{C}$ is an eigenfunction of $L_{-\lambda}$ with μ as eigenvalue if and only if there exists a sequence $(a_{p,q})$ such that*

$$f(r\omega)=\sum_{p,q\geq 0}r^{p+q}{}_1F_1(\frac{\mu+\lambda n}{2\lambda}+q,n+p+q,-\frac{\lambda}{2}r^2)a_{p,q}h^{p,q}(\omega)e^{\frac{\lambda}{4}r^2}in\mathcal{C}^\infty(\mathbb{C}^n).$$

Proof Let f be an eigenfunction of $L_{-\lambda}$ with μ as eigenvalue, i.e.,

$$L_{-\lambda}f(z)=\mu f(z).$$

Using 3.1, we have

$$[\frac{d^2}{dr^2}+\frac{2n-1}{r}\frac{d}{dr}+\frac{1}{r^2}\Delta_s-\frac{\lambda^2}{4}r^2-2\lambda E_\omega+\mu]f(r\omega)=0.$$

Since the operator Laplacian is an elliptic differential operator on $\mathbb{C}^n$, the solutions of $L_{-\lambda}f=\mu f$ are in $\mathcal{C}^\infty(\mathbb{C}^n)$. Thus, the function $f(r\omega)$ can be expanded as a function of ω into its trigonometrical series in terms of the spherical harmonics as follows

$$f(r\omega)=\sum_{p,q\geq 0}b^{p,q}(r)h^{p,q}(\omega)in\mathcal{C}^\infty([0,+\infty[\times S^{2n-1}).$$

Using the fact $\Delta_s(h^{p,q})=-(p+q)(p+q+2n-2)h^{p,q}$ and $E_\omega h^{p,q}=\frac{p-q}{2}h^{p,q}$ because $h^{p,q}\in\mathcal{H}^{p,q}$, we see that $b^{p,q}(r)$ must satisfy

$$[\frac{d^2}{dr^2}+\frac{2n-1}{r}\frac{d}{dr}+\frac{[\mu-\lambda(p-q)]r^2-(p+q)(p+q+2n-2)-\frac{\lambda^2}{4}r^4}{r^2}]b^{p,q}(r)=0. \tag{3.2}$$

Put $b^{p,q}(r)=r^{p+q}\varphi(r),\ M=\mu-\lambda(p-q),\ d=n+p+q$ and $a=-\frac{\lambda^2}{4}$.

Then the equation 3.2 becomes

$$\varphi''(r)+\frac{2d-1}{r}\varphi'(r)+(M+ar^2)\varphi(r)=0. \tag{3.3}$$

Putting $\varphi(r)=\psi(r^2)$, the equation 3.3 reduces to

$$t\psi''(t)+d\psi'(t)+(\frac{M}{4}-(\frac{\lambda}{4})^2t)\psi(t)=0. \tag{3.4}$$

Another substitution $\psi(t)=e^{\frac{\lambda}{4}t}g(t)$, reduces the equation 3.4 to the following

$$tg''(t)+(d+\frac{\lambda}{2}t)g'(t)+(\frac{M}{4}+\frac{\lambda}{4}d)g(t)=0. \tag{3.5}$$

Using the change of variable $t = -\frac{2s}{\lambda}$ and $h(s) = g(-\frac{2s}{\lambda})$ in the equation 3.5, we obtain

$$sh''(s) + (d - s)h'(s) - \frac{M + \lambda d}{2\lambda}h(s) = 0. \tag{3.6}$$

But this is the hypergeometric equation with parameters $\left(d, \frac{M + \lambda d}{2\lambda}\right)$ and the only solutions which are smooth at $s = 0$ are multiples of ${}_1F_1(\frac{M + \lambda d}{2\lambda}, d, s)$.

Retracing the various substitutions made in the reduction of equation 3.2 to equation 3.6, an eigenfunction of $L_{-\lambda}$ corresponding to eigenvalue μ is therefore given by

$$f_\lambda(r\omega) = \sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{M + \lambda d}{2\lambda}, d, -\frac{\lambda}{2}r^2)a_{p,q}h^{p,q}(\omega)e^{\frac{\lambda}{4}r^2} in\ \mathcal{C}^\infty(\mathbb{C}^n).$$

□

As a consequence, we have

Corollary 3.1 *Let $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C}$. Then $f_\lambda : \mathbb{C}^n \to \mathbb{C}$ is an eigenfunction of $L_{-\lambda}$ corresponding to eigenvalue $\mu\lambda$ if and only if*

$$f_\lambda(r\omega) = \sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{n - \mu}{2} + p, n + p + q, \frac{\lambda}{2}r^2)a_{p,q}h^{p,q}(\omega)e^{-\frac{\lambda}{4}r^2} in\ \mathcal{C}^\infty(\mathbb{C}^n).$$

Proof Using Theorem 3.1 with μ will be changed by $\mu\lambda$ we get

$$f_\lambda(r\omega) = \sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{\mu + n}{2} + q, n + p + q, -\frac{\lambda}{2}r^2)a_{p,q}h^{p,q}(\omega)e^{\frac{\lambda}{4}r^2} in\ \mathcal{C}^\infty(\mathbb{C}^n).$$

Since ${}_1F_1(a, b, z) = e^z{}_1F_1(b - a, b, -z)$ we have

$$f_\lambda(r\omega) = \sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{n - \mu}{2} + p, n + p + q, \frac{\lambda}{2}r^2)a_{p,q}h^{p,q}(\omega)e^{-\frac{\lambda}{4}r^2} in\ \mathcal{C}^\infty(\mathbb{C}^n).$$

□

In the following, we give our characterization of $L^2(\mathbb{C}^n)-$ eigenfunctions of $L_{-\lambda}$, $\lambda \in \mathbb{C}^*$, we denote for $p \in [1, \infty[$ by

$$A^p_\mu(\mathbb{C}^n) = \left\{f : \mathbb{C}^n \to \mathbb{C} : L_{-\lambda}f = \lambda\mu f, \|f\|_p = \left(\int_{\mathbb{C}^n} |f_\lambda(z)|^p dz\right)^{\frac{1}{p}} < +\infty\right\},$$

and

$$A^\infty_\mu(\mathbb{C}^n) = \left\{f : \mathbb{C}^n \to \mathbb{C} : L_{-\lambda}f = \lambda\mu f, \|f\|_\infty = \operatorname{ess\,sup}_{z\in\mathbb{C}} |f_\lambda(z)| < +\infty\right\}.$$

We assert as in [1, Theorem 2.1] that,

Theorem 3.2 *Let $\lambda \in \mathbb{C}^*$ such that $Re\ (\lambda) > 0$ and let $\mu \in \mathbb{C}$, then*

1. *For $\mu \neq n + 2m$, $m \in \mathbb{N}$ the space $A^2_\mu(\mathbb{C}^n)$ is trivial.*
2. *For $\mu = n + 2m, m \in \mathbb{N}$, then $f_\lambda \in A^2_{n+2m}(\mathbb{C}^n)$ if and only if $f_\lambda(r\omega) = \sum_{\substack{0\leq p\leq m\\ q\geq 0}} r^{p+q}{}_1F_1(p - m, d, \frac{\lambda}{2}r^2)e^{\frac{-\lambda}{4}r^2}a_{p,q}.h^{p,q}(\omega)$ in $\mathbb{C}^\infty(\mathbb{C}^n)$ such that*

$$\|f_\lambda\|_2 = \sqrt{\sum_{0\leq p\leq m; q\geq 0} \gamma_\lambda(m, n, p, q) \sum_{j=1}^{d(p,q)} |a^{p,q}_j|^2} < +\infty.$$

With $\gamma_\lambda(m, n, p, q) = \int_{\mathbb{C}^n} |{}_1F_1(p - m, d, \frac{\lambda}{2}|z|^2)|^2 e^{-Re(\frac{\lambda}{2})|z|^2}|z|^{2(p+q)}dz.$

Proof 1. Assume $\mu \neq n+2m, m \in \mathbb{N}$. Let f_λ be a nonzero eigenfunction $L_{-\lambda}$ with $\mu\lambda$ as eigenvalue. Then, by Corollary 3.1, the function f_λ can be expanded as

$$f_\lambda(r\omega) = \sum_{p,q\geq 0} r^{p+q} {}_1F_1(\frac{n-\mu}{2}+p, n+p+q, \frac{\lambda}{2}r^2) a_{p,q} h^{p,q}(\omega) e^{-\frac{\lambda}{4}r^2} in \mathcal{C}^\infty(\mathbb{C}^n).$$

If $f_\lambda \in A^2_\mu(\mathbb{C}^n)$ then its norm in the Hilbert space $L^2(\mathbb{C}^n, dz)$ is

$$\|f_\lambda\|_2^2 = \sum_{p,q\geq 0} \sum_{j=1}^{d(p,q)} |a_j^{p,q}|^2 \int_{\mathbb{C}^n} |{}_1F_1(\frac{-\mu+2p+n}{2}, d, \frac{\lambda}{2}|z|^2)|^2 e^{-Re(\frac{\lambda}{2})|z|^2} |z|^{2(p+q)} dz < \infty.$$

So, $\forall p, q \geq 0$, $\int_{\mathbb{C}^n} |{}_1F_1(\frac{-\mu+2p+n}{2}, d, \frac{\lambda}{2}|z|^2)|^2 e^{-Re(\frac{\lambda}{2})|z|^2} |z|^{2(p+q)} dz < +\infty$.
But by 2.2, it follows that

$$\left|{}_1F_1(\frac{-\mu+2p+n}{2}, n+p+q, \frac{\lambda}{2}r^2)\right|^2 e^{-Re(\frac{\lambda}{2})r^2} r^{2(p+q)} \underset{+\infty}{\sim} \left|\frac{\Gamma(n+p+q)}{\Gamma(\frac{-\mu+2p+n}{2})}(\frac{\lambda}{2}r^2)^{\frac{-\mu+2p+n}{2}-(n+p+q)}\right|^2 \times e^{Re(\frac{\lambda}{2})t^2} r^{2(p+q)}$$

Therefore ,

$$\int_{\mathbb{C}^n} |{}_1F_1(\frac{-\mu+2p+n}{2}, n+p+q, \frac{\lambda}{2}|z|^2)|^2 e^{-Re(\frac{\lambda}{2})|z|^2} |z|^{2(p+q)} dz = +\infty.$$

This yields a contradiction.
2. According to the above, $f_\lambda \in A^2_\mu(\mathbb{C}^n)$ implies $\frac{-\mu+2p+n}{2}$ is a pole of Γ i.e., $\frac{-\mu+2p+n}{2} \in (-\mathbb{N})$. Hence, $\mu = 2m+n$.
Notice that if $0 \leq p \leq m$ then $\Gamma(p-m) = \infty$, so $f_\lambda \in A^2_{2m+n}(\mathbb{C}^n)$ implies

$$\|f_\lambda\|_2 = \sqrt{\sum_{0\leq p\leq m; q\geq 0} \gamma_\lambda(m, n, p, q) \sum_{j=1}^{d(p,q)} |a_j^{p,q}|^2} < +\infty.$$

Conversely we have the result by applying Corollary 3.1.

□

As a consequence, we have

Corollary 3.2 *Let* $\lambda \in \mathbb{C}^*$ *such that* $Re\ (\lambda) > 0$, *then*

$$\sigma_p(L_{-\lambda}) = \{(2m+n)\lambda, m \in \mathbb{N}\}.$$

where $\sigma_p(L_{-\lambda})$ denotes the point spectrum of $L_{-\lambda}$.

The result of Theorem 3.2 motivate to give a necessary condition so that $f_\lambda(r\omega) = \sum_{p,q\geq 0} r^{p+q} {}_1F_1(\frac{\mu+n}{2} + q, n+p+q, -\frac{\lambda}{2}r^2) a_{p,q} h^{p,q}(\omega) e^{-\frac{\lambda}{4}r^2}$ with $a_{0,0} \neq 0$ are not only in $L^2(\mathbb{C}^n)$ but also in $L^s(\mathbb{C}^n)$, $s \in [1, \infty]$. We claim

Theorem 3.3 *Let* $s \in [1, +\infty]$ *and let* $\lambda \in \mathbb{C}$ *such that* $Re\ (\lambda) > 0$.
If $\frac{-\mu+n}{2} \notin (-\mathbb{N})$, *then* $f_\lambda \notin L^s(\mathbb{C}^n)$.

Proof The proof of this theorem is by contradiction. Assume $\lambda \in \mathbb{C}^*$, Re $(\lambda) > 0$.

1. For $s \in [1, +\infty[$. Suppose that $\| f_\lambda\|_s^s < \infty$, then using polar coordinates we have

$$\| f_\lambda\|_s^s = \int_0^\infty \int_{S^{2n-1}} r^{2n-1}\Big|\Psi_r(w)\Big|^s dr d\sigma(w) < +\infty$$

where $\Psi_r(w) = \sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{-\mu+n+2p}{2}, d, \frac{\lambda}{2}r^2)e^{-\frac{\lambda}{4}r^2} a_{p,q}h^{p,q}(w)$.
Hence, by Fubini's theorem for a.e r, we have

$$\int_0^\infty \int_{S^{2n-1}} r^{2n-1}|\Psi_r(w)|^s d\sigma(w)dr < +\infty$$

and

$$r^{2n-1}\int_{S^{2n-1}} |\Psi_r(w)|^s d\sigma(w) < +\infty.$$

Then, by Hölder's inequality we get

$$\sigma(S^{2n-1})^{1-s}|\int_{S^{2n-1}} \Psi_r(\omega)d\sigma(\omega)|^s \leq \int_{S^{2n-1}} |\Psi_r(\omega)|^s d\sigma(\omega).$$

Therefore.

$$\Big|\int_{S^{2n-1}} \sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{-\mu+n+2p}{2}, n+p+q, \frac{\lambda}{2}r^2)e^{-\frac{\lambda}{4}r^2} a_{p,q}h^{p,q}(w)d\sigma(w)\Big|^s \leq \|\Psi_r\|^s_{L^s(S^{2n-1})}.$$

Since the series $\sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{-\mu+n+2p}{2}, n+p+q, \frac{\lambda}{2}r^2)e^{-\frac{\lambda}{4}r^2} a_{p,q}h^{p,q}(w)$ converges uniformly in $z = rw$ on each compact set of $\mathbb{C}^n$, then the series

$$\sum_{p,q\geq 0} r^{p+q}{}_1F_1(\frac{-\mu+n+2p}{2}, d, \frac{\lambda}{2}r^2)e^{-\frac{\lambda}{4}r^2} a_{p,q}h^{p,q}(w)$$

converges in w uniformly on S^{2n-1}. So, we can reverse the order of sum and integration, we obtain

$$|c \quad {}_1F_1(\frac{-\mu+n}{2}, n, \frac{\lambda}{2}r^2)e^{-\frac{\lambda}{4}r^2} \quad |^s \leq \|\Psi_r\|^s_{L^s(S^{2n-1})}.$$

With another use of the asymptotic behavior 2.2, we obtain

$$c\Big|\frac{\Gamma(n)}{\Gamma(\frac{-\mu+n}{2})}e^{\frac{\lambda}{4}r^2}(\frac{\lambda}{2}r^2)^{(\frac{-\mu+n}{2}-n)}\Big|^s \underset{r\to+\infty}{\leq} \Big\|\Psi_r\Big\|^s_{L^s(S^{2n-1})}.$$

So

$$\int_0^\infty r^{2n-1}\|\Psi_r\|^s_{L^s(S^{2n-1})}dr = \infty$$

and this is impossible.
2. For $s = +\infty$. If $|f(z)| \leq C$, then as before we have

$$\begin{aligned} c|\frac{\Gamma(n)}{\Gamma(\frac{-\mu+n}{2})}e^{\frac{\lambda}{4}r^2}(\frac{\lambda}{2}r^2)^{(\frac{-\mu+n}{2}-n)}| &\underset{r\to+\infty}{\leq} |\int_{S^{2n-1}} f(\omega)d\sigma(\omega)| \\ &\underset{r\to+\infty}{\leq} \int_{S^{2n-1}} |f(\omega)|d\sigma(\omega) \\ &\underset{r\to+\infty}{\leq} C\sigma(S^{2n-1}). \end{aligned}$$

Now, if we take the limit $r \to +\infty$, the member who is left goes to $+\infty$, this yields a contradiction.

$\square$

4 Poisson transform for $\mathbb{H}^n$

In this section, for each $\lambda > 0$ and $m \in \mathbb{N}$, we define the Poisson kernel on the Heisenberg group by

$$\mathcal{P}_m^{\lambda}(z,\omega) = \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} \sum_{j=1}^{d(p,q)} c_m^{\lambda}(p,q)^{-1}(-1)^p 2^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) h_j^{p,q}(z)\overline{h_j^{p,q}}(\omega)$$

where $w \in S^{2n-1}$, $z \in \mathbb{C}^n$ and $c_m^{\lambda}(p,q)^2 = (\frac{4}{\lambda})^{p+q} 2^{n+p+q-1} \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)}$.

Given $g \in L^2(S^{2n-1})$, its Poisson integral

$$\mathcal{P}_m^{\lambda}(g)(z) = \int_{S^{2n-1}} \mathcal{P}_m^{\lambda}(z,\omega) g(\omega) d(\omega).$$

Remark 4.1 1. Note that for $m = 0, n = 1, \lambda = 2$ and $\mid z \mid= 1$ it is impossible to express $\mathcal{P}_m^{\lambda}(z,z)$ in a close formula, so it seems hard to give an estimate of $\mathcal{P}_m^{\lambda}(z,\omega)$.

2. A simple computation shows that $L_{-\lambda}\mathcal{P}_m^{\lambda}(z,\omega) = (2m+n)\lambda \mathcal{P}_m^{\lambda}(z,\omega)$ and

$$z \to \mathcal{P}_m^{\lambda}(z,\omega) \notin A_{n+2m}^2.$$

Let Λ_m denote the subspace of $L^2(S^{2n-1})$ defined by

$$\Lambda_m = \Big\{ g(z) = \sum_{\substack{0\le p\le m\\ q\ge 0}} a_{p,q} h_{p,q}(z), \quad \|g\|^2_{L^2(S^{2n-1})} = \sum_{\substack{0\le p\le m\\ q\ge 0}} \sum_{j=1}^{d(p,q)} |a_j^{p,q}|^2 < \infty \Big\}.$$

Proposition 4.1 *For every $g \in \Lambda_m$, we have*

$$\| P_m^{\lambda}(g)\|_{L^2(\mathbb{C}^n)} = \lambda^{-\frac{n}{2}} \|g\|_{L^2(S^{2n-1})}.$$

Proof Let $g \in L^2(S^{2n-1})$, we have

$$\begin{aligned}
\mathcal{P}_m^{\lambda}(g)(z) &= \int_{S^{2n-1}} \mathcal{P}_m^{\lambda}(z,\omega) g(\omega) d\sigma(\omega) \\
&= \int_{S^{2n-1}} \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} \sum_{j=1}^{d(p,q)} c_m^{\lambda}(p,q)^{-1}(-1)^p 2^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) h_j^{p,q}(z)\overline{h_j^{p,q}}(\omega) g(\omega) d\sigma(\omega) \\
&= \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^{\lambda}(p,q)^{-1}(-1)^p 2^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) \sum_{j=1}^{d(p,q)} h_j^{p,q}(z) \int_{S^{2n-1}} \overline{h_j^{p,q}}(\omega) g(\omega) d\sigma(\omega) \\
&= \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^{\lambda}(p,q)^{-1}(-1)^p 2^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) \sum_{j=1}^{d(p,q)} a_{p,q}^{j} h_j^{p,q}(z) \\
&= \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^{\lambda}(p,q)^{-1}(-1)^p 2^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) a_{p,q} h^{p,q}(z) \\
&= \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^{\lambda}(p,q)^{-1}(-1)^p 2^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) a_{p,q} h^{p,q}(z)
\end{aligned}$$

Hence, the $L^2(\mathbb{C}^n, dz)$ norm of $P_m^\lambda(g)$ is

$$\|\mathcal{P}_m^\lambda(g)\|_2^2 = \lambda^{-n} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^\lambda(p,q)^{-2} 4^{p+q} \sum_{j=1}^{d(p,q)} |a_j^{p,q}|^2$$
$$\times \Big[\int_0^{+\infty} |L_{m-p}^{n+p+q-1}(\frac{\lambda}{2}r^2)|^2 r^{2(n+p+q)-1} dr e^{-\frac{\lambda}{2}r^2} dr\Big]. \tag{4.1}$$

By using the formula

$$\int_0^{+\infty} |L_k^\alpha(t)|^2 t^\alpha e^{-t} dt = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)},$$

the last integral in 4.1 is

$$\int_0^{+\infty} |L_{m-p}^{n+p+q-1}(\frac{\lambda}{2}r^2)|^2 r^{2(n+p+q)-1} e^{-\frac{\lambda}{2}r^2} dr = \frac{1}{2}(\frac{2}{\lambda})^{p+q+n} \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)}.$$

Since

$$c_m^\lambda(p,q)^2 = (\frac{4}{\lambda})^{p+q} 2^{n+p+q-1} \frac{\Gamma(m+n+q)}{\Gamma(m-p+1)}$$

we obtain

$$\|\mathcal{P}_m^\lambda(g)\|_{L^2(\mathbb{C}^n)}^2 = \lambda^{-n} \sum_{\substack{0\le p\le m\\ q\ge 0}} \sum_{j=1}^{d(p,q)} |a_j^{p,q}|^2. \tag{4.2}$$

In particular, for every $g \in \Lambda_m$

$$\|P_m^\lambda(g)\|_{L^2(\mathbb{C}^n)} = \lambda^{-\frac{n}{2}} \|g\|_{L^2(S^{2n-1})}.$$

□

With the above, we assert

Theorem 4.1 *$\mathcal{P}_m^\lambda$ is an isometric isomorphism of $(\Lambda_m, \lambda^{-\frac{n}{2}}\|.\|_{L^2(S^{2n-1})})$ onto $A^2_{2m+n}(\mathbb{C}^n)$.*

Proof By Proposition 4.1, it sufficient to show that $\mathcal{P}_m^\lambda$ is surjective.

Let $f_\lambda \in A^2_{2m+n}(\mathbb{C}^n)$.

Since ${}_1F_1(-k, \alpha+1, x) = \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(x)$ and $\varphi_{k,\lambda}^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right) e^{-\frac{1}{4}|\lambda||z|^2}$.

Then by Theorem 3.2, there exists a sequence $(b_{p,q})$ such that

$$f_\lambda(z) = \sum_{\substack{0\le p\le m\\ q\ge 0}} |z|^{p+q} \varphi_{m-p,\lambda}^{n+p+q-1}(z) \frac{\Gamma(p+q+n)\Gamma(m-p+1)}{\Gamma(m+n+q)} b_{p,q}.h^{p,q}(\omega)$$

with $\sum_{0\le p\le m q\ge 0} \gamma_\lambda(m,n,p,q) \sum_{j=1}^{d(p,q)} |b_j^{p,q}|^2 < +\infty$.

Next, set $a_j^{p,q} = \lambda^{\frac{n}{2}} c_m^\lambda(p,q)(-1)^p 2^{-(p+q)} \frac{\Gamma(p+n+q)\Gamma(m-p+1)}{\Gamma(n+m+q)} b_j^{p,q}$.

So, we have $\mathcal{P}_m^\lambda(g)(z) = f_\lambda(z)$, with $g(z) = \sum_{0\le p\le m q\ge 0} a_{p,q} h_{p,q}(z) \in \Lambda_m$ because

$$\lambda^{-2n} \sum_{\substack{0\le p\le m\\ q\ge 0}} \sum_{j=1}^{d(p,q)} |a_j^{p,q}|^2 = \sum_{\substack{0\le p\le m\\ q\ge 0}} \gamma_\lambda(m,n,p,q) \sum_{j=1}^{d(p,q)} |b_j^{p,q}|^2 < +\infty.$$

i.e., $\mathcal{P}_m^\lambda$ is surjective. □

We are now able to prove a first injection from $A_m^2(\mathbb{C}^n)$ to $A_m^1(\mathbb{C}^n)$.

Theorem 4.2 *The injection $A_m^2(\mathbb{C}^n) \hookrightarrow A_m^1(\mathbb{C}^n)$ is continuous.*

Proof Let $f_\lambda(|z|w) \in A_m^2(\mathbb{C}^n)$, by Theorem 3.2, there exists $(a_{p,q})$ such that
$f_\lambda(|z|w) = \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^\lambda(p,q)^{-1}(-1)^p 2^{p+q}\varphi_{m-p,\lambda}^{n+p+q-1}(z)|z|^{p+q}a_{p,q}h^{p,q}(w).$

Applying Cauchy-Schwarz , we have

$$|f_\lambda(|z|w)| \le \sqrt{\Big(\sum_{\substack{0\le p\le m\\ q\ge 0}} \gamma_\lambda(m,n,p,q) \sum_{\substack{0\le p\le m\\ q\ge 0}} |a_j^{p,q}|^2\Big)}$$
$$\times \lambda^{-\frac{n}{2}} \sqrt{\sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^\lambda(p,q)^{-2}(4)^{p+q}|z|^{2(p+q)}(\varphi_{m-p,\lambda}^{n+p+q-1}(z))^2 \frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}.$$

Using $\sqrt{\Big(\sum_{\substack{0\le p\le m\\ q\ge 0}} \gamma_\lambda(m,n,p,q) \sum_{\substack{0\le p\le m\\ q\ge 0}} |a_j^{p,q}|^2\Big)} = \|f_\lambda\|_{L^2(\mathbb{C}^n)}$ and

$$\sqrt{\sum_{\substack{0\le p\le m\\ q\ge 0}} u_{p,q}} \le \sum_{\substack{0\le p\le m\\ q\ge 0}} \sqrt{u_{p,q}}$$

we obtain

$$|f_\lambda(z)| \le \lambda^{-\frac{n}{2}} \|f_\lambda\|_{L^2(\mathbb{C}^n)}$$
$$\times \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^\lambda(p,q)^{-1} \sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}(2)^{p+q}|z|^{(p+q)}|\varphi_{m-p,\lambda}^{n+p+q-1}(z)|$$

(because the series $\sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^\lambda(p,q)^{-1} \sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}(2)^{p+q}|z|^{(p+q)}|\varphi_{m-p,\lambda}^{n+p+q-1}(z)|$ converges see below).
Now, by integrating on $\mathbb{C}^n$ we obtain

$$\|f_\lambda\|_{L^1(\mathbb{C}^n)} \le \lambda^{-\frac{n}{2}} \|f_\lambda\|_{L^2(\mathbb{C}^n)}$$
$$\times \int_{\mathbb{C}^n} \sum_{\substack{0\le p\le m\\ q\ge 0}} c_m^\lambda(p,q)^{-1} \sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}(2)^{p+q}|z|^{(p+q)}|\varphi_{m-p,\lambda}^{n+p+q-1}(z)|dz.$$

On the other hand, since $L_n^\alpha(t) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}(-1)^k \frac{x^k}{k!}$ we have

$$\int_{\mathbb{C}^n} |z|^{p+q}|\varphi_{m-p,\lambda}^{n+p+q-1}(z)|dz = \int_{\mathbb{C}^n} |z|^{p+q}\Big|\sum_{k=0}^{m-p} \binom{n+m+q-1}{m-p-k}(-1)^k \frac{(\frac{\lambda}{2})^k}{k!}|z|^{2k}\Big| \exp(-\frac{\lambda}{4}|z|^2)dz$$
$$\le \sum_{k=0}^{m-p} \binom{n+m+q-1}{m-p-k} \frac{(\frac{\lambda}{2})^k}{k!} \int_{\mathbb{C}^n} |z|^{2k+p+q} \exp(-\frac{\lambda}{4}|z|^2)dz$$

we make the change of variables $\frac{\lambda}{4}|z|^2 = t$, then the integral $\int_{\mathbb{C}^n} |z|^{2k+p+q} \exp(-\frac{\lambda}{4}|z|^2)dz$ is equal

$$\sigma(S^{2n-1})\frac{2^{2(k+n)+p+q-1}}{\sqrt{\lambda}^{2(k+n)+p+q}}\Gamma(k+n+\frac{p+q}{2}).$$

So

$$\int_{\mathbb{C}^n} |z|^{p+q}|\varphi^{n+p+q-1}_{m-p,\lambda}(z)|dz \leq \sigma(S^{2n-1})\sum_{k=0}^{m-p}\binom{n+m+q-1}{m-p-k}\frac{(\frac{\lambda}{2})^k}{k!}\frac{2^{2(k+n)+p+q-1}}{\sqrt{\lambda}^{2(k+n)+p+q}}\Gamma(k+n+\frac{p+q}{2}).$$

Therefore,

$$\begin{aligned}\|f_\lambda\|_{L^1(\mathbb{C}^n)} &\leq \lambda^{-\frac{n}{2}}\sigma(S^{2n-1})\|f_\lambda\|_{L^2(\mathbb{C}^n)}\\ &\times \sum_{\substack{0\leq p\leq m\\ q\geq 0\\ 0\leq k\leq m-p}} c^\lambda_m(p,q)^{-1}\sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}(2)^{p+q}\frac{1}{k!}\binom{n+m+q-1}{m-p-k}\\ &\times (\frac{\lambda}{2})^k\frac{2^{2(k+n)+p+q-1}}{\sqrt{\lambda}^{2(k+n)+p+q}}\Gamma(k+n+\frac{p+q}{2})\end{aligned}$$

where we have used the following facts

1. by Stirling's formula $\Gamma(t+1) \underset{t\to+\infty}{\sim} \sqrt{2\pi t}(\frac{t}{e})^t$ we have

$$\frac{\Gamma(k+n+\frac{p+q+1}{2})}{\Gamma(k+n+\frac{p+q}{2})} \underset{q\to+\infty}{\sim} \sqrt{\frac{q}{2}}$$

2. by ratio test, the series

$$\sum_{\substack{0\leq p\leq m\\ q\geq 0\\ 0\leq k\leq m-p}}\Big[c^\lambda_m(p,q)^{-1}\sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}(2)^{p+q}\frac{1}{k!}\binom{n+m+q-1}{m-p-k}(\frac{\lambda}{2})^k\frac{2^{2(k+n)+p+q-1}}{\sqrt{\lambda}^{2(k+n)+p+q}}$$
$$\times\Gamma(k+n+\frac{p+q}{2})\Big]$$

converges in q. This completes the proof of the Theorem.

□

To establish another injection of the space $A^2_m(\mathbb{C}^n)$ we need the following lemma.

Lemma 4.1 *Let $\lambda > 0$. For all $(m,r) \in \mathbb{N}\times\mathbb{R}^+$, we have*

$$r^m\exp(-\frac{\lambda}{4}r^2) \leq \sqrt{(\frac{2m}{\lambda})^m}\exp(-\frac{m}{2}).$$

Proof It suffices to study the function $r \to r^m\exp(-\frac{\lambda}{4}r^2)$ on $\mathbb{R}^+$. □

Theorem 4.3 *The injection $A^2_m(\mathbb{C}^n) \hookrightarrow A^\infty_m(\mathbb{C}^n)$ is continuous.*

Proof Let $f_\lambda(|z|w) \in A^2_m(\mathbb{C}^n)$. As in the proof of Theorem 4.2, there exists a sequence $(a^{p,q})$ such that
$$f_\lambda(|z|w) = \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c^\lambda_m(p,q)^{-1}(-1)^p 2^{p+q}\varphi^{n+p+q-1}_{m-p,\lambda}(z)|z|^{p+q}a_{p,q}h^{p,q}(w)$$
with

$$|f_\lambda(|z|w)| \le \lambda^{-\frac{n}{2}} \sum_{\substack{0\le p\le m\\ q\ge 0}} c^\lambda_m(p,q)^{-1}(2)^{p+q}|z|^{(p+q)}|\varphi^{n+p+q-1}_{m-p,\lambda}(z)|\sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}} \times \|f_\lambda\|_{L^2(\mathbb{C}^n)}$$

Set $\Theta(z) = |z|^{(p+q)}\left|\varphi^{n+p+q-1}_{m-p,\lambda}(z)\right| = r^{(p+q)}|L^{n+p+q-1}_{m-p}(\frac{\lambda}{2}r^2)|\exp(-\frac{\lambda}{4}r^2), \quad z = r\theta.$
So

$$\begin{aligned}|\Theta(z)| &= r^{p+q}|\sum_{k=0}^{m-p}\binom{n+m+q-1}{m-p-k}(-1)^k\frac{(\frac{\lambda}{2})^k}{k!}|z|^{2k}|\exp(-\frac{\lambda}{4}|z|^2)\\ &\le \sum_{k=0}^{m-p}\binom{n+m+q-1}{m-p-k}\frac{1}{k!}(\frac{\lambda}{2})^k\sqrt{\frac{2(p+q+2k)}{\lambda}}^{p+q+2k}\exp(-k-\frac{p+q}{2}).\end{aligned}$$

Using Lemma 4.1 in the last inequality we get

$$|\Theta(z)| \le \sum_{k=0}^{m-p}\binom{n+m+q-1}{m-p-k}\frac{1}{k!}(\frac{\lambda}{2})^k\sqrt{\frac{2(p+q+2k)}{\lambda}}^{p+q+2k}\exp(-k-\frac{p+q}{2}).$$

Hence

$$\begin{aligned}\sum_{\substack{0\le p\le m\\ q\ge 0}} c^\lambda_m(p,q)^{-1}2^{p+q}|z|^{(p+q)}|\varphi^{n+p+q-1}_{m-p,\lambda}(z)|\sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}} \le \sum_{\substack{0\le p\le m\\ q\ge 0\\ 0\le k\le m-p}} \Bigg[c^\lambda_m(p,q)^{-1}(2)^{p+q}\\ \times\sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}\\ \times\binom{n+m+q-1}{m-p-k}\frac{1}{k!}(\frac{\lambda}{2})^k\\ \times\sqrt{\frac{2(p+q+2k)}{\lambda}}^{p+q+2k}\\ \times\exp(-k-\frac{p+q}{2})\Bigg]\end{aligned}$$

where we have used the following fact, the series

$$\sum_{\substack{0\le p\le m\\ q\ge 0\\ 0\le k\le m-p}} \left[c^\lambda_m(p,q)^{-1}(2)^{p+q}\sqrt{\frac{d(p,q)}{\gamma_\lambda(m,n,p,q)}}\binom{n+m+q-1}{m-p-k}\frac{1}{k!}(\frac{\lambda}{2})^k\sqrt{\frac{2(p+q+2k)}{\lambda}}^{p+q+2k}\times\exp(-k-\frac{p+q}{2})\right]$$

converges by the ratio test.
This proves the Theorem. □

In fact, we have the following injection

Corollary 4.1 *Let $p \in [1, +\infty]$, then the injection $A^2_m(\mathbb{C}^n) \hookrightarrow A^p_m(\mathbb{C}^n)$ is continuous.*

Proof By Theorems 4.2 and 4.3, it suffices to show

$$\|f\|_p \le \|f\|_1^{\frac{1}{p}} \|f\|_\infty^{1-\frac{1}{p}}.$$

Indeed $\|f\|_p^p = \int_{\mathbb{C}^n} |f(z)||f(z)|^{p-1}dz \le \int_{\mathbb{C}^n} |f(z)|\|f\|_\infty^{p-1}dz = \|f\|_1\|f\|_\infty^{p-1}$. □

Now we are in position to give a refinement of Theorem [3.5, [10]], we have

Theorem 4.4 *Let $p \in [1, +\infty]$. Then a function $f_\lambda \in L^p(\mathbb{C}^n)$ satisfies $L_{-\lambda} f_\lambda = (2m+n) f_\lambda$ if and only if $f_\lambda = P_m^\lambda(g)$ for some $g \in L^2(S^{2n-1})$.*

Moreover, the following estimates hold,

$$C'_\lambda \|g\|_{L^2(S^{2n-1})} \le \|\mathcal{P}_m^\lambda(g)\|_{L^p(\mathbb{C}^n)} \le C_\lambda \|g\|_{L^2(S^{2n-1})}.$$

where C_λ and C'_λ are two positive constants.

Proof The necessary condition follows from Corollary4.1.

We now proceed to prove the converse. Let $f_\lambda \in A^P_{n+2m}$, note that by the global hypoellipticity of L_{-1} the function $f_1(z) = f_\lambda(\frac{z}{\sqrt{\lambda}})$ which is an eigenfunction of L_{-1} belongs to $\mathcal{S}(\mathbb{C}^n)$, and hence $f_\lambda \in A^2_{n+2m}$. Therefore by Theorem 4.1 there exists $g \in \Lambda_m$ such that $\mathcal{P}_m^\lambda(g)(z) = f_\lambda(z)$.

Again by Corollary 4.1, it remains to show that $C'_\lambda\|g\|_{L^2(S^{2n-1})} \le \|\mathcal{P}_m^\lambda(g)\|_{L^p(\mathbb{C}^n)}$. From the equality $\|P_m^\lambda(g)\|_{L^2(\mathbb{C}^n)} = \lambda^{-\frac{n}{2}}\|g\|_{L^2(S^{2n-1})}$ we get by polarization

$$< f, g >_{L^2(S^{2n-1})} = \lambda^{-n} < P_m^\lambda(f), P_m^\lambda(g) >_{L^2(\mathbb{C}^n)}$$

for $g \in \Lambda_m$ and $f \in L^2(S^{2n-1})$ and by Hôlder's inequality we get

$$|< f, g >_{L^2(S^{2n-1})}| \le \lambda^{-n}\|P_m^\lambda(f)\|_{L^q(\mathbb{C}^n)}\|P_m^\lambda(g)\|_{L^p(\mathbb{C}^n)}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Since $\|\mathcal{P}_m^\lambda(f)\|_{L^q(\mathbb{C}^n)} \le C'_\lambda\|f\|_{L^2(S^{2n-1})}$ we have

$$|< f, g >_{L^2(S^{2n-1})}| \le C'_\lambda\|f\|_{L^2(S^{2n-1})}\|P_m^\lambda(g)\|_{L^p(\mathbb{C}^n)}.$$

Taking the supremum over all $f \in L^2(S^{2n-1})$ with $\|f\|_{L^2(S^{2n-1})} \le 1$, we obtain

$$\|g\|_{L^2(S^{2n-1})} \le C'_\lambda\|P_m^\lambda(g)\|_{L^p(\mathbb{C}^n)}.$$

This finishes the proof of Theorem. □

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