Indian J Pure Appl Math (2023) 54:980–995
https://doi.org/10.1007/s13226-022-00308-8

ORIGINAL RESEARCH

On conformal minimal immersions with constant curvature from two-spheres into the complex hyperquadrics

Hong Li · Xiaoxiang Jiao

Received: 19 September 2021 / Accepted: 4 August 2022 / Published online: 2 September 2022

Abstract In this paper, firstly we study the geometry of conformal minimal two-spheres immersed in the complex hyperquadric Q_{n-2}. Then we classify the linearly full irreducible conformal minimal immersions with constant curvature from S^2 to Q_{n-2} ($n \geqslant 7$) of isotropy order $r = n - 6$ under some conditions, which shows that all such immersions can be expressed by Veronese surfaces in $\mathbb{C}P^{n-1}$ only under some conditions.

Keywords Conformal minimal surface · Isotropy order · Constant curvature · Linearly full

Mathematics Subject Classification 53C42 · 53C55

1 Introduction

It is an interesting problem in differential geometry to study the geometry of minimal surfaces in symmetric spaces. This problem is extensively investigated in the literatures by using various methods (see the seminal work of Calabi [4], Burstall and Wood [3], Uhlenbeck [13]). In this paper, we consider conformal minimal immersions from S^2 into the complex hyperquadric Q_{n-2}.

Minimal two-spheres with constant curvature in $\mathbb{C}P^n$ were classified in [2]. The similar classification problems in complex Grassmannian $G(k, n; \mathbb{C})$ and the complex hyperquadric Q_n are sophisticated. In 2013, Jiao and Wang proved that every minimal surface of constant Kaehler angle in Q_2 is holomorphic, anti-holomorphic, or totally real and classified all the totally real conformal minimal two-spheres with constant curvature in Q_2, Q_3, Q_4 and Q_5 ([10] and [14]). In 2014, Li, Jiao and He [11] gave a classification theorem of linearly full totally unramified conformal minimal immersions with constant curvature from S^2 to Q_3. In 2016, Peng, Wang and Xu [12] determined all minimal homogeneous two-spheres in Q_n, and they also calculated their geometric quantities: the Gauss curvature, the cosine of K*ä*hler angle and the length of their second fundamental form. Recently, We [7] classified the linearly full irreducible conformal minimal immersions with constant curvature from S^2 to Q_5 under some conditions. We also completely determined the Gaussian curvature of all linearly full totally unramified irreducible and all linearly full reducible conformal minimal immersions from S^2 to Q_5 with constant curvature. For reducible case, we gave some examples, up to SO(7) equivalence, in which none

Communicated by Rahul Roy.

H. Li
Department of Mathematics, Yunnan Normal University, Kunming 650500, China
E-mail: lihong16@mails.ucas.ac.cn

X. Jiao (✉)
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
E-mail: xxjiao@ucas.ac.cn

of the spheres are congruent, with the same Gaussian curvature. For general linearly full totally unramified conformal minimal two-spheres immersed in the complex hyperquadric Q_{n-2}, in 2017, Jiao and Li determined all linearly full reducible conformal minimal immersions from S^2 to Q_{n-2} with constant curvature of isotropy order $r \geqslant n-4$ and all linearly full totally unramified irreducible conformal minimal immersions from S^2 to Q_{n-2} with constant curvature of finite isotropy order $r \geqslant n-5$ (see Theorem 4.6 in [8]). A natural and important question is to classify linearly full reducible conformal minimal immersions from S^2 to Q_{n-2} with constant curvature of isotropy order $r \leqslant n-5$ and linearly full totally unramified irreducible conformal minimal immersions from S^2 to Q_{n-2} with constant curvature of isotropy order $r \leqslant n-6$. But answering this question is very difficult. Now we have given the explicit expressions of linearly full irreducible conformal minimal immersions from S^2 to Q_{n-2} of lower isotropy order $r = n-6$. Compare with irreducible conformal minimal immersions from S^2 to Q_{n-2} of isotropy order $r \geqslant n-5$ in [8], the expressions of $r = n-6$ have two cases while the expressions of $r \geqslant n-5$ just have one case. Jiao and Li [8] showed that all linearly full totally unramified irreducible conformal minimal immersions from S^2 to Q_{n-2} with constant curvature of isotropy order $r \geqslant n-5$ can be expressed by Veronese surfaces in $\mathbb{C}P^{n-1}$, while the immersions of $r = n-6$ can be expressed by Veronese surfaces in $\mathbb{C}P^{n-1}$ only under some conditions.

It is well known that $G(2, n; \mathbb{R})$ can be identified with the complex hyperquadric Q_{n-2} in $\mathbb{C}P^{n-1}$ ([15]). In [1], Bahy-El-Dien and Wood gave the construction of all harmonic two-spheres in $G(2, n; \mathbb{R})$ of finite isotropy order. In this paper, we classify the conformal minimal immersions from S^2 to $G(2, n; \mathbb{R})$ with constant curvature of isotropy order $r = n-6$ by theory of harmonic maps.

This paper is organized as below. In the second section of this paper, firstly we identify Q_{n-2} with $G(2, n; \mathbb{R})$, then we collect some fundamental formulas and facts concerning $G(k, n; \mathbb{C})$ from the view of harmonic sequences, finally we review Veronese sequence and the rigidity theorem of conformal minimal immersions with constant curvature from S^2 to $\mathbb{C}P^n$. In the third section, we apply Bahy-El-Dien and Wood's (see [1]) results to study some properties of the harmonic sequence generated by a harmonic map from S^2 to $G(2, n; \mathbb{R})$ and obtain some characteristics of the corresponding harmonic map in $G(2, n; \mathbb{R})$. In the last section, we study the geometry of linearly full irreducible conformal minimal 2-spheres immersed in $G(2, n; \mathbb{R})$ ($n \geqslant 7$) with constant curvature of isotropy order $r = n-6$ and give a classification theorem of them under some conditions (see Theorem 1).

2 Preliminaries

(**A**) Let $G(k, n; \mathbb{R})$ (resp., $G(k, n; \mathbb{C})$) be the real (resp., complex) Grassmann manifold consisting of all real (resp., complex) k-dimensional subspaces in $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) and

$$\tau : G(k, n; \mathbb{C}) \to G(k, n; \mathbb{C})$$

denote the complex conjugation of $G(k, n; \mathbb{C})$. It is not difficult to verify τ is an isometry with respect to the standard Riemannian metric of $G(k, n; \mathbb{C})$. Its fixed point set is $G(k, n; \mathbb{R})$, hence, $G(k, n; \mathbb{R})$ is a totally geodesic submanifold of $G(k, n; \mathbb{C})$.

A map

$$f : Q_{n-2} \to G(2, n; \mathbb{R})$$

defined by

$$q \mapsto \frac{\sqrt{-1}}{2} Z \wedge \overline{Z},$$

where $q \in Q_{n-2}$ and Z is a homogeneous coordinate vector of q. This map is obviously well-defined. It's not difficult to check that the map is two-to-one and onto, and it is an isometry. Thus we can identify Q_{n-2} with $G(2, n; \mathbb{R})$. Here, suppose that the metric on $G(2, n; \mathbb{R})$ is given by Section 2 of [6], then the metric is twice as much as the standard metric on Q_{n-2} induced by the inclusion map $\sigma : Q_{n-2} \to \mathbb{C}P^{n-1}$, where the latter space is given the Fubini-Study metric of constant holomorphic sectional curvature 4.

(**B**) In this section, we review some fundamental formulas and facts concerning $G(k, n; \mathbb{C})$ from the view of harmonic sequences.

Let M be a Riemann surface and let $\phi : M \to G(k, n; \mathbb{C})$ be a map. We shall always use one-to-one correspondence between maps $\phi : M \to G(k, n; \mathbb{C})$ and rank k subbundles $\underline{\phi}$ of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$

given by setting the fibre $\underline{\phi}_x = \phi(x)$ for all $x \in M$. Then $\underline{\phi}$ is called a harmonic subbundle of $\underline{\mathbb{C}}^n$ whenever ϕ is a harmonic map (see [3]).

Let $(z, \overline{z})$ be a complex coordinate on M. We take the metric $ds_M^2 = dzd\overline{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}.$$

Let $\underline{\varphi}$, $\underline{\psi}$ be two subbundles of $\underline{\mathbb{C}}^n$ which are perpendicular to each other. Under the local coordinate z, we define ∂'-second fundamental form (for more details see [1]) $A'_{\underline{\varphi},\underline{\psi}} : \underline{\varphi} \to \underline{\psi}$ by $A'_{\underline{\varphi},\underline{\psi}}(\nu) = \pi_{\underline{\psi}}(\partial\nu)$, where ν is a local smooth vector field of $\underline{\varphi}$, $\pi_{\underline{\psi}}$ denotes orthogonal projection onto $\underline{\psi}$. Similarly we have ∂''-second fundamental form $A''_{\underline{\varphi},\underline{\psi}}$. In particular, $A'_{\underline{\varphi}} = A'_{\underline{\varphi},\underline{\varphi}^\perp}$, $A''_{\underline{\varphi}} = A''_{\underline{\varphi},\underline{\varphi}^\perp}$ denote the second fundamental forms of $\underline{\varphi}$ in $\underline{\mathbb{C}}^n$. The second fundamental forms $A'_{\underline{\varphi}}$ and $A''_{\underline{\varphi}}$ represent the holomorphic (i.e., (1, 0)) and antiholomorphic (i.e., (0, 1)) parts of the differential of φ (see [3]).

Let $\phi : S^2 \to G(k, n; \mathbb{C})$ be a smooth harmonic map. Then using ϕ, two harmonic sequences are derived as follows:

$$\underline{\phi} = \underline{\phi}_0 \xrightarrow{\partial'} \underline{\phi}_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\phi}_\alpha \xrightarrow{\partial'} \cdots, \tag{1}$$

$$\underline{\phi} = \underline{\phi}_0 \xrightarrow{\partial''} \underline{\phi}_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} \underline{\phi}_{-\alpha} \xrightarrow{\partial''} \cdots, \tag{2}$$

where $\underline{\phi}_\alpha = \partial'\underline{\phi}_{\alpha-1} = \underline{ImA'_{\underline{\phi}_{\alpha-1}}}$ and $\underline{\phi}_{-\alpha} = \partial''\underline{\phi}_{-\alpha+1} = \underline{ImA''_{\underline{\phi}_{-\alpha+1}}}$ are harmonic subbundles of $\underline{\mathbb{C}}^n$ respectively, $\alpha = 1, 2, \cdots$.

We call a harmonic map $\phi : S^2 \to G(k, n; \mathbb{C})$ strongly isotropic if $\phi_\alpha \perp \phi$, $\forall \alpha \in \mathbb{Z}$, $\alpha \neq 0$.

Consider a harmonic map $\phi : S^2 \to G(k, n; \mathbb{C})$, define its isotropy order (see [3]) to be the greatest integer r such that $\phi_\alpha \perp \phi$ for all α with $1 \leqslant \alpha \leqslant r$; if ϕ is strongly isotropic, then we write $r = \infty$.

Let $\phi : S^2 \to G(k, n; \mathbb{C})$ be a harmonic map of finite isotropy order r, we set

$$c'_r(\phi) = A'_{\underline{\phi}_r,\underline{\phi}} \circ A'_{\underline{\phi}_{r-1},\underline{\phi}_r} \circ \cdots \circ A'_{\underline{\phi}_1,\underline{\phi}_2} \circ A'_{\underline{\phi},\underline{\phi}_1},$$

then $c'_r(\phi)$ is called the first ∂'-return map of ϕ. Similarly we have first ∂''-return map of ϕ.

Lemma 1 *([1]) Let $\phi : S^2 \to G(k, n; \mathbb{R})$ be a harmonic map of finite isotropy order r. Then r is odd. Further, writing $r = 2s + 1$, we have $0 \leqslant s \leqslant [(n-3)/2]$. (Here [ ] denotes integer part of.)*

Definition 1 Let $\phi : S^2 \to G(k, n; \mathbb{C})$ be a map. ϕ is linearly full if $\underline{\phi}$ cannot be contained in any proper trivial subbundle $S^2 \times \mathbb{C}^m$ of $S^2 \times \mathbb{C}^n$ $(m < n)$.

Let $\phi : S^2 \to G(k, n; \mathbb{C})$ be a linearly full harmonic map and belongs to the following harmonic sequence:

$$\underline{\phi}_0 \xrightarrow{\partial'} \underline{\phi}_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\phi} = \underline{\phi}_\alpha \xrightarrow{\partial'} \underline{\phi}_{\alpha+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\phi}_{\alpha_0} \xrightarrow{\partial'} 0 \tag{3}$$

for $\alpha = 0, \cdots, \alpha_0$. Choose the local unit orthogonal frame $e_1^\alpha, e_2^\alpha, \cdots, e_{k_\alpha}^\alpha$ so that they locally span subbundle $\underline{\phi}_\alpha$ of $S^2 \times \mathbb{C}^n$, where $k_\alpha = \text{rank } \underline{\phi}_\alpha$.

Suppose that $W_\alpha = (e_1^\alpha, e_2^\alpha, \cdots, e_{k_\alpha}^\alpha)$ is an $(n \times k_\alpha)$-matrix. Then we have

$$\begin{aligned} \phi_\alpha &= W_\alpha W_\alpha^*, \\ W_\alpha^* W_\alpha &= I_{k_\alpha \times k_\alpha},\ W_\alpha^* W_{\alpha+1} = 0,\ W_\alpha^* W_{\alpha-1} = 0. \end{aligned} \tag{4}$$

By (4) and a direct calculation we obtain

$$\begin{cases} \partial W_\alpha = W_{\alpha+1}\Omega_\alpha + W_\alpha \Psi_\alpha, \\ \overline{\partial} W_\alpha = -W_{\alpha-1}\Omega_{\alpha-1}^* - W_\alpha \Psi_\alpha^*, \end{cases} \tag{5}$$

where Ω_α is a $(k_{\alpha+1} \times k_\alpha)$-matrix and Ψ_α is a $(k_\alpha \times k_\alpha)$-matrix.

Evidently, integrability conditions for (5) are

$$\begin{cases} \overline{\partial}\Omega_\alpha = \Psi^*_{\alpha+1}\Omega_\alpha - \Omega_\alpha\Psi^*_\alpha, \\ \overline{\partial}\Psi_\alpha + \partial\Psi^*_\alpha = \Omega^*_\alpha\Omega_\alpha + \Psi^*_\alpha\Psi_\alpha - \Omega_{\alpha-1}\Omega^*_{\alpha-1} - \Psi_\alpha\Psi^*_\alpha. \end{cases} \tag{6}$$

Set $L_\alpha = tr(\Omega_\alpha\Omega^*_\alpha)$. By a straightforward computation, the metric induced by ϕ_α is given by

$$ds^2_\alpha = (L_{\alpha-1} + L_\alpha)dzd\overline{z}. \tag{7}$$

The Laplacian Δ_α and the curvature K_α of ds^2_α are given by

$$\Delta_\alpha = \frac{4}{L_{\alpha-1}+L_\alpha}\partial\overline{\partial}, \quad K_\alpha = -\frac{2}{L_{\alpha-1}+L_\alpha}\partial\overline{\partial}\log(L_{\alpha-1}+L_\alpha). \tag{8}$$

Set

$$\delta_\alpha = \frac{1}{2\pi\sqrt{-1}}\int_{S^2} L_\alpha d\overline{z}\wedge dz. \tag{9}$$

Definition 2 If $det(\Omega_\alpha\Omega^*_\alpha)dz^{k_{\alpha+1}}d\overline{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (3), we call that $\phi_\alpha : S^2 \to G(k, n; \mathbb{C})$ is unramified. If $det(\Omega_\alpha\Omega^*_\alpha)dz^{k_{\alpha+1}}d\overline{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (1) (resp. (2)) for each $\alpha = 0, 1, \cdots$, we say that the harmonic sequence (1) (resp. (2)) is totally unramified. If (1) and (2) are both totally unramified, we say that ϕ_α is totally unramified.

We recall ([3], Section 3A) that a harmonic map $\phi : S^2 \to G(k, n; \mathbb{C})$ in (1) (resp. (2)) is said to be ∂'-irreducible (resp. ∂''-irreducible) if rank $\underline{\phi}$ = rank $\underline{\phi}_1$ (resp. rank $\underline{\phi}$ = rank $\underline{\phi}_{-1}$), and ∂'-reducible (resp. ∂''-reducible) otherwise. In particular, let ϕ be a harmonic map from S^2 to $G(2, n; \mathbb{R})$, then ϕ is ∂'-irreducible (resp. ∂''-reducible) if and only if ϕ is ∂''-irreducible (resp. ∂'-reducible). In this case, we simply call that ϕ is irreducible (resp. reducible). If ϕ_α in (3) is unramified and ∂'-irreducible, then $|det\Omega_\alpha|^2dz^{k_\alpha}d\overline{z}^{k_\alpha}$ is a well-defined invariant and has no zeros on S^2, then we have

$$\frac{1}{2\pi\sqrt{-1}}\int_{S^2}\partial\overline{\partial}\log|det\Omega_\alpha|^2 d\overline{z}\wedge dz = -2k_\alpha. \tag{10}$$

Lemma 2 *([5]) Let $f : S^2 \to N$ be a conformal immersion, where N is a Riemannian manifold, then f is minimal if and only if f is harmonic.*

(**C**) We recall in this section the rigidity theorem of conformal minimal immersions with constant curvature from S^2 to $\mathbb{C}P^n$.

Let $\varphi : S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion, a harmonic sequence is derived as follows

$$0 \xrightarrow{\partial'} \underline{\varphi}^{(n)}_0 \xrightarrow{\partial'} \underline{\varphi}^{(n)}_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi} = \underline{\varphi}^{(n)}_q \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi}^{(n)}_n \xrightarrow{\partial'} 0, \tag{11}$$

for some $q = 0, 1, \cdots, n$.

Define a sequence $f^{(n)}_0, \cdots, f^{(n)}_n$ of local sections of $\underline{\varphi}^{(n)}_0, \cdots, \underline{\varphi}^{(n)}_n$ inductively so that $f^{(n)}_0$ is a nowhere zero local section of $\underline{\varphi}^{(n)}_0$ (we assume without loss of generality that $\overline{\partial}f^{(n)}_0 \equiv 0$) and $f^{(n)}_{q+1} = \pi_{\underline{\varphi}^{(n)\perp}_q}(\partial f^{(n)}_q)$ for $q = 0, 1, \cdots, n-1$. Then we have some formulas as follows:

$$\partial f^{(n)}_q = f^{(n)}_{q+1} + \frac{\langle\partial f^{(n)}_q, f^{(n)}_q\rangle}{|f^{(n)}_q|^2}f^{(n)}_q, \qquad q = 0, 1, \cdots, n-1.$$

$$\overline{\partial}f^{(n)}_q = -\frac{|f^{(n)}_q|^2}{|f^{(n)}_{q-1}|^2}f^{(n)}_{q-1}, \qquad q = 1, 2, \cdots, n.$$

Define $l^{(n)}_q$ by putting

$$l^{(n)}_q = \frac{|f^{(n)}_{q+1}|^2}{|f^{(n)}_q|^2}, \qquad q = 0, 1, \cdots, n-1, \qquad l^{(n)}_{-1} = l^{(n)}_n = 0. \tag{12}$$

Then Bolton et al ([2]) proved the following unintegrated *Plücker* formula

$$\partial\overline{\partial}\log l_q^{(n)} = l_{q+1}^{(n)} - 2l_q^{(n)} + l_{q-1}^{(n)}, \qquad q = 0, 1, \cdots, n-1.$$

Definition 3 The degree $\sigma_q^{(n)}$ of $f_q^{(n)}$ is defined by $\sigma_q^{(n)} = 1/(2\pi\sqrt{-1})\int_{S^2}\partial\overline{\partial}\log|f_q^{(n)}|^2 d\overline{z}\wedge dz$, where $q = 0, 1, 2, \cdots, n$.

Let $F_q^{(n)} = f_0^{(n)}\wedge\cdots\wedge f_q^{(n)}$ be a local lift of the q-th osculating curve, where $q = 0, \cdots, n$. We write $F_q^{(n)} = f(z)\tilde{F}_q^{(n)}$, where $f(z)$ is the greatest common divisor of the $\binom{n+1}{q+1}$ components of $F_q^{(n)}$. Then $\tilde{F}_q^{(n)}$ is a nowhere zero holomorphic curve, and the degree $\delta_q^{(n)}$ of $F_q^{(n)}$ is given by $\delta_q^{(n)} = 1/(2\pi\sqrt{-1})\int_{S^2}\partial\overline{\partial}\log|F_q^{(n)}|^2 d\overline{z}\wedge dz$, which is equal to the degree of the polynomial function $\tilde{F}_q^{(n)}$. By a simple calculation we have

$$\delta_q^{(n)} = \frac{1}{2\pi\sqrt{-1}}\int_{S^2} l_q^{(n)} d\overline{z}\wedge dz, \tag{13}$$

which is consistent with (9) in the case $k = 1$.

Moreover, if (11) is a totally unramified harmonic sequence, then (see [2])

$$\delta_q^{(n)} = (q+1)(n-q). \tag{14}$$

Let

$$0 \xrightarrow{\partial'} \underline{V}_0^{(n)} \xrightarrow{\partial'} \underline{V}_1^{(n)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{V}_n^{(n)} \xrightarrow{\partial'} 0,$$

which is called the Veronese sequence, defined by $V_q^{(n)} = (v_{q,0}, \cdots, v_{q,n})^T$, where, for $z \in S^2$,

$$v_{q,r}(z) = \frac{q!}{(1+z\overline{z})^q}\sqrt{\binom{n}{r}}z^{r-q}\sum_k(-1)^k\binom{r}{q-k}\binom{n-r}{k}(z\overline{z})^k,$$

$\max\{0, q-r\} \leqslant k \leqslant \min\{q, n-r\}$, and $|V_q^{(n)}|^2 = (n!q!/(n-q)!)(1+z\overline{z})^{n-2q}$. Each map $V_q^{(n)}$ has induced metric

$$ds_q^2 = \frac{n+2q(n-q)}{(1+z\overline{z})^2}dzd\overline{z}, \tag{15}$$

the corresponding constant curvature K_q is given by

$$K_q = \frac{4}{n+2q(n-q)}. \tag{16}$$

From the rigidity theorem of Calabi, Bolton et al proved the following rigidity result (see [2]).

Lemma 3 *([2]) Let $\varphi : S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion with constant curvature. Then, up to a holomorphic isometry of $\mathbb{C}P^n$, the harmonic sequence determined by φ is the Veronese sequence.*

3 Characterization of irreducible harmonic maps from S^2 to $G(2, n; \mathbb{R})$

It follows from [8] that all linearly full totally unramified irreducible harmonic maps from S^2 to Q_{n-2} with constant curvature of finite isotropy order $r \geqslant n-5$ have been characterized. In the following of this section we investigate the irreducible harmonic maps $\phi : S^2 \to G(2, n; \mathbb{R})$ $(n \geqslant 7)$ of isotropy order $r = n-6$. By Lemma 1, we have that n is odd. Theorem 4.7 of [1] implies that

Lemma 4 *Let $\phi : S^2 \to G(2, n; \mathbb{R})$ $(n \geqslant 7)$ be an irreducible harmonic map of isotropy order $r = n - 6$. Then there is a unique sequence of harmonic maps $\phi^i : S^2 \to G(2, n; \mathbb{C})$ $(i = 0, 1, 2, \cdots, 2l)$, where $l \in \{1, 2\}$ such that*

(1) $\underline{\phi}^0$ is real mixed pair, in fact $\underline{\phi}^0 = \underline{f}_0^{(m)} \oplus \underline{\overline{f}}_0^{(m)}$, where $f_0^{(m)} \in H_n^{l+(n-7)/2}$;

(2) $\underline{\phi}^{2l} = \underline{\phi}$;

(3) for $j = 0, 1, \cdots, l-1$, $\underline{\phi}^{2j+1}$ is obtained from $\underline{\phi}^{2j}$ by forward replacement of the image $\underline{\delta}_j$ of the first ∂''-return map of ϕ^{2j} viz

$$\underline{\phi}^{2j+1} = \underline{\delta}_j^{\perp} \cap \underline{\phi}^{2j} \oplus \underline{Im}(A'_{\underline{\phi}^{2j}}|\underline{\delta}_j);$$

(4) for $j = 0, 1, \cdots, l-1$, $\underline{\phi}^{2j+2}$ is obtained from $\underline{\phi}^{2j+1}$ by backward replacement of $\underline{\gamma}_{j+1}^{\perp} \cap \underline{\phi}^{2j+1}$, where $\underline{\gamma}_{j+1}$ is a holomorphic line subbundle of $\underline{\phi}^{2j+1}$ not equal to the image of the first ∂'-return map of ϕ^{2j+1}.

Here, $H_n^{l+(n-7)/2}$ denote the set of all holomorphic maps $f_0^{(m)} : S^2 \to \mathbb{C}P^m \subset \mathbb{C}P^{n-1}$, $2l + n - 5 \leqslant m \leqslant n - 1$ satisfying

$$\begin{cases} \langle f_i^{(m)}, \overline{f}_0^{(m)} \rangle = 0, \ (0 \leqslant i \leqslant 2l + n - 6), \\ \langle f_{2l+n-5}^{(m)}, \overline{f}_0^{(m)} \rangle \neq 0, \end{cases}$$

where $0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{2l+n-6}^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0$ *is a harmonic sequence in* $\mathbb{C}P^m \subset \mathbb{C}P^{n-1}$.

Let $\phi : S^2 \to G(2, n; \mathbb{R})$ $(n \geqslant 7)$ be a linearly full irreducible harmonic map of isotropy order $r = n - 6$. In the following we apply Lemma 4 to characterize ϕ explicitly. By Lemma 4, we need to consider two cases: $l = 1$ and $l = 2$.

(**1**) When $l = 1$, $i = 0, 1, 2$. $\underline{\phi}^0$ with isotropy order $n - 4$ belongs to the following harmonic sequence:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_m^{(m)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \underline{\overline{f}}_1^{(m)} \xleftarrow{\partial''} \underline{\phi}^0 \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0, \tag{17}$$

where $\underline{\phi}^0 = \underline{f}_0^{(m)} \oplus \underline{\overline{f}}_0^{(m)}$, $n - 3 \leqslant m \leqslant n - 1$, $f_0^{(m)} \in H_n^{(n-5)/2}$ and

$$0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \underline{f}_2^{(m)} \xrightarrow{\partial'} \underline{f}_3^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0, \tag{18}$$

(18) is a harmonic sequence in $\mathbb{C}P^m \subset \mathbb{C}P^{n-1}$. Since $f_0^{(m)} \in H_n^{(n-5)/2}$, we obtain

$$\begin{cases} \langle f_i^{(m)}, \overline{f}_0^{(m)} \rangle = 0, \ (0 \leqslant i \leqslant n - 4), \\ \langle f_{n-3}^{(m)}, \overline{f}_0^{(m)} \rangle \neq 0. \end{cases} \tag{19}$$

By (3) of Lemma 4, $\underline{\phi}^1$ is obtained from $\underline{\phi}^0$ by forward replacement of $\underline{f}_0^{(m)}$. Using (17) we have

$$\underline{\phi}^1 = \underline{\overline{f}}_0^{(m)} \oplus \underline{f}_1^{(m)}.$$

The isotropy order of ϕ^1 is $n - 5$, and a harmonic sequence is derived as follows:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_m^{(m)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \underline{\overline{f}}_2^{(m)} \xleftarrow{\partial''} \underline{\overline{\phi}}^1 = \underline{\overline{f}}_1^{(m)} \oplus \underline{f}_0^{(m)} \xleftarrow{\partial''} \underline{\phi}^1 \xrightarrow{\partial'} \underline{f}_2^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0. \tag{20}$$

From (20), the image of the first ∂'-return map of ϕ^1 is $\underline{\overline{f}}_0^{(m)}$. By (4) of Lemma 4, let $V = f_1^{(m)} + x_0 \overline{f}_0^{(m)}$, where x_0 is a smooth function on S^2 except some isolated points. Since $\underline{V}$ is a holomorphic line subbundle of $\underline{\phi}^1$, we have

$$\pi_{\underline{\phi}^1} \overline{\partial} V \in \underline{V}. \tag{21}$$

By a straightforward calculation, (21) is equivalent to the following equation

$$\overline{\partial}(x_0 |f_0^{(m)}|^2) = 0. \tag{22}$$

Set

$$h = x_0 |f_0^{(m)}|^2, \tag{23}$$

(22) is equivalent to the following equation

$$\overline{\partial} h = 0. \tag{24}$$

Let $X = -\overline{x}_0 |f_0^{(m)}|^2 f_1^{(m)} + |f_1^{(m)}|^2 \overline{f}_0^{(m)}$, it satisfies $\underline{X} = \underline{V}^{\perp} \cap \underline{\phi}^1$. A direct computation yields $\underline{Im}(A''_{\underline{\phi}^1}|\underline{X}) = \overline{\underline{V}}$. Since $\underline{\phi}^2$ is obtained from $\underline{\phi}^1$ by backward replacement of $\underline{X}$, we have $\underline{\phi}^2 = \underline{V} \oplus \overline{\underline{V}}$. Moreover, $\underline{\phi} = \underline{\phi}^2$ with isotropy order $n-6$ belongs to the following harmonic sequence:

$$0 \xleftarrow{\partial''} \overline{\underline{f}}_m^{(m)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \overline{\underline{f}}_3^{(m)} \xleftarrow{\partial''} \overline{\underline{X}} \oplus \overline{\underline{f}}_2^{(m)} \xleftarrow{\partial''} \underline{\phi} \xrightarrow{\partial'} \underline{X} \oplus \underline{f}_2^{(m)} \xrightarrow{\partial'} \underline{f}_3^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0. \tag{25}$$

Therefore we have

Proposition 1 *The map $\phi : S^2 \to G(2, n; \mathbb{R})$ $(n \geqslant 7)$ is a linearly full irreducible harmonic map with isotropy order $r = n-6$ and $l = 1$ if and only if n is odd, and $\underline{\phi} = \overline{\underline{V}} \oplus \underline{V}$ with $V = f_1^{(m)} + x_0 \overline{f}_0^{(m)}$, where $f_0^{(m)} \in H_n^{(n-5)/2}$ and the corresponding coefficient x_0 satisfies (22).*

Proof It follows from the construction of ϕ as shown above that the necessity is obvious. Since $f_0^{(m)} \in H_n^{(n-5)/2}$, then by (22), we have $(I - 2\phi)\partial\overline{\partial}\phi = \partial\phi\overline{\partial}\phi + \overline{\partial}\phi\partial\phi$, which implies that ϕ is harmonic (see Proposition 3.2 in [9]). Thus the sufficiency follows. □

(**2**) When $l = 2,\ i = 0, 1, 2, 3, 4$. $\underline{\phi}^0$ with isotropy order $n-2$ belongs to the following harmonic sequence:

$$0 \xleftarrow{\partial''} \overline{\underline{f}}_{n-1}^{(n-1)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \overline{\underline{f}}_1^{(n-1)} \xleftarrow{\partial''} \underline{\phi}^0 \xrightarrow{\partial'} \underline{f}_1^{(n-1)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{n-1}^{(n-1)} \xrightarrow{\partial'} 0, \tag{26}$$

where $\underline{\phi}^0 = \underline{f}_0^{(n-1)} \oplus \overline{\underline{f}}_0^{(n-1)}$, $f_0^{(n-1)} \in H_n^{(n-3)/2}$ and

$$0 \xrightarrow{\partial'} \underline{f}_0^{(n-1)} \xrightarrow{\partial'} \underline{f}_1^{(n-1)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{n-1}^{(n-1)} \xrightarrow{\partial'} 0 \tag{27}$$

is a harmonic sequence in $\mathbb{C}P^{n-1}$. Since $f_0^{(n-1)} \in H_n^{(n-3)/2}$, we obtain

$$\begin{cases} \langle f_i^{(n-1)}, \overline{f}_0^{(n-1)} \rangle = 0,\ (0 \leqslant i \leqslant n-2), \\ \langle f_{n-1}^{(n-1)}, \overline{f}_0^{(n-1)} \rangle \neq 0. \end{cases} \tag{28}$$

Thus we have

$$\overline{\underline{f}}_0^{(n-1)} = \underline{f}_{n-1}^{(n-1)},\ \overline{\underline{f}}_1^{(n-1)} = \underline{f}_{n-2}^{(n-1)},\ \cdots,\ \overline{\underline{f}}_{(n-1)/2}^{(n-1)} = \underline{f}_{(n-1)/2}^{(n-1)},$$

and

$$l_0^{(n-1)} = l_{n-2}^{(n-1)},\ l_1^{(n-1)} = l_{n-3}^{(n-1)},\ \cdots,\ l_{(n-3)/2}^{(n-1)} = l_{(n-1)/2}^{(n-1)}. \tag{29}$$

By (3) of Lemma 4, $\underline{\phi}^1$ is obtained from $\underline{\phi}^0$ by forward replacement of $\underline{f}_0^{(n-1)}$, using (26) we have $\underline{\phi}^1 = \overline{\underline{f}}_0^{(n-1)} \oplus \underline{f}_1^{(n-1)}$. The isotropy order of ϕ^1 is $n-3$, and a harmonic sequence is derived as follows:

$$0 \xleftarrow{\partial''} \overline{\underline{f}}_{n-1}^{(n-1)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \overline{\underline{f}}_2^{(n-1)} \xleftarrow{\partial''} \overline{\underline{\phi}}^1 = \overline{\underline{f}}_1^{(n-1)} \oplus \underline{f}_0^{(n-1)} \xleftarrow{\partial''} \underline{\phi}^1 \xrightarrow{\partial'} \underline{f}_2^{(n-1)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{n-1}^{(n-1)} \xrightarrow{\partial'} 0. \tag{30}$$

From (30), the image of the first ∂'-return map of ϕ^1 is $\overline{\underline{f}}_0^{(n-1)}$. By (4) of Lemma 4, let $V = f_1^{(n-1)} + x_0 \overline{f}_0^{(n-1)}$, where x_0 is a smooth function on S^2 except some isolated points. Since $\underline{V}$ is a holomorphic line subbundle of $\underline{\phi}^1$, we have

$$\pi_{\underline{\phi}^1} \overline{\partial} V \in \underline{V}. \tag{31}$$

A straightforward calculation shows that (31) is equivalent to

$$\overline{\partial}(x_0|f_0^{(n-1)}|^2)=0. \tag{32}$$

Set

$$h=x_0|f_0^{(n-1)}|^2, \tag{33}$$

(31) is equivalent to

$$\overline{\partial}h=0. \tag{34}$$

Let $X=-\overline{x}_0|f_0^{(n-1)}|^2f_1^{(n-1)}+|f_1^{(n-1)}|^2\overline{f}_0^{(n-1)}$, it satisfies $\underline{X}=\underline{V}^{\perp}\cap\underline{\phi}^1$. By a direct computation, we have $\underline{Im}(A''_{\underline{\phi}^1}|\underline{X}))=\underline{\overline{V}}$. Since $\underline{\phi}^2$ is obtained from $\underline{\phi}^1$ by backward replacement of $\underline{X}$, we have $\underline{\phi}^2=\underline{V}\oplus\underline{\overline{V}}$. Moreover, $\underline{\phi}^2$ with isotropy order $n-4$ belongs to the following harmonic sequence:

$$0\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_{n-1}^{(n-1)}\overset{\partial''}{\longleftarrow}\cdots\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_3^{(n-1)}\overset{\partial''}{\longleftarrow}\underline{\overline{X}}\oplus\underline{\overline{f}}_2^{(n-1)}\overset{\partial''}{\longleftarrow}\underline{\phi}^2\overset{\partial'}{\longrightarrow}\underline{X}\oplus\underline{f}_2^{(n-1)}\overset{\partial'}{\longrightarrow}\underline{f}_3^{(n-1)}\overset{\partial'}{\longrightarrow}\cdots\overset{\partial'}{\longrightarrow}\underline{f}_{n-1}^{(n-1)}\overset{\partial'}{\longrightarrow}0. \tag{35}$$

From (35), the image of the first ∂''-return map of ϕ^2 is $\underline{V}$. By (3) of Lemma 4, $\underline{\phi}^3$ is obtained from $\underline{\phi}^2$ by forward replacement of $\underline{V}$. Since $\underline{Im}(A'_{\underline{\phi}^2}|\underline{V})=\underline{Z}$, where $Z=f_2^{(n-1)}+(C/|V|^2)X$, $C=\partial x_0-x_0\partial\log|f_1^{(n-1)}|^2$, we have $\underline{\phi}^3=\underline{\overline{V}}\oplus\underline{Z}$. Moreover, $\underline{\phi}^3$ with isotropy order $n-5$ belongs to the harmonic sequence as follows:

$$\begin{aligned}&0\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_{n-1}^{(n-1)}\overset{\partial''}{\longleftarrow}\cdots\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_4^{(n-1)}\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_3^{(n-1)}\oplus\underline{\overline{Y}}_1\overset{\partial''}{\longleftarrow}\underline{\overline{\phi}}^3\overset{\partial''}{\longleftarrow}\underline{\phi}^3\overset{\partial'}{\longrightarrow}\underline{f}_3^{(n-1)}\\&\oplus\underline{Y}_1\overset{\partial'}{\longrightarrow}\underline{f}_4^{(n-1)}\overset{\partial'}{\longrightarrow}\cdots\overset{\partial'}{\longrightarrow}\underline{f}_{n-1}^{(n-1)}\overset{\partial'}{\longrightarrow}0,\end{aligned} \tag{36}$$

where $\underline{\overline{\phi}}^3=\underline{V}\oplus\underline{\overline{Z}}$, $Y_1=\overline{C}|f_0^{(n-1)}|^2|f_1^{(n-1)}|^2f_2^{(n-1)}-|f_2^{(n-1)}|^2X$. From (36), the image of the first ∂'-return map of ϕ^3 is $\underline{\overline{V}}$. By (4) of Lemma 4, let $V_1=Z+x_1\overline{V}$, where x_1 is a smooth function on S^2 except some isolated points. Since $\underline{V}_1$ is a holomorphic line subbundle of $\underline{\phi}^3$, we obtain

$$\pi_{\underline{\phi}^3}\overline{\partial}V_1\in\underline{V}_1. \tag{37}$$

Through a straightforward calculation, (37) is equivalent to

$$\overline{\partial}(x_1|V|^2)+l_0^{(n-1)}(\partial h-h\partial\log|f_0^{(n-1)}|^2-h\partial\log|f_1^{(n-1)}|^2)=0. \tag{38}$$

Moreover, let $X_1=-\overline{x}_1|V|^2Z+|Z|^2\overline{V}$, it satisfies $\underline{X}_1=\underline{V}_1^{\perp}\cap\underline{\phi}^3$. Through a direct computation, we have $\underline{Im}(A''_{\underline{\phi}^3}|\underline{X}_1)=\underline{\overline{V}}_1$. Since $\underline{\phi}^4$ is obtained from $\underline{\phi}^3$ by backward replacement of $\underline{X}_1$, we have $\underline{\phi}^4=\underline{V}_1\oplus\underline{\overline{V}}_1$. Moreover, $\underline{\phi}=\underline{\phi}^4$ with isotropy order $n-6$ belongs to the following harmonic sequence:

$$0\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_{n-1}^{(n-1)}\overset{\partial''}{\longleftarrow}\cdots\overset{\partial''}{\longleftarrow}\underline{\overline{f}}_5^{(n-1)}\overset{\partial''}{\longleftarrow}\underline{\overline{\phi}}_2\overset{\partial''}{\longleftarrow}\underline{\overline{\phi}}_1\overset{\partial''}{\longleftarrow}\underline{\phi}\overset{\partial'}{\longrightarrow}\underline{\phi}_1\overset{\partial'}{\longrightarrow}\underline{\phi}_2\overset{\partial'}{\longrightarrow}\underline{f}_5^{(n-1)}\overset{\partial'}{\longrightarrow}\cdots\overset{\partial'}{\longrightarrow}\underline{f}_{n-1}^{(n-1)}\rightarrow 0, \tag{39}$$

where $\underline{\phi}_1=\underline{X}_1\oplus\underline{Y}_2$, $\underline{\phi}_2=\underline{f}_4^{(n-1)}\oplus\underline{Y}_3$, $Y_2=f_3^{(n-1)}-\frac{D}{|Z|^2}Y_1$, $D=(\partial C-C\partial\log|f_2^{(n-1)}|^2)/|V|^2-x_1/|f_0^{(n-1)}|^2$ and $Y_3=\overline{D}|f_2^{(n-1)}|^2|X|^2f_3^{(n-1)}+|f_3^{(n-1)}|^2Y_1$.

Therefore we have

Proposition 2 *The map $\phi:S^2\to G(2,n;\mathbb{R})$ $(n\geqslant 7)$ is a linearly full irreducible harmonic map with isotropy order $r=n-6$ and $l=2$ if and only if n is odd, and $\underline{\phi}=\underline{\overline{V}}_1\oplus\underline{V}_1$ with $V_1=f_2^{(n-1)}+\frac{\partial x_0-x_0\partial\log|f_1^{(n-1)}|^2}{|f_1^{(n-1)}|^2+|x_0|^2|f_0^{(n-1)}|^2}(-\overline{x}_0|f_0^{(n-1)}|^2f_1^{(n-1)}+|f_1^{(n-1)}|^2\overline{f}_0^{(n-1)})+x_1(\overline{f}_1^{(n-1)}+\overline{x}_0f_0^{(n-1)})$, where $f_0^{(n-1)}\in H_7^{(n-3)/2}$ and the corresponding coefficients x_0 and x_1 satisfy equations (32) and (38), respectively.*

Proof It follows from the construction of ϕ as shown above that the necessity is obvious. Since $f_0^{(n-1)}\in H_7^{(n-3)/2}$, then by (32) and (38), we have $(I-2\phi)\partial\overline{\partial}\phi=\partial\phi\overline{\partial}\phi+\overline{\partial}\phi\partial\phi$, which implies that ϕ is harmonic (see Proposition 3.2 in [9]). Thus the sufficiency follows. □

4 Irreducible conformal minimal immersions with constant curvature from S^2 to $G(2, n; \mathbb{R})$

In this section, we can regard harmonic maps from S^2 to $G(2,n;\mathbb{R})$ as conformal minimal immersions of S^2 in $G(2,n;\mathbb{R})$ from Lemma 2. Then we discuss linearly full irreducible harmonic maps from S^2 to $G(2,n;\mathbb{R})$ ($n \geqslant 7$) with constant curvature of isotropy order $r = n-6$ in Section 3.

Let $\phi : S^2 \to G(2,n;\mathbb{R})$ ($n \geqslant 7$) be a linearly full irreducible harmonic map of isotropy order $r = n-6$.

(**1**) When $l = 1$, by Proposition 1 and (25), we choose

$$e_1 = \frac{\overline{V}}{|V|}, e_2 = \frac{V}{|V|}, e_3 = \frac{X}{|X|}, e_4 = \frac{f_2^{(m)}}{|f_2^{(m)}|}, e_5 = \frac{\overline{X}}{|X|}, e_6 = \frac{\overline{f}_2^{(m)}}{|f_2^{(m)}|}, e_7 = \frac{f_3^{(m)}}{|f_3^{(m)}|},$$

where $V = f_1^{(m)} + x_0 \overline{f}_0^{(m)}$ and x_0 is a smooth function on S^2 except some isolated points. Let $W_0 = (e_1, e_2)$, $W_1 = (e_3, e_4)$, $W_{-1} = (e_5, e_6)$ and $W_2 = (e_7)$, using (5), we obtain

$$\Omega_0 = \begin{pmatrix} -\frac{|f_1^{(m)}|}{|f_0^{(m)}|} & \frac{\langle \partial V, X\rangle}{|X||V|} \\ 0 & \frac{|f_2^{(m)}|}{|V|} \end{pmatrix}, \Omega_{-1} = \begin{pmatrix} -\frac{\langle \partial V, X\rangle}{|X||V|} & -\frac{|f_2^{(m)}|}{|V|} \\ \frac{|f_1^{(m)}|}{|f_0^{(m)}|} & 0 \end{pmatrix}, \Omega_1 = \begin{pmatrix} 0 & \frac{|f_3^{(m)}|}{|f_2^{(m)}|} \end{pmatrix}, \tag{40}$$

from (40), by applying $L_\alpha = tr(\Omega_\alpha \Omega_\alpha^*)$ ($\alpha = -1, 0, 1$), a direct calculation shows

$$L_0 = L_{-1} = \frac{\langle \partial V, X\rangle \langle X, \partial V\rangle}{|X|^2|V|^2} + \frac{|f_2^{(m)}|^2}{|V|^2} + l_0^{(m)}, \tag{41}$$

$$L_1 = l_2^{(m)}, \tag{42}$$

$$|det\Omega_0|^2 dz^2 d\overline{z}^2 = \frac{|f_2^{(m)}|^2}{|V|^2} l_0^{(m)} dz^2 d\overline{z}^2, \tag{43}$$

$$det(\Omega_1\Omega_1^*) dz d\overline{z} = l_2^{(m)} dz d\overline{z}. \tag{44}$$

Using (41), (42) and (43), by a direct calculation, we also obtain

$$\partial\overline{\partial} \log |det\Omega_0|^2 = L_{-1} - 2L_0 + L_1. \tag{45}$$

If ϕ is totally unramified, then $|det\Omega_0|^2 dz^2 d\overline{z}^2 \neq 0$, $det(\Omega_1\Omega_1^*) dz d\overline{z} \neq 0$ everywhere on S^2 and $f_i^{(m)} : S^2 \to \mathbb{C}P^m \subset \mathbb{C}P^{n-1}$ ($i = 3, \cdots, m$) are all unramified. It follows from (43) and (44) that $l_p^{(m)} dz d\overline{z} \neq 0$ ($p = 0, 1, 2$) everywhere on S^2. Since $f_i^{(m)} : S^2 \to \mathbb{C}P^m \subset \mathbb{C}P^{n-1}$ ($i = 3, \cdots, m$) are all unramified, we obtain $l_i^{(m)} dz d\overline{z} \neq 0$ ($i = 3, \cdots, m-1$) everywhere on S^2. From the above discussion, we conclude that $l_j^{(m)} dz d\overline{z} \neq 0$ ($j = 0, 1, 2, \cdots, m-1$) everywhere on S^2, i.e., the harmonic sequence

$$0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0 \tag{46}$$

is also totally unramified.

Since $l_0^{(m)} \neq 0$ and $l_1^{(m)} \neq 0$, using (43), we obtain

$$|det\Omega_0|^2 dz^2 d\overline{z}^2 = \frac{|f_0^{(m)}|^2}{|V|^2} (l_0^{(m)})^2 l_1^{(m)} dz^2 d\overline{z}^2. \tag{47}$$

In this case, we prove

Proposition 3 *There does not exist a linearly full totally unramified irreducible harmonic map with constant curvature of isotropy order $r = n-6$ and $l = 1$ from S^2 to $G(2,n;\mathbb{R})$ ($n \geqslant 7$).*

Proof Since the harmonic sequence (46) is totally unramified, it follows from (9), (14), (41) and (42) that

$$\delta_0 = \delta_{-1}, \delta_1 = \delta_2^{(m)} = 3(m-2), \delta_0^{(m)} = m, \delta_1^{(m)} = 2(m-1). \tag{48}$$

From (9), (10) and (45), we obtain

$$\delta_1 - 2\delta_0 + \delta_{-1} = -4, \tag{49}$$

hence, we have

$$\delta_0 = 3m - 2. \tag{50}$$

Since ϕ is of constant curvature K, using (50) we have $K = 2/(3m-2)$, and we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that the induced metric $ds^2 = 2L_0 dz d\overline{z}$ of ϕ is given by

$$ds^2 = \frac{6m-4}{(1+z\overline{z})^2} dz d\overline{z}.$$

So

$$L_0 = \frac{3m-2}{(1+z\overline{z})^2}. \tag{51}$$

Consider the local lift of the p-th osculating curve (see [2]) $F_p^{(m)} = f_0^{(m)} \wedge \cdots \wedge f_p^{(m)}$ $(p = 0, \cdots, m)$. We choose a nowhere zero holomorphic $\mathbb{C}^n$-valued function $f_0^{(m)}$, then $F_p^{(m)}$ is a nowhere zero holomorphic curve of degree $\delta_p^{(m)}$ satisfying $\partial\overline{\partial} \log |F_p^{(m)}|^2 = l_p^{(m)}$ and $|F_p^{(m)}|^2$ is a polynomial function. Thus we can apply (41), (42), (45), (47) and (51) to obtain

$$\partial\overline{\partial} \log \frac{(1+z\overline{z})^{3m-2}}{|f_0^{(m)}|^4 |V|^2} = \partial\overline{\partial} \log \frac{(1+z\overline{z})^{3m-2}}{|F_0^{(m)}|^2 |F_1^{(m)}|^2} \frac{|f_0^{(m)}|^2}{|V|^2} l_0^{(m)} = 0. \tag{52}$$

By (47), we have $|f_0^{(m)}|^2 l_0^{(m)}/|V|^2$ is a globally defined function without zeros on S^2, then it follows from (48) that $(1+z\overline{z})^{3m-2}/(|f_0^{(m)}|^4|V|^2)$ is globally defined on $\mathbb{C}$ and has positive constant limit $1/c$ as $z \to \infty$. Thus from (52) we have

$$\frac{(1+z\overline{z})^{3m-2}}{|f_0^{(m)}|^4 |V|^2} = \frac{1}{c}, \tag{53}$$

so

$$|f_0^{(m)}|^2 |V|^2 = \frac{c(1+z\overline{z})^{3m-2}}{|f_0^{(m)}|^2}. \tag{54}$$

Applying the equation $V = f_1^{(m)} + x_0 \overline{f}_0^{(m)}$, (23) and (54), we obtain

$$|F_1^{(m)}|^2 + |h|^2 = \frac{c(1+z\overline{z})^{3m-2}}{|f_0^{(m)}|^2}. \tag{55}$$

Observing (55) and (24), we can obtain that h is a holomorphic function on $\mathbb{C}$ at most with the pole $z = \infty$. Hence, it is a polynomial function about z. Since $|F_1^{(m)}|^2$ and $|f_0^{(m)}|^2$ are polynomial functions and $\delta_0^{(m)} = m$, we obtain

$$|f_0^{(m)}|^2 = \mu(1+z\overline{z})^m, \tag{56}$$

where μ is a positive parameter.

Since $n-3 \leqslant m \leqslant n-1$ $(n \geqslant 7)$, if $h \neq 0$, then $1 + z\overline{z}$ is a factor of it, which contradicts with the fact that h is a holomorphic polynomial function. Therefore $h = 0$, which means that $x_0 = 0$. Then we obtain

$$V = f_1^{(m)}, \ \underline{\phi} = \underline{\overline{f}}_{-1}^{(m)} \oplus \underline{f}_{-1}^{(m)}.$$

From (56) and Lemma 3, up to a holomorphic isometry of $\mathbb{C}P^{n-1}$, $f_0^{(m)}$ is a Veronese curve. We may choose a complex coordinate z on $\mathbb{C} = S^2\backslash\{pt\}$ such that $f_0^{(m)} = UV_0^{(m)}$, where $U \in U(n)$ and $V_0^{(m)}$ has the standard expression given in part (C) of Section 2 (adding zeros to $V_0^{(m)}$ such that $V_0^{(m)} \in \mathbb{C}^n$). Hence, we have $\underline{\phi} = \underline{\overline{UV}_1^{(m)}} \oplus \underline{UV_1^{(m)}}$. In order to determine ϕ, we only need to determine the matrix U. To do this, consider the linearly full reducible harmonic map $\phi^0 : S^2 \to G(2, n; \mathbb{R})$ $(n \geqslant 7)$, which is of constant curvature and isotropy order $n-4$, where $\underline{\phi}^0 = \underline{\overline{UV}_0^{(m)}} \oplus \underline{UV_0^{(m)}}$. By Proposition 3.2 (III) of [8], we can obtain that there doesn't exist such U, so there does not exist a linearly full totally unramified irreducible harmonic map with constant curvature of isotropy order $r = n-6$ and $l = 1$ from S^2 to $G(2, n; \mathbb{R})$ $(n \geqslant 7)$. □

(**2**) When $l = 2$, by Proposition 2 and (39), we choose

$$e_1 = \frac{\overline{V_1}}{|V_1|}, e_2 = \frac{V_1}{|V_1|}, e_3 = \frac{X_1}{|X_1|}, e_4 = \frac{Y_2}{|Y_2|}, e_5 = \frac{\overline{X}_1}{|X_1|}, e_6 = \frac{\overline{Y_2}}{|Y_2|}, e_7 = \frac{f_4^{(n-1)}}{|f_4^{(n-1)}|},$$
$$e_8 = \frac{Y_3}{|Y_3|}, e_9 = \frac{\overline{f}_4^{(n-1)}}{|f_4^{(n-1)}|}, e_{10} = \frac{\overline{Y_3}}{|Y_3|}, e_{11} = \frac{f_5^{(n-1)}}{|f_5^{(n-1)}|},$$

where $V_1 = Z + x_1\overline{V}$ and x_1 is a smooth function on S^2 except some isolated points. Let $W_0 = (e_1, e_2)$, $W_1 = (e_3, e_4)$, $W_{-1} = (e_5, e_6)$, $W_2 = (e_7, e_8)$, $W_{-2} = (e_9, e_{10})$ and $W_3 = (e_{11})$, then it follows from (5) that

$$\Omega_0 = \begin{pmatrix} -\frac{|Z|}{|V|} & \frac{\langle \partial V_1, X_1\rangle}{|X_1||V_1|} \\ 0 & \frac{|Y_2|}{|V_1|} \end{pmatrix}, \Omega_{-1} = \begin{pmatrix} -\frac{\langle \partial V_1, X_1\rangle}{|X_1||V_1|} & -\frac{|Y_2|}{|V_1|} \\ \frac{|Z|}{|V|} & 0 \end{pmatrix}, \tag{57}$$

$$\Omega_1 = \begin{pmatrix} 0 & \frac{|f_4^{(n-1)}|}{|Y_2|} \\ \frac{|F_3^{(n-1)}|^2|V_1||V|}{|f_0^{(n-1)}|^2|Z||Y_3|} & \frac{\langle \partial Y_2, Y_3\rangle}{|Y_2||Y_3|} \end{pmatrix}, \Omega_2 = \begin{pmatrix} \frac{|f_5^{(n-1)}|}{|f_4^{(n-1)}|} & 0 \end{pmatrix}, \tag{58}$$

$$\Psi_0 = \begin{pmatrix} -\frac{\partial \log |V_1|^2}{2} & 0 \\ 0 & \frac{\partial \log |V_1|^2}{2} \end{pmatrix}. \tag{59}$$

From (57) and (58), by applying $L_\alpha = tr(\Omega_\alpha\Omega_\alpha^*)$ $(\alpha = -1, 0, 1, 2)$, a direct computation yields

$$L_0 = L_{-1} = \frac{\langle \partial V_1, X_1\rangle\langle X_1, \partial V_1\rangle}{|X_1|^2|V_1|^2} + \frac{|Z|^2}{|V|^2} + \frac{|Y_2|^2}{|V_1|^2}, \tag{60}$$

$$L_1 = \frac{|f_4^{(n-1)}|^2}{|Y_2|^2} + \frac{\langle \partial Y_2, Y_3\rangle\langle Y_3, \partial Y_2\rangle}{|Y_2|^2|Y_3|^2} + \frac{|F_3^{(n-1)}|^4|V_1|^2|V|^2}{|f_0^{(n-1)}|^4|Z|^2|Y_3|^2}, \tag{61}$$

$$L_2 = l_4^{(n-1)}, \tag{62}$$

$$|det\Omega_0|^2 dz^2 d\overline{z}^2 = \frac{|Z|^2}{|V|^2}\frac{|Y_2|^2}{|V_1|^2} dz^2 d\overline{z}^2, \tag{63}$$

$$|det\Omega_1|^2 dz^2 d\overline{z}^2 = det(\Omega_1\Omega_1^*) dz^2 d\overline{z}^2 = \frac{|f_2^{(n-1)}|^2|f_3^{(n-1)}|^2|f_4^{(n-1)}|^2|V_1|^2}{|Z|^4|Y_2|^4} l_0^{(n-1)} dz^2 d\overline{z}^2, \tag{64}$$

$$det(\Omega_2\Omega_2^*) dz d\overline{z} = l_4^{(n-1)} dz d\overline{z}. \tag{65}$$

Using (60)-(64), by a straightforward computation, we obtain

$$\partial\overline{\partial} \log |det\Omega_0|^2 = L_{-1} - 2L_0 + L_1, \tag{66}$$

$$\partial\overline{\partial} \log |det\Omega_1|^2 = L_0 - 2L_1 + L_2. \tag{67}$$

If ϕ is totally unramified, then $det(\Omega_1\Omega_1^*) dz^2 d\overline{z}^2 \neq 0$, $det(\Omega_2\Omega_2^*) dz d\overline{z} \neq 0$ everywhere on S^2 and $f_\beta^{(n-1)}$ $(5 \leqslant \beta \leqslant n-1)$ are all unramified. It follows from (64) and (65) that $l_p^{(n-1)} dz d\overline{z} \neq 0$ $(p = 0, 1, 2, 3, 4)$

everywhere on S^2. Since $f_\beta^{(n-1)}$ $(5 \leqslant \beta \leqslant n-1)$ are unramified, we obtain $l_j^{(n-1)} dz d\bar{z} \neq 0$ $(j = 5, \cdots, n-2)$ everywhere on S^2, hence the harmonic sequence

$$0 \xrightarrow{\partial'} \underline{f}_0^{(n-1)} \xrightarrow{\partial'} \underline{f}_1^{(n-1)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{n-1}^{(n-1)} \xrightarrow{\partial'} 0 \tag{68}$$

is also totally unramified.

Since $l_p^{(n-1)} dz d\bar{z} \neq 0$ $(p = 0, 1, 2, 3, 4)$ everywhere on S^2, from (63) and (64), we have

$$|det\Omega_0|^2 dz^2 d\bar{z}^2 = \frac{|Z|^2}{|V|^2} \frac{|Y_2|^2}{|V_1|^2} dz^2 d\bar{z}^2 = \frac{|f_1^{(n-1)}|^2}{|f_3^{(n-1)}|^2} \frac{|Z|^2}{|V|^2} \frac{|Y_2|^2}{|V_1|^2} l_1^{(n-1)} l_2^{(n-1)} dz^2 d\bar{z}^2, \tag{69}$$

$$|det\Omega_1|^2 dz^2 d\bar{z}^2 = det(\Omega_1 \Omega_1^*) dz^2 d\bar{z}^2 = \frac{|f_3^{(n-1)}|^4 |f_2^{(n-1)}|^2 |V_1|^2}{|Z|^4 |Y_2|^4} l_0^{(n-1)} l_3^{(n-1)} dz^2 d\bar{z}^2. \tag{70}$$

In this case, we prove

Proposition 4 *Let $\phi : S^2 \to G(2, n; \mathbb{R})$ $(n \geqslant 7)$ be a linearly full irreducible totally unramified harmonic map with constant curvature K of isotropy order $r = n - 6$. If $l = 2$ and $|f_2^{(n-1)}|^2$ is a polynomial function, then n must be odd, and up to an isometry of $G(2, n; \mathbb{R})$, $\underline{\phi} = \underline{\overline{f}_2^{(n-1)}} \oplus \underline{f}_2^{(n-1)} = \underline{\overline{UV}_2^{(n-1)}} \oplus \underline{UV}_2^{(n-1)}$ with $K = 2/(5n-13)$ for some $U \in G := \{U \in U(n) | U^T U = W\}$, where $W = antidiag\{1, -1, 1, -1, \cdots, 1\}$, which is an $(n \times n)-$matrix.*

Proof Since the harmonic sequence (68) is totally unramified, it follows from (9), (14), (60) and (62) that

$$\delta_0 = \delta_{-1}, \delta_2 = \delta_4^{(n-1)} = 5(n-5), \delta_0^{(n-1)} = n-1, \delta_1^{(n-1)} = 2(n-2), \delta_2^{(n-1)} = 3(n-3). \tag{71}$$

From (9), (10), (66) and (67), we have

$$\delta_{-1} - 2\delta_0 + \delta_1 = -4, \tag{72}$$

$$\delta_0 - 2\delta_1 + \delta_2 = -4, \tag{73}$$

hence, we obtain

$$\delta_0 = 5n - 13, \delta_1 = 5n - 17. \tag{74}$$

Since ϕ is of constant curvature K, using (74) we have $K = 2/(5n-13)$, and we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that the induced metric $ds^2 = 2L_0 dz d\bar{z}$ of ϕ is given by

$$ds^2 = \frac{10n - 26}{(1 + z\bar{z})^2} dz d\bar{z},$$

so

$$L_0 = \frac{5n - 13}{(1 + z\bar{z})^2}. \tag{75}$$

Consider the local lift of the p-th osculating curve (see [2]) $F_p^{(n-1)} = f_0^{(n-1)} \wedge \cdots \wedge f_p^{(n-1)}$ $(p = 0, \cdots, n-1)$. We choose a nowhere zero holomorphic $\mathbb{C}^n$-valued function $f_0^{(n-1)}$, then $F_p^{(n-1)}$ is a nowhere zero holomorphic curve of degree $\delta_p^{(n-1)}$ satisfying $\partial\bar{\partial} \log |F_p^{(n-1)}|^2 = l_p^{(n-1)}$ and $|F_p^{(n-1)}|^2$ is a polynomial function. Thus we may apply (60), (62), (66), (67), (69), (70) and (75) to obtain that

$$\partial\bar{\partial} \log \frac{(1 + z\bar{z})^{5n-13}}{|f_0^{(n-1)}|^4 |V|^4 |V_1|^2} = \partial\bar{\partial} \log \frac{(1 + z\bar{z})^{5n-13}}{|F_1^{(n-1)}|^2 |F_2^{(n-1)}|^2} \frac{|F_1^{(n-1)}|^2 |F_2^{(n-1)}|^2}{|f_0^{(n-1)}|^4 |V|^4 |V_1|^2} = 0. \tag{76}$$

By applying (69), we have $(|f_1^{(n-1)}|^2 |Z|^2 |Y_2|^2)/(|f_3^{(n-1)}|^2 |V|^2 |V_1|^2)$ is a globally defined function without zeros on S^2. Similarly, by using (70), we obtain $(|f_3^{(n-1)}|^4 |f_2^{(n-1)}|^2 |V_1|^2)/(|Z|^4 |Y_2|^4)$ is a globally defined function without zeros on S^2, so

$$\left(\frac{|f_1^{(n-1)}|^2}{|f_3^{(n-1)}|^2} \frac{|Z|^2}{|V|^2} \frac{|Y_2|^2}{|V_1|^2}\right)^2 \frac{|f_3^{(n-1)}|^4 |f_2^{(n-1)}|^2 |V_1|^2}{|Z|^4 |Y_2|^4} = \frac{|F_1^{(n-1)}|^2 |F_2^{(n-1)}|^2}{|f_0^{(n-1)}|^4 |V|^4 |V_1|^2}$$

is a globally defined function without zeros on S^2. Then it follows from (71) that

$$\frac{(1+z\overline{z})^{5n-13}}{|f_0^{(n-1)}|^4|V|^4|V_1|^2} = \frac{(1+z\overline{z})^{5n-13}}{|F_1^{(n-1)}|^2|F_2^{(n-1)}|^2}\frac{|F_1^{(n-1)}|^2|F_2^{(n-1)}|^2}{|f_0^{(n-1)}|^4|V|^4|V_1|^2}$$

is globally defined on $\mathbb{C}$ and has positive constant limit $1/c$ as $z \to \infty$. Thus from (76) we have

$$\frac{(1+z\overline{z})^{5n-13}}{|f_0^{(n-1)}|^4|V|^4|V_1|^2} = \frac{1}{c}. \tag{77}$$

Applying

$$V = f_1^{(n-1)} + x_0\overline{f}_0^{(n-1)}$$

and

$$V_1 = Z + x_1\overline{V} \;= f_2^{(n-1)} + \frac{\partial h - h\partial\log|F_1^{(n-1)}|^2}{|f_0^{(n-1)}|^2|V|^2}(-\overline{x}_0|f_0^{(n-1)}|^2 f_1^{(n-1)} + |f_1^{(n-1)}|^2\overline{f}_0^{(n-1)}) + x_1(\overline{f}_1^{(n-1)} + \overline{x}_0 f_0^{(n-1)}),$$

we have

$$|f_0^{(n-1)}|^2|V|^2 = |F_1^{(n-1)}|^2 + |h|^2, \tag{78}$$

$$|f_0^{(n-1)}|^2|V|^2|V_1|^2 = |F_2^{(n-1)}|^2 + |h|^2|f_2^{(n-1)}|^2 + |f_1^{(n-1)}|^2|\partial h - h\partial\log|F_1^{(n-1)}|^2|^2 + |x_1|^2|V|^4|f_0^{(n-1)}|^2. \tag{79}$$

From (77) and (78), we obtain

$$|F_1^{(n-1)}|^2 + |h|^2 = \frac{c(1+z\overline{z})^{5n-13}}{|f_0^{(n-1)}|^2|V|^2|V_1|^2}. \tag{80}$$

From (34), (71), (78), (4) and (80), we can obtain that h is a holomorphic function on $\mathbb{C}$ at most with the pole $z=\infty$. Hence, it is a polynomial function about z. Set

$$g = hf_2^{(n-1)} - (\partial h - h\partial\log|F_1^{(n-1)}|^2)f_1^{(n-1)} + x_1|V|^2 f_0^{(n-1)}, \tag{81}$$

from (4), we deduce that

$$|f_0^{(n-1)}|^2|V|^2|V_1|^2 = |F_2^{(n-1)}|^2 + |g|^2. \tag{82}$$

Combining (34) and (38), we obtain $\bar{\partial} g = 0$, so g is a holomorphic $\mathbb{C}^n$-valued function. From (80), we have

$$|F_2^{(n-1)}|^2 + |g|^2 = \frac{c(1+z\overline{z})^{5n-13}}{|F_1^{(n-1)}|^2 + |h|^2}. \tag{83}$$

Observing (83), using $\bar{\partial} g = 0$ and (71), we find that $|g|^2$ a polynomial function about z and $\bar{z}$ on $\mathbb{C}$ at most with the pole $z=\infty$. Since $|F_2^{(n-1)}|^2 + |g|^2$ and $|F_1^{(n-1)}|^2 + |h|^2$ are both polynomial functions, from (71), (78), (82) and (83), we obtain

$$|F_2^{(n-1)}|^2 + |g|^2 = \frac{c}{\sqrt{\lambda_1}}(1+z\overline{z})^{3n-9}, \tag{84}$$

$$|F_1^{(n-1)}|^2 + |h|^2 = \sqrt{\lambda_1}(1+z\overline{z})^{2n-4}, \tag{85}$$

where λ_1 is a positive parameter. Then it follows from (78) and (82) that

$$|V_1|^2 = \frac{c}{\lambda_1}(1+z\overline{z})^{n-5}. \tag{86}$$

From (57), (59) and (6), we obtain

$$-\partial\overline{\partial}\log|V_1|^2=\frac{|Z|^2}{|V|^2}-\frac{\langle\partial V_1,X_1\rangle\langle X_1,\partial V_1\rangle}{|X_1|^2|V_1|^2}-\frac{|Y_2|^2}{|V_1|^2}. \tag{87}$$

Applying (60), (75), (86) and (87), we have

$$\begin{cases}\dfrac{|Z|^2}{|V|^2}=\dfrac{2n-4}{(1+z\overline{z})^2}, & (88)\\[2ex] \dfrac{\langle\partial V_1,X_1\rangle\langle X_1,\partial V_1\rangle}{|X_1|^2|V_1|^2}+\dfrac{|Y_2|^2}{|V_1|^2}=\dfrac{3n-9}{(1+z\overline{z})^2}. & (89)\end{cases}$$

Using (88), we have

$$(2n-4)|V|^2=(1+z\overline{z})^2|Z|^2, \tag{90}$$

$$(2n-4)|f_0^{(n-1)}|^2|V|^2=|f_0^{(n-1)}|^2(1+z\overline{z})^2|Z|^2, \tag{91}$$

applying $|Z|^2=|f_2^{(n-1)}|^2+(l_0^{(n-1)}|\partial h-h\partial\log|F_1^{(n-1)}|^2|^2)/|V|^2$, (85) and (91), we obtain

$$|f_0^{(n-1)}|^2|F_2^{(n-1)}|^2=(1+z\overline{z})^{2n-6}A, \tag{92}$$

where $A=\sqrt{\lambda_1}[(2n-4)|F_1^{(n-1)}|^2-|\partial h(1+z\overline{z})-(2n-4)h\overline{z}|^2]$ is a polynomial function, from (71), we know that $|F_2^{(n-1)}|^2$ has at least a factor $(1+z\overline{z})^{n-5}$. From $|F_2^{(n-1)}|^2=|F_1^{(n-1)}|^2|f_2^{(n-1)}|^2$, if $|f_2^{(n-1)}|^2$ is a polynomial function (We define $|f_2^{(n-1)}|^2$ at $z=\infty$ to be $|\widetilde{f}_2^{(n-1)}|^2$, where $|f_2^{(n-1)}|^2=P(z,\overline{z})|\widetilde{f}_2^{(n-1)}|^2$ and $P(z,\overline{z})$ is the highest power term of $|f_2^{(n-1)}|^2$), then we obtain the following two cases:

(**I**) $1+z\overline{z}$ is not a factor of $|F_1^{(n-1)}|^2$, since $|F_2^{(n-1)}|^2$, $|f_2^{(n-1)}|^2$ and $|f_0^{(n-1)}|^2$ are polynomial functions, from (71) and Definition 3, we obtain

$$|f_0^{(n-1)}|^2=\lambda_2(1+z\overline{z})^{n-1}, \tag{93}$$

where λ_2 is a positive parameter. By Lemma 3, up to a holomorphic isometry of $\mathbb{C}P^{n-1}$, $f_0^{(n-1)}$ is a Veronese curve, which is a contradiction, thus this case does't exist.

(**II**) $1+z\overline{z}$ is a factor of $|F_1^{(n-1)}|^2$, from (85), if $|h|^2\neq 0$, then $1+z\overline{z}$ is a factor of it, which contradicts with the fact that h is a holomorphic polynomial function. Therefore $h=0$, which implies that $x_0=0$. Then we obtain

$$V=f_1^{(n-1)},\quad V_1=f_2^{(n-1)}+x_1\overline{f}_1^{(n-1)}. \tag{94}$$

By (85), we have

$$|F_1^{(n-1)}|^2=\sqrt{\lambda_1}(1+z\overline{z})^{2n-4}. \tag{95}$$

From (92) and (95), we have

$$|F_2^{(n-1)}|^2=\frac{\lambda_1(2n-4)(1+z\overline{z})^{4n-10}}{|f_0^{(n-1)}|^2}. \tag{96}$$

Since $|f_0^{(n-1)}|^2$ and $|F_2^{(n-1)}|^2$ are both polynomial functions, by (71), we have

$$|f_0^{(n-1)}|^2=\lambda_2(1+z\overline{z})^{n-1}, \tag{97}$$

where λ_2 is a positive parameter.

It follows from (84), (94) and (97) that we obtain

$$|F_2^{(n-1)}|^2+|x_1|^2|f_1^{(n-1)}|^4|f_0^{(n-1)}|^2=\frac{c}{\sqrt{\lambda_1}}(1+z\overline{z})^{3n-9}. \tag{98}$$

By (38) we obtain $\overline{\partial}(x_1|f_1^{(n-1)}|^2)=0$. Observing (98), from the fact that both $|F_2^{(n-1)}|^2$ and $|f_0^{(n-1)}|^2$ have no singular points except $z=\infty$, we have that $x_1|f_1^{(n-1)}|^2$ is a holomorphic function on $\mathbb{C}$ at most with the pole $z=\infty$. So it is a polynomial function about z. Without loss of generality, we set

$$x_1|f_1^{(n-1)}|^2=H(z), \tag{99}$$

from (98), we obtain

$$|F_2^{(n-1)}|^2+|f_0^{(n-1)}|^2|H|^2=\frac{c}{\sqrt{\lambda_1}}(1+z\overline{z})^{3n-9}. \tag{100}$$

Since both sides of (100) are polynomial functions, $\delta_0^{(n-1)}=n-1$ and (97), if $H\neq 0$, from (100) $1+z\overline{z}$ is a factor of it, which contradicts with the fact that H is a holomorphic polynomial function. Thus we have $H=0$, which implies that $x_1=0$. Then we obtain

$$V_1=f_2^{(n-1)}, \tag{101}$$

from (86) and (101), we obtain

$$|V_1|^2=|f_2^{(n-1)}|^2=\frac{c}{\lambda_1}(1+z\overline{z})^{n-5}. \tag{102}$$

From (97) and Lemma 3, up to a holomorphic isometry of $\mathbb{C}P^{n-1}$, $f_0^{(n-1)}$ is a Veronese curve. We can choose a complex coordinate z on $\mathbb{C}=S^2\backslash\{pt\}$ such that $f_0^{(n-1)}=UV_0^{(n-1)}$, where $U\in U(n)$ and $V_0^{(n-1)}$ has the standard expression given in part (C) of Section 2. Hence, we obtain $\underline{\phi}=\overline{\underline{UV}_2^{(n-1)}}\oplus\underline{UV}_2^{(n-1)}$. In order to determine ϕ, we just need to determine the matrix U. To do this, consider the linearly full reducible harmonic map $\phi^0:S^2\to G(2,n;\mathbb{R})$, which is of constant curvature and isotropy order $n-2$, where $\underline{\phi}^0=\overline{\underline{UV}_0^{(n-1)}}\oplus\underline{UV}_0^{(n-1)}$. By Proposition 3.2 (II) and Remark 3.3 of [8] we can obtain $U^TU=antidiag\{1,-1,1,-1,\cdots,1\}$. □

Remark 1 In Proposition 4, $G\neq\emptyset$. Since n is odd, let $n=2s+1$.

If s is even, choose

$$U_0=\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}\\
\frac{\sqrt{-1}}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}\\
0 & \frac{1}{\sqrt{2}} & \cdots & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{2}} & 0\\
0 & \frac{\sqrt{-1}}{\sqrt{2}} & \cdots & 0 & 0 & 0 & \cdots & \frac{\sqrt{-1}}{\sqrt{2}} & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & \cdots & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \cdots & 0 & 0\\
0 & 0 & \cdots & \frac{\sqrt{-1}}{\sqrt{2}} & 0 & \frac{\sqrt{-1}}{\sqrt{2}} & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix};$$

if s is odd, choose

$$U_0=\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}\\
\frac{\sqrt{-1}}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}\\
0 & \frac{1}{\sqrt{2}} & \cdots & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{2}} & 0\\
0 & \frac{\sqrt{-1}}{\sqrt{2}} & \cdots & 0 & 0 & 0 & \cdots & \frac{\sqrt{-1}}{\sqrt{2}} & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & \cdots & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \cdots & 0 & 0\\
0 & 0 & \cdots & \frac{\sqrt{-1}}{\sqrt{2}} & 0 & -\frac{\sqrt{-1}}{\sqrt{2}} & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & \sqrt{-1} & 0 & \cdots & 0 & 0
\end{pmatrix}.$$

Let u_i denote the i-th column of U_0. For each $1\leqslant i\leqslant s$, all the elements of u_i are zero except the $(2i-1)$-th and $(2i)$-th elements, which are $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{-1}}{\sqrt{2}}$ respectively. Moreover, the column vectors of U_0 satisfy equation $\overline{u}_j+(-1)^ju_{n+1-j}=0$, $1\leqslant j\leqslant n$. Thus we have $U_0\in G$ and for an arbitrary $U\in G$ can be obtained from U_0 by an $SO(n)$-motion.

By Proposition 3 and Proposition 4, we conclude a classification theorem of linearly full totally unramified irreducible conformal minimal immersions with constant curvature from S^2 to $G(2,n;\mathbb{R})$ $(n \geqslant 7)$ of isotropy order $r = n-6$ as follows:

Theorem 1 *Let $\phi : S^2 \to G(2,n;\mathbb{R})$ $(n \geqslant 7)$ be a linearly full totally unramified irreducible conformal minimal immersions with constant curvature K of isotropy order $r = n-6$. If $|f_2^{(n-1)}|^2$ is a polynomial function, then n must be odd, and up to an isometry of $G(2,n;\mathbb{R})$, $\underline{\phi} = \overline{\underline{f}_2^{(n-1)}} \oplus \underline{f}_2^{(n-1)} = \overline{\underline{UV}_2^{(n-1)}} \oplus \underline{UV}_2^{(n-1)}$ with $K = 2/(5n-13)$ for some $U \in G := \{U \in U(n) | U^T U = W\}$, where $W = antidiag\{1,-1,1,-1,\cdots,1\}$, which is an $(n \times n)-$matrix.*

Acknowledgements This work was partially supported by the National Natural Science Foundation of China (Grant No. 11871450), the Ph.D. Research Startup Foundation of Yunnan Normal University 2019XJLK16, Scientific Research Foundation Project of Yunnan Education Department (2020J0093) and Yunnan Fundamental Research Projects (LS21022).

References

1. Bahy-El-Dien A, Wood J C. The explicit construction of all harmonic two-spheres in $G_2(\mathbb{R}^n)$. J Reine Angew Math, 1989, 1989(398), 36–66
2. Bolton J, Jensen G R, Rigoli M, Woodward L M. On conformal minimal immersions of S^2 into $\mathbb{C}P^n$. Math Ann, 1988, 279(4): 599–620
3. Burstall F E, Wood J C. The construction of harmonic maps into complex Grassmannians. J Differential Geom, 1986, 23(3): 255–297
4. Calabi E. Minimal immersions of surfaces in Euclidean spheres. J Differential Geom, 1967, 1(1–2): 111–125
5. Eells J, Sampson J H. Harmonic mappings of Riemannian manifolds. Amer J Math, 1964, 86(1): 109–160
6. Jiao X X. Pseudo-holomorphic curves of constant curvature in complex Grassmannians. Israel J Math, 2008, 163(1): 45–60
7. Jiao X X, Li H. Conformal minimal immersions with constant curvature from S^2 to Q_5. Front Math China, 2019, 14(2): 315–348
8. Jiao X X, Li M Y. Classification of conformal minimal immersions of constant curvature from S^2 to Q_n. Ann Mat Pura Appl, 2017, 196(3): 1001–1023
9. Jiao X X, Peng J G. Minimal two-sphere in $G(2,4)$. Front Math China, 2010, 5(2): 297–310
10. Jiao X X, Wang J. Minimal surfaces in a complex hyperquadric Q_2. Manuscripta Math, 2013, 140(3–4): 597–611
11. Li M Y, Jiao X X, He L. Classification of conformal minimal immersions of constant curvature from S^2 to Q_3. J Math Soc Japan, 2016, 68(2): 863–883
12. Peng C K, Wang J, Xu X W. Minimal two-spheres with constant curvature in the complex hyperquadric. J Math pures Appl, 2016, 106(3): 453–476
13. Uhlenbeck K. Harmonic maps into Lie groups (classical solutions of the chiral model). J Differential Geom, 1989, 30(1): 1–50
14. Wang J, Jiao X X. Totally real minimal surfaces in the complex hyperquadrics. Differential Geom Appl, 2013, 31(4): 540–555
15. Wolfson J G. Harmonic maps of the two-sphere into the complex hyperquadric. J Differential Geom, 1986, 24(2): 141–152