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ORIGINAL RESEARCH

On Schatten p-norm of the distance matrices of graphs

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Abstract For a connected simple graph G, the generalized distance matrix is defined by $D_\alpha := \alpha Tr(G) + (1-\alpha)D(G),\ 0 \leq \alpha \leq 1$, where $Tr(G)$ is the diagonal matrix of vertex transmissions and $D(G)$ is the distance matrix. For particular values of α, we obtain the distance matrix, the distance Laplacian matrix and the distance signless Laplacian matrix and other uncountable distance based matrices. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ be the D_α eigenvalues of G and $p \geq 2$ a real number, the Schatten p-norm is the p-th root of the sum of p-th powers of eigenvalues of $D_\alpha,\ \alpha \in [\frac{1}{2}, 1]$, that is, $\|D_\alpha\|_p^p = \partial_1^p + \partial_2^p + \cdots + \partial_n^p$. In this paper, we obtain various bounds for $\|D_\alpha\|_p^p$ in terms of different graph parameters and characterize the corresponding extremal graphs.

Keywords Distance matrix · Distance Laplacian matrix · Distance Signless Laplacian matrix · D_α matrix · Schatten p-norm · Ky Fan k norm

Mathematics Subject Classification 05C50 · 05C12 · 15A18.

1 Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where n is its *order* and m is its *size*. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighbourhood* (open neighbourhood) of v. The *degree* d_v of v is $|N(v)|$. A graph is called *r-regular*, if degree of each vertex is *r*. We use standard terminology and denote by K_n, the complete graph, by $K_{a,n-a}$, the complete bipartite graph, by $CS_{t,n-t}$, the complete split graph with independence number $n-t$ and clique number t.

The adjacency matrix $A(G) = (a_{ij})$ of G is a square matrix of order n whose (i, j)-th entry is 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_{v_i}$ of G, for $i = 1, 2, \ldots, n$. The positive semi-definite matrices $L(G) = A(G) - D(G)$ and $Q(G) = A(G) + D(G)$ are known as the Laplacian matrix and the signless Laplacian matrix, respectively. More literature about these matrices including their norms can be found in [6,7,12,13,21].

In a connected graph G, the *distance* between two vertices $v_i, v_j \in V(G)$, denoted by $d(v_i, v_j)$, is defined as the length of a shortest path between them. The *diameter* of G is the maximum distance between any two

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vertices of G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is defined as $\mathcal{D}(G) = (d(u, v))_{n\times n}, u, v \in V(G)$. The *transmission* $Tr_G(v_i)$ of a vertex v_i is the i^{th} row sum of $\mathcal{D}(G)$, which is same as the sum of the distances from v_i to all other vertices of G, that is, $Tr_G(v_i) = \sum\limits_{v_j\in V(G)} d(v_i, v_j)$. A graph G is said to be *k-transmission regular*, if $Tr_G(v_i) = k$, for each $v_i \in V(G)$. The *transmission* or Wiener index of a graph G, denoted by $W(G)$, (shortly W) is the sum of distances between all unordered pairs of vertices in G. Clearly, $2W(G) = \sum\limits_{v\in V(G)} Tr_G(v)$. For the vertex $v_i \in V(G)$, the transmission $Tr_G(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is called the *transmission degree sequence* of G. More about the distance matrix can be seen in the survey [2].

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [3] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$ is called the *distance Laplacian matrix* of G and the matrix $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$ is called the *distance signless Laplacian matrix of* G. These matrices are real symmetric, $\mathcal{D}^L(G)$ is positive semi-definite and $\mathcal{D}^Q(G)$ is positive definite, for $n \geq 3$. These matrices are very well studied, for further readings we refer to [3,4,19,24,27].

Recently, Cui et al. [8] introduced the *generalized distance matrix* $D_\alpha(G)$ as a convex combinations of $Tr(G)$ and $D(G)$, and is defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, for $0 \leq \alpha \leq 1$. Since $D_0(G) = \mathcal{D}(G)$, $2D_{\frac{1}{2}}(G) = \mathcal{D}^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)\mathcal{D}^L(G)$. This matrix reduces to merging the spectral theories of the distance matrix, the distance Laplacian matrix and the distance signless Laplacian matrix. Since the matrix $D_\alpha(G)$ is real symmetric, all its eigenvalues ∂_i are real and can be indexed from the largest to the smallest as: $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. The largest eigenvalue ∂_1 is called the *generalized distance spectral radius* of G. The D_α matrix has received attention of researchers, wealthy literature has been published and can be seen in [9,10,12,18,25,26].

In Section 2, we obtain the upper and the lower bounds for $\|D_\alpha\|_p^p = \sum\limits_{i=1}^{n} \partial_i^p$ in terms of different parameters related to graphs like the maximum transmission, the order of G, trace and the spectral radius of D_α matrix, the Wiener index, and other parameters. Section 3 discuss $\sum\limits_{i=1}^{n} \partial_i^p$ for some special classes of graphs.

2 Lower and upper bounds for $\sum\limits_{i=1}^{n} \partial_i^p$

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all square matrices of order n with entries from complex field $\mathbb{C}$. For $M \in \mathbb{M}_n(\mathbb{C})$, the square roots of the eigenvalues of MM^* or M^*M, where M^* is the complex conjugate of M are known as *singular values*. As MM^* is positive semi-definite, so the singular values of M are non negative, denoted by $s_1(M) \geq s_2(M) \geq \cdots \geq s_n(M)$. The *Schatten p-norm* of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ is the p-th root of the sum of the p-th powers of the singular values, that is

$$\|M\|_p = (s_1^p(M) + s_2^p(M) + \cdots + s_n^p(M))^{\frac{1}{p}},$$

where $n \geq p \geq 1$. For $p = 2$, The Schatten 2-norm is the *Frobenius norm*, denoted by $\|M\|_F$, defined by

$$\|M\|_F^2 = s_1^2(M) + s_2^2(M) + \cdots + s_n^2(M).$$

The Schatten 1-norm is the sum of all the singular values and is known as the *trace norm* of M. The sum of the first k largest singular values is the *Ky Fan k-norm* or *nuclear norm*, that is for $p = 1$ and $n \geq k \geq 1$, we have

$$\|M\|_k = s_1(M) + s_2(M) + \cdots + s_k(M).$$

$\|M\|_1 = s_1(M)$ is the largest singular value of M and is called the *spectral norm*. It is well known that for a *Hermitian matrix* M, $s_i(M) = |\lambda_i(M)|$, and for positive semi-definite matrix M, $s_i(M) = \lambda_i(M)$, where $\lambda_i(M)$, $i = 1, 2, \ldots, n$, are the eigenvalues of M. More about Schatten p-norm and Ky Fan k-norm can be seen in [14,15,20].

Since D_α is a real, symmetric and positive semi-definite matrix [8] provided $\alpha \in \left[\frac{1}{2}, 1\right]$, so we assume now onwards that $\alpha \in \left[\frac{1}{2}, 1\right]$, unless other wise stated. The following lemma can be found in [12].

Lemma 1 *([12]) Let G be a connected graph of order n having transmission degrees $Tr_1, Tr_2, \ldots, Tr_n$ and Wiener index W. Then*

(1) $\sum\limits_{i=1}^{n} \partial_i = 2\alpha W$. **(2)** $\sum\limits_{i=1}^{n} \partial_i^2 = \alpha^2 \sum\limits_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 \|D(G)\|_F^2 = \|D_\alpha(G)\|_F^2$.

(3) $\sum\limits_{i=1}^{n} \left(\partial_i - \frac{2\alpha W}{n}\right)^2 = \alpha^2 \sum\limits_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 \|D(G)\|_F^2 - \frac{4\alpha^2 W^2}{n}$.

(4) $\partial(G) \geq \frac{2W}{n}$, **(5)** $\partial(G) \geq \sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}}$. *Equality holds in* **(4)** *and* **(5)** *if and only if G is transmission regular graph.*

It is easy to validate the following lemma.

Lemma 2 *A connected graph G has two distinct D_α eigenvalues if and only if $G \cong K_n$.*

Now, we give some bounds for Ky Fan k-norm for the matrix D_α.

Let G be a connected graph with D_α eigenvalues $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. For $1 \leq k \leq n-1$, let $M_k = \sum\limits_{i=1}^{k} \partial_i$ and $m_k = \sum\limits_{i=1}^{k} \partial_{n-i+1}$, that is, M_k is the sum of the k largest eigenvalues (Ky Fan k-norm) and m_k is the sum of the smallest k eigenvalues of D_α matrix of G. If G is connected without isolated vertices and $\alpha \in [\frac{1}{2}, 1)$, then

$$M_k \geq \alpha \sum_{i=1}^{k} 1 = \alpha k, \text{ for } 1 \leq k \leq n-1,$$

which is a consequence of **Schur's Inequality** stating that *the spectrum of any positive definite symmetric matrix majorizes its main diagonal.* This can be further improved as follows:

$$\frac{M_k}{k} = \frac{\sum\limits_{i=1}^{k} \partial_i}{k} \geq \frac{\sum\limits_{i=k+1}^{n} \partial_i}{n-k} = \frac{2\alpha W - M_k}{n-k} \tag{1}$$

which after simplification gives $M_k \geq \frac{2\alpha W k}{n}$. It can be easily verified that equality holds if and only if $G \cong K_n$.

Now, we give some upper bounds for the spectral graph invariant M_k.

Proposition 1 *Let G be a connected graph with $m \geq n$ edges. Then*

$$M_k \leq \frac{2\alpha W k + \sqrt{k(n-k)\Big(n\|D_\alpha(G)\|_F^2 - 4\alpha^2 W^2\Big)}}{n}, \tag{2}$$

with equality holding if and only if $G \cong K_n$.

Proof Using Cauchy-Schwarz inequality and Lemma 1, we have

$$\begin{aligned}
\Big(2\alpha W - M_k\Big)^2 &= \left(\sum_{i=k+1}^{n} \partial_i\right)^2 \leq (n-k)\left(\sum_{i=k+1}^{n} \partial_i^2\right) = (n-k)\left(\sum_{i=1}^{n} \partial_i^2 - \sum_{i=1}^{k} \partial_i^2\right) \\
&= (n-k)\left(\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 \|D(G)\|_F^2 - \sum_{i=1}^{k} \partial_i^2\right) \\
&\leq (n-k)\left(\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 \|D(G)\|_F^2 - \frac{M_k^2}{k}\right).
\end{aligned}$$

After simplifications, we obtain

$$nM_k^2 - 4\alpha W k M_k + 4\alpha^2 W^2 k - k\left(\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 \|D(G)\|_F^2\right)(k-n) \leq 0.$$

Therefore, Inequality (2) follows.

Assume that equality holds in (2). Then all above inequalities must be equalities. So $\partial_1 = \partial_2 = \cdots = \partial_k$ and $\partial_{k+1} = \partial_{k+2} = \cdots = \partial_n$, that is, G has exactly two distinct D_α eigenvalues. So, by Lemma 2, $G \cong K_n$. Similarly it is easy to check equality other way round. □

Proceeding in a similar manner, we can show that

$$m_k \geq \frac{2\alpha Wk + \left\{k(k-n)\left[4\alpha^2 W^2 - n\|D_\alpha(G)\|_F^2\right]\right\}^{\frac{1}{2}}}{n} \tag{3}$$

with equality holding if and only if $G \cong K_n$.

We note that for $k = 1$ in Equations (2) and (3), we obtain the upper and the lower bound for the largest and the smallest D_α eigenvalues of G.

Next, we establish another bound for the spectral graph invariant M_k.

Proposition 2 *Let G be a connected graph of order n. Then*

$$M_k \leq 2\alpha W - (n-k)\left(\frac{det(D_\alpha)}{\partial_1 \partial_2^{k-1}}\right)^{\frac{1}{n-k}}, \tag{4}$$

where $det = \prod_{i=1}^{n} \partial_i$ *is the determinant of the* D_α *matrix. Equality holds in 4 if and only if* $G \cong K_n$.

Proof Applying arithmetic-geometric mean inequality and Lemma 1, we have

$$\begin{aligned}
M_k = \sum_{i=1}^{n} \partial_i &= 2\alpha W - (\partial_{k+1} + \partial_{k+2} + \cdots + \partial_n) \\
&\leq 2\alpha W - (n-k)\left(\prod_{i=k+1}^{n} \partial_i^2\right)^{\frac{1}{n-k}} = 2\alpha W - (n-k)\left(\frac{det(D_\alpha)}{\prod_{i=1}^{k} \partial_i}\right)^{\frac{1}{n-k}} \\
&\leq 2\alpha W - (n-k)\left(\frac{det(D_\alpha)}{\partial_1 \partial_2^{k-1}}\right)^{\frac{1}{n-k}}
\end{aligned}$$

Which is the Inequality (4) and equality is as in Proposition (1). □

Before, proceeding further, we recall some facts about the matrix D_α. For $0 \leq \beta \leq \alpha \leq 1$, $\partial_i(D_\beta(G)) \leq \partial_i(D_\alpha(G))$, for all $i = 1, 2, \ldots, n$ [8]. The equality holds if and only if G is transmission regular and $k = 1$. Using this fact, we have

$$M_k(D_\beta(G)) \leq M_k(D_\alpha(G)),$$

with equality holding if and only if $k = 1$ and G is a transmission regular graph.

The D_α energy (generalized distance energy) of G is the mean deviation of the values of the D_α eigenvalues of G, that is,

$$E^{D_\alpha}(G) = \sum_{i=1}^{n} \left|\partial_i - \frac{2\alpha W(G)}{n}\right|,$$

which is the trace norm of the matrix $\left(D_\alpha - \frac{2\alpha W}{n} I_n\right)$, where I_n is the identity matrix. Let σ be the positive integer satisfying $\partial_\sigma \geq \frac{2\alpha W}{n}$. Then using $\sum_{i=1}^{n} \partial_i = 2\alpha W$ and the definition of generalized distance energy [12], we have

$$E^{D_\alpha}(G) = 2\left(\sum_{i=1}^{\sigma} \partial_i - \frac{2\sigma\alpha W}{n}\right) = 2 \max_{1\leq j\leq n} \left\{\sum_{i=1}^{j} \partial_i - \frac{2\alpha j W(G)}{n}\right\}.$$

Therefore, we see that the Ky Fan k norm M_k has direct relation with the D_α energy or with the trace norm of the matrix $D_\alpha - \frac{2\alpha W}{n} I_n$. The energy of matrices and graphs is very deeply studied topic, since it has its origin in theoretical chemistry where it helps in approximating the π electron energy of molecules [16,21]. It is very long standing problem in spectral graph theory to characterize the graphs with extremal energy(D_α energy). Thus, it follows that the Ky fan k-norm $M_\sigma = \sum_{i=1}^{\sigma} \partial_i$ (for $k = \sigma$, where $1 \leq \sigma \leq n-1$) will helps us in finding the extremal D_α energy graphs.

Now, we find the bounds for the graph invariant $\sum_{i=1}^{n} \partial_i^p$ for D_α matrix of graphs. Similar type of problems for distance and distance signless Laplacian matrices were studied in [17,24]. All ours results are valid for distance matrix and distance Laplacian (signless) matrices and other uncountably many variations of these matrices.

Since the largest eigenvalue of a non negative matrix lies between the smallest row sum and the largest row sum, so we have

$$\sum_{i=1}^{n} \partial_i^p \leq \partial_1^p + \partial_2^p + \partial_2^p + \cdots + \partial_2^p = \partial_1^p + (n-1)\partial_2^p \leq Tr_{\max}^p + (n-1)\partial_2^p,$$

where $Tr_{\max}$ is the maximum transmission. Thus, any upper bound of largest and second largest eigenvalue gives the upper bound for Schatten p-norm for D_α matrix.

Similarly, we see that

$$\sum_{i=1}^{n} \partial_i^p \geq \partial_1^p + (n-1)\partial_n^p \geq \partial_1^p + (n-1)\partial_n^p$$

with equality holding if and only $G \cong K_n$. Also using any lower bounds for ∂_1 given in Lemma 1, we get the corresponding bounds for $\sum_{i=1}^{n} \partial_i^p$. Besides using the lower bound [10] $\partial_1 \geq \alpha Tr_{\max}$, we get

$$\sum_{i=1}^{n} \partial_i^p \geq \alpha Tr_{\max}^p + (n-1)\partial_n^p.$$

In particular for k transmission regular graphs we have

$$D_\alpha(G) = k\alpha I_n + (1-\alpha)\mathcal{D}(G),$$

where I_n is the identity matrix. Therefore,

$$\sum_{i=1}^{n} \partial_i^p = \sum_{i=1}^{n} (k\alpha + (1-\alpha)\rho_i)^p,$$

where ρ_i is the eigenvalue of the distance matrix $\mathcal{D}(G)$.

Next, we state an important inequality and can be found in any standard textbook.

Jensen's inequality[Theorem 3.9 [5], also see P. 454 [30]]: Let f be a convex function on an interval $\mathcal{I}$ and let $x_1, x_2, \ldots, x_n$ be points of $\mathcal{I}$ and let $b_1, b_2, \ldots, b_n$ be positive real numbers satisfying $\sum_{k=1}^{n} b_k = 1$. Then

$$f\left(\sum_{k=1}^{n} b_k x_k\right) \leq \sum_{k=1}^{n} b_k f(x_k).$$

If f is strictly convex, then equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

We note that, the above inequality is reversed if f is concave function.

The following result establishes the lower bounds for $\sum_{i=1}^{n} \partial_i^p$ in terms of various graph invariants.

Theorem 3 *Let G be a connected graph of order $n \geq 2$. Then the following hold.*

(1)
$$\sum_{i=1}^{n} \partial_i^p \geq n\left(\frac{2W}{n}\right)^p,$$
with equality holding if and only if $G \cong K_1$.

(2)
$$\sum_{i=1}^{n} \partial_i^p \geq \left(\frac{2W}{n}\right)^p + \frac{\left(2W(\alpha - \frac{1}{n})\right)^p}{(n-1)^{p-1}},$$
with equality holding if and only if $G \cong K_n$.

(3)
$$\sum_{i=1}^{n} \partial_i^p \geq \left(\frac{T}{n}\right)^p + \frac{\left(2\alpha W - \sqrt{\frac{T}{n}}\right)^p}{(n-1)^{p-1}},$$
where $T = \sum\limits_{i=1}^{n} Tr_i$ and equality holding holds if and only if $G \cong K_n$.

Proof **(i).** Since $f(x) = x^p$ is strictly convex for $x > 0$, so by Jensen's inequality, we have

$$\sum_{i=1}^{n} \partial_i^p \geq n\left(\frac{\sum\limits_{i=1}^{n} \partial_i}{n}\right)^p = n\left(\frac{2W}{n}\right)^p,$$

with equality holding if and only if $\partial_1 = \partial_2 = \cdots = \partial_n$, which is possible only for $G \cong K_1$.

(ii). Again, by Jensen's inequality, we have

$$\sum_{i=2}^{n} \partial_i^p \geq (n-1)\left(\sum_{i=2}^{n} \frac{\partial_i}{n-1}\right)^p \geq \frac{1}{(n-1)^{p-1}}\left(\sum_{i=2}^{n} \partial_i\right)^p.$$

with equality holding if and only if $\partial_2 = \partial_3 = \cdots = \partial_n$.

Now,

$$\partial_1^p + \partial_2^p + \cdots + \partial_n^p \geq \partial_1^p + \frac{1}{(n-1)^{p-1}}\left(\sum_{i=2}^{n} \partial_i\right)^p = \partial_1^p + \frac{1}{(n-1)^{p-1}}\left(2\alpha W - \partial_1\right)^p.$$

Let $f(x) = x^p + \frac{(2\alpha W - x)^p}{(n-1)^{p-1}}$. By evaluating $f^{'}(x) \geq 0$, we see that $f(x)$ is increasing for $x \geq \frac{2\alpha W}{n}$. By Lemma 1, we have $\partial_1 \geq \frac{2W}{n} \geq \frac{2\alpha W}{n}$ and

$$\sum_{i=1}^{n} \partial_i^p \geq f\left(\frac{2W}{n}\right) = \left(\frac{2W}{n}\right)^p + \frac{\left(2W(\alpha - \frac{1}{n})\right)^p}{(n-1)^{p-1}},$$

with equality holding if and only if $\partial_2 = \partial_3 = \cdots = \partial_n$ and $\partial_1 = \frac{2W}{n}$. Therefore, G has exactly two distinct D_α eigenvalues. So, by Lemma 2, G is the complete graph K_n.

(iii) Following steps of part **(2)** and using part **(5)** of Lemma 1, the result follows. □

Our next result gives the bounds for $\sum\limits_{i=1}^{n} \partial_i^p$ in terms of the determinant of the D_α matrix.

Theorem 4 *Let G be a graph of order $n \geq 2$ and $det(D_\alpha)$ be the determinant of D_α. Then the following holds.*

(1)
$$\sum_{i=1}^{n} \partial_i^p \geq n\Big(det(D_\alpha)\Big)^{\frac{p}{n}},$$
with equality holding if and only if $G \cong K_1$.

(2)
$$\sum_{i=1}^{n} \partial_i^p \geq n\Big(det(D_\alpha)\Big)^{\frac{p}{n}},$$

with equality holding if and only if $G \cong K_n$.

Proof **(i).** Consider the function $f(x) = \log(x)$, which is concave in $[2, \infty)$. Thus by Jensen's inequality, we have

$$\log\left(\sum_{i=1}^{n} \frac{\partial_i^p}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n} \log \partial_i^p \geq \frac{1}{n}\log \prod_{i=1}^{n} \partial_i^p \geq \log\left(\prod_{i=1}^{n} \partial_i^p\right)^{\frac{1}{n}},$$

which implies that

$$\sum_{i=1}^{n} \partial_i^p \geq n\left(\prod_{i=1}^{n} \partial_i^p\right)^{\frac{1}{n}} = n\Big(\partial_1\ \partial_2\ \ldots\ \partial_{n-1}\ \partial_n\Big)^{\frac{p}{n}} = \Big(det(D_\alpha)\Big)^{\frac{p}{n}},$$

equality holds if and only if $\partial_1 = \partial_2 = \cdots = \partial_n$.

(2). As

$$\sum_{i=1}^{n} \partial_i^p = \partial_i^p + \sum_{i=2}^{n} \partial_i^p.$$

Applying arithmetic-geometric mean inequality to the right side, we have

$$\sum_{i=1}^{n} \partial_i^p \geq \partial_i^p + (n-1)\left(\prod_{i=2}^{n} \partial_i^p\right)^{\frac{1}{n-1}} = \partial_i^p + (n-1)\left(\frac{det(D_\alpha)}{\partial_1}\right)^{\frac{p}{n-1}},$$

with equality holding if and only if $\partial_2 = \partial_3 = \cdots = \partial_n$.

Consider the function

$$f(x) = x^p + (n-1)\Big(det(D_\alpha)\Big)^{\frac{p}{n-1}}\ x^{\frac{-p}{n-1}}.$$

Clearly, $f(x)$ is increasing for $x \geq det(D_\alpha)^{\frac{1}{n}}$. Also, by Lemma 1 and arithmetic-geometric mean inequality, we have

$$\partial_1 \geq \frac{2w}{n} \geq \frac{2\alpha W}{n} = \frac{\sum_{i=1}^{n} \partial_i}{n} \geq \left(\prod_{i=1}^{n} \partial_i\right)^{\frac{1}{n}} = det(D_\alpha)^{\frac{1}{n}}.$$

This, further implies that

$$\sum_{i=1}^{n} \partial_i^p \geq f\left(\frac{2W}{n}\right) = \left(\frac{2W}{n}\right)^p + (n-1)det(D_\alpha)^{\frac{p}{n-1}}\left(\frac{2W}{n}\right)^{\frac{-p}{n-1}},$$

with equality holding if and only if $\partial_1 = \frac{2W}{n}$ and $\partial_2 = \partial_3 = \cdots = \partial_n$. That means G is degree regular with two distinct D_α eigenvalues, which is possible only if $G \cong K_n$. □

The following power mean inequality can be seen in [21].

Power mean inequality ([21]): If $q > p > 0$, and $x_1, x_2, \ldots, x_n$ are non negative real numbers, then

$$\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{x_1^q + x_2^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}},$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

As an application of above inequality, we have the following lower bound for $\sum_{i=1}^{n} \partial_i^p$.

Theorem 5 *Let G be a connected graph with $n \geq 3$ and $1 \leq k < n$. Then*

$$\sum_{i=1}^{n} \partial_i^p \geq trace(D_\alpha)^p \left(\frac{k}{n^p} + \left(\frac{n-k}{n(n-k)}\right)^p (n-k)\right),$$

where trace is the sum of D_α eigenvalues of G. Equality holds if and only if $G \cong K_1$.

Proof By *power mean inequality*, we have

$$\left(\frac{\sum_{i=1}^{k} \partial_i^p}{k}\right)^{\frac{1}{p}} \geq \frac{M_k}{k},$$

that is,

$$\sum_{i=1}^{k} \partial_i^p \geq \frac{M_k^p}{k^{p-1}},$$

with equality holding if and only if $\partial_1 = \partial_2 = \cdots = \partial_k$.

Similarly,

$$\sum_{i=k+1}^{n} \partial_i^p \geq \frac{(2\alpha W - M_k)^p}{(n-k)^{p-1}},$$

with equality holding if and only $\partial_{k+1} = \partial_{k+2} = \cdots = \partial_n$. Therefore, we have

$$\sum_{i=1}^{n} \partial_i^p = \sum_{i=1}^{k} \partial_i^p + \sum_{i=k+1}^{n} \partial_i^p \geq \frac{M_k^p}{k^{p-1}} + \frac{(2\alpha W - M_k)^p}{(n-k)^{p-1}}.$$

Let $x = M_k$ and consider the function

$$f(x) = \frac{x^p}{k^{p-1}} + \frac{(2\alpha W - x)^p}{(n-k)^{p-1}}.$$

Now, $f^{'}(x) \geq 0$ implies that $p\left(\frac{x}{k}\right)^{p-1} - p\left(\frac{2\alpha W - x}{n-k}\right)^{p-1} \geq 0$, which is true for $x \geq \frac{2\alpha W k}{n}$. Thus,

$$\begin{aligned}\sum_{i=1}^{n} \partial_i^p &\geq (2\alpha W)^p \left(\frac{k}{n^p} + \left(\frac{n-k}{n(n-k)}\right)^p (n-k)\right)\\ &= \mathrm{trace}(D_\alpha)^p \left(\frac{k}{n^p} + \left(\frac{n-k}{n(n-k)}\right)^p (n-k)\right).\end{aligned}$$

Suppose equality occurs, then above all inequalities are equalities, that is $\partial_1 = \partial_2 \cdots = \partial_k$, $\partial_{k+1} = \cdots = \partial_n$ and $M_k = \frac{2\alpha W k}{n}$. This implies that $\partial_1 = \partial_2 = \cdots = \partial_n = \frac{2\alpha W}{n}$, which is true for $G \cong K_1$. □

Lemma 3 *([11]) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non negative real numbers satisfying $ra_i \leq b_i \leq Ra_i$, for $1 \leq i \leq n$. Then,*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\sum_{i=1}^{n} a_i b_i,$$

equality holds if and only if $b_i = Ra_i$ or $b_i = ra_i$ for $1 \leq i \leq n$.

The next shows that $\sum_{i=1}^{n} \partial_i^p$ can be bounded between the smallest and the largest eigenvalue of D_α matrix.

Theorem 6 *Let G be a connected graph with $n \geq 3$ and $1 \leq k < n$. Then*

$$n\partial_n^p \leq \sum_{i=1}^{n} \partial_i^p \leq n\partial_1^p,$$

with equality holding if and only if G is transmission regular.

Proof Choosing $b_i = \partial_i^{\frac{p}{2}},\ a_i = 1,\ r = \partial_n^{\frac{p}{2}}$ and $R = \partial_1^{\frac{p}{2}}$, then we have

$$\partial_n^{\frac{p}{2}} \leq \partial_i^{\frac{p}{2}} \leq \partial_1^{\frac{p}{2}}.$$

By Lemma 3, we have

$$\sum_{i=1}^{n} \partial_i^p + (\partial_1\partial_n)^{\frac{p}{2}} \sum_{i=1}^{n} 1 \leq (\partial_n^{\frac{p}{2}} + \partial_1^{\frac{p}{2}}) \sum_{i=1}^{n} \partial_i^{\frac{p}{2}},$$

which implies

$$\sum_{i=1}^{n} \partial_i^p + (\partial_1\partial_n)^{\frac{p}{2}} n \leq (\partial_n^{\frac{p}{2}} + \partial_1^{\frac{p}{2}}) n\partial_1^{\frac{p}{2}},$$

which further equals to

$$\sum_{i=1}^{n} \partial_i^p \leq n\partial_1^p.$$

Similarly, choosing $b_i = \partial_i^p,\ a_i = 1,\ r = \partial_n^p$ and $R = \partial_1^p$ in Lemma 3, we have

$$\sum_{i=1}^{n} \partial_i^{2p} + (\partial_1\partial_n)^p n \leq (\partial_n^p + \partial_1^p) \sum_{i=1}^{n} \partial_i^p,$$

which implies that

$$\begin{aligned}\sum_{i=1}^{n} \partial_i^p &\geq \frac{1}{\partial_n^p + \partial_1^p} \left(\sum_{i=1}^{n} \partial_i^{2p} + n(\partial_1\partial_n)^p \right) \\ &\geq \frac{1}{\partial_n^p + \partial_1^P} \left(nt^{2p} + n(\partial_1\partial_n)^p \right) = n\partial_n^p.\end{aligned}$$

Equality holds if and only if G is transmission regular. □

Lemma 4 *([22]) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non negative real numbers. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} \big(M_1 M_2 - m_1 m_2\big)^2,$$

where $M_1 = \max a_i,\ M_2 = \max b_i,\ m_1 = \min a_i$ and $m_2 = \min b_i,\ i = 1, 2, \ldots, n$.

Theorem 7 *Let G be a connected graph with $n \geq 3$. Then*

$$\left(n^2\partial_1^{2p} - \frac{n^2}{4}(\partial_1^p - \partial_n^p) \right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \partial_i^p \leq \frac{n^2}{4}(\partial_1^{\frac{p}{2}} - \partial_n^{\frac{p}{2}})^2 + n^2\partial_1^p,$$

with equality holding if and only if G is transmission regular.

Proof Choosing $a_i = \partial_i^{\frac{p}{2}},\ b_i = 1,\ M_1 = \partial_1^{\frac{p}{2}},\ m_1 = \partial_n^{\frac{p}{2}}$ and $M_2 = m_2 = 1$ in Lemma 4, we have

$$n\sum_{i=1}^{n}\partial_i^p - \left(\sum_{i=1}^{n}\partial_i^{\frac{p}{2}}\right)^2 \leq \frac{n^2}{4}\big(\partial_1^{\frac{p}{2}} - \partial_n^{\frac{p}{2}}\big)^2,$$

which implies

$$\begin{aligned}
n\sum_{i=1}^{n}\partial_i^p &\leq \frac{n^2}{4}\big(\partial_1^{\frac{p}{2}} - \partial_n^{\frac{p}{2}}\big)^2 + \left(\sum_{i=1}^{n}\partial_i^{\frac{p}{2}}\right)^2, \\
&\leq \frac{n^2}{4}\big(\partial_1^{\frac{p}{2}} - \partial_n^{\frac{p}{2}}\big)^2 + \left(\sum_{i=1}^{n}\partial_1^{\frac{p}{2}}\right)^2 \\
&= \frac{n^2}{4}\big(\partial_1^{\frac{p}{2}} - \partial_n^{\frac{p}{2}}\big)^2 + n^2\partial_1^p.
\end{aligned}$$

Thus,

$$\sum_{i=1}^{n}\partial_i^p \leq \frac{n}{4}\big(\partial_1^{\frac{p}{2}} - \partial_n^{\frac{p}{2}}\big)^2 + n\partial_1^p.$$

Similarly, choosing $a_i = \partial_i^p,\ b_i = 1,\ M_1 = \partial_1^p, m_1 = \partial_n^p$ and $m_2 = M_2 = 1$ in Lemma 4, we have

$$n\sum_{i=1}^{n}\partial_i^{2p} \leq \frac{n^2}{4}(\partial_1^p - \partial_n^p)^2 + \left(\sum_{i=1}^{n}\partial_i^p\right)^2,$$

which implies

$$\left(\sum_{i=1}^{n}\partial_i^p\right)^2 \geq n\sum_{i=1}^{n}\partial_1^{2p} - \frac{n^2}{4}(\partial_1^p - \partial_n^p)^2.$$

Finally, we get

$$\begin{aligned}
\sum_{i=1}^{n}\partial_i^p &\geq \left(n^2\partial_1^{2p} - \frac{n^2}{4}(\partial_1^p - \partial_n^p)^2\right)^{\frac{1}{2}}, \\
&\geq \left(n^2\partial_1^{2p} - \frac{n^2}{4}(\partial_1^p - \partial_n^p)\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Lemma 5 *([28]) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non negative real numbers. Then*

$$\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\sum_{i=1}^{n}a_ib_i\right)^2,$$

where $M_1 = \max a_i,\ M_2 = \max b_i,\ m_1 = \min a_i$ and $m_2 = \min b_i,\ i = 1, 2, \ldots, n$.

The following results gives the bounds for $\sum_{i=1}^{n}\partial_i^p$ in terms of the smallest and the largest D_α eigenvalues of G.

Theorem 8 *Let G be a connected graph of order $n \geq 3$. Then*

$$\frac{n\partial_1^p}{\frac{1}{2}\sqrt{\frac{\partial_1}{\partial_n}} + \sqrt{\frac{\partial_n}{\partial_1}}} \leq \sum_{i=1}^{n}\partial_i^p \leq \frac{n\partial_1^{\frac{p}{2}}}{4}\left(\sqrt{\frac{\partial_1}{\partial_n}} + \sqrt{\frac{\partial_n}{\partial_1}}\right)^2.$$

Proof Let $a_i = \partial_i^{\frac{p}{2}},\ b_i = 1$, so that $M_1 = \partial_1,\ m_n = \partial_n,\ M_2 = m_2 = 1$. Now from Lemma 5, we have

$$n\sum_{i=1}^{n}\partial_i^p \leq \frac{1}{4}\left(\sqrt{\frac{\partial_1}{\partial_n}}+\sqrt{\frac{\partial_n}{\partial_1}}\right)^2\left(\sum_{i=1}^{n}\partial_i^{\frac{p}{2}}\right)^2$$
$$\leq \frac{1}{4}\left(\sqrt{\frac{\partial_1}{\partial_n}}+\sqrt{\frac{\partial_n}{\partial_1}}\right)^2\partial_1^{\frac{p}{2}}n^2.$$

Thus,

$$\sum_{i=1}^{n}\partial_i^p \leq \frac{1}{4}\left(\sqrt{\frac{\partial_1}{\partial_n}}+\sqrt{\frac{\partial_n}{\partial_1}}\right)^2\partial_1^{\frac{p}{2}}n.$$

Again, choosing $a_i = \partial_i^p,\ b_i = 1$ in Lemma 5, we have

$$n\sum_{i=1}^{n}\partial_i^{2p} \leq \frac{1}{4}\left(\sqrt{\frac{\partial_1}{\partial_n}}+\sqrt{\frac{\partial_n}{\partial_1}}\right)^2\left(\sum_{i=1}^{n}\partial_i^{p}\right)^2,$$

this implies

$$n\sum_{i=1}^{n}\partial_i^{2p} \leq n^2\partial_1^{2p} \leq \frac{1}{4}\left(\sqrt{\frac{\partial_1}{\partial_n}}+\sqrt{\frac{\partial_n}{\partial_1}}\right)^2\left(\sum_{i=1}^{n}\partial_i^{p}\right)^2.$$

Therefore

$$\sum_{i=1}^{n}\partial_i^p \geq \frac{n\partial_1^p}{\frac{1}{2}\sqrt{\frac{\partial_1}{\partial_n}}+\sqrt{\frac{\partial_n}{\partial_1}}}.$$

□

Therefore we have seen that any bounds on the spectral radius and the smallest eigenvalue of D_α can be used to the bounds for the Schatten p-norm of D_α matrix.

3 Schatten p-norm for some special class of graphs

In this section, we discuss $\sum\limits_{i=1}^{n}\partial_i^p$ for some special classes of graphs and for $\alpha = \frac{1}{2}$, we obtain a special spectral invariant known as the D_α energy like invariants, which has various applications [29].

The following lemma can be found in [8].

Lemma 6 *([8]) Let G be a connected graph of order n and $\frac{1}{2} \leq \alpha \leq 1$. If G' is a connected graph obtained from G by deleting an edge, then $\partial_i(D_\alpha(G')) \geq \partial_i(D_\alpha(G))$, for all $1 \leq i \leq n$.*

If $\mathcal{B}$ is the class of all connected bipartite graphs, then it is clear that any bipartite graph G is spanning subgraph of complete bipartite graph $K_{a,n-a}$. By Lemma 6, we have

$$\sum_{i=1}^{n}\partial_i^p(G) \geq \sum_{i=1}^{n}\partial_i^p(K_{a,n-a}),$$

with equality holding if and only if $G \cong K_{a,n-a}$. Now using the D_α spectra of $K_{a,n-a}$ [1], we have

$$\sum_{i=1}^{n}\partial_i^p(K_{a,n-a}) = (a-1)(\alpha(a+n)-2)^p + (n-a-1)(\alpha(2n-a)-2)^p + x_1^p + x_2^p,$$

where $x_1 = \frac{\alpha n+2n-4-\sqrt{D}}{2}$, $x_2 = \frac{\alpha n+2n-4+\sqrt{D}}{2}$ and $D = (\alpha-2)^2(2a^2+n^2-2na)+2(na-a^2)(\alpha^2-2)$.

A complete split graph, denoted by $CS_{t,n-t}$, is the graph consisting of a clique on t vertices and an independent set (a subset of vertices of a graph is said to be an independent set if the subgraph induced by them is an empty graph) on the remaining $n-t$ vertices, such that each vertex of the clique is adjacent to every vertex of the independent set.

The D_α spectra of $CS_{t,n-t}$ [1] is

$$\left\{(\alpha n-1)^{[t-1]}, (\alpha(2n-t)-2)^{[n-t-1]}, y_1, y_2\right\},$$

where $y_i = \frac{2n-t+\alpha n-3\pm\sqrt{\theta}}{2}$, $i=1,2$ and $\theta = (5-4\alpha)t^2+(6\alpha n-8n-4\alpha+6)t+n^2(\alpha-2)^2+2n\alpha-4n+1$.

Among the class $\mathcal{C}$ of all split graphs, the complete split graph $CS_{t,n-t}$ has the minimum Schatten p-norm and

$$\sum_{i=1}^{n}\partial_i^p(CS_{t,n-t}) = (t-1)(\alpha n-1)^p+(n-t-1)(\alpha(2n-t)-2)^p+y_1^p+y_2^p,$$

with y_1 and y_2 defined above.

The vertex connectivity of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices of G, whose deletion disconnects G. Let $\mathcal{F}_n$ be the family of simple connected graphs on n vertices and let $\mathcal{V}_n^k = \{G\in\mathcal{F}_n : \kappa(G)\le k\}$.

Let G_1 and G_2 be two connected graphs, the join of G_1 and G_2 is denoted by $G_1\vee G_2$, is defined as the graph G in which each vertex of G_1 is joined to every vertex of G_2. By putting $n_0=k$, $n_1=t$ and $n_3=n-k-t$ in the Corollary 4 of [1], we obtain the D_α spectrum of $K_k\vee(K_t\cup K_{n-k-t})$.

The D_α spectrum of $K_k\vee(K_t\cup K_{n-k-t})$ is

$$\left\{(\alpha n-1)^{[k-1]}, (\alpha(2n-k-t)-1)^{[t-1]}, (\alpha(n+t))^{n-k-t}, z_1, z_2, z_3\right\},$$

where z_i's are the zeros of the characteristic polynomial of the following matrix

$$\begin{pmatrix} \alpha(n-k)+k-1 & (1-\alpha)t & (1-\alpha)(n-k-t)\\ (1-\alpha)k & \alpha(2n-k-2t)+t-1 & 2(1-\alpha)(n-k-t)\\ (1-\alpha)k & 2(1-\alpha)t & \alpha(2t+k)+n-k-t-1 \end{pmatrix}.$$

Among all connected graphs with given vertex connectivity k, the graph $K_k\vee(K_t\cup K_{n-k-t})$ has the minimum Schattan p-norm and is given by

$$\begin{aligned}\sum_{i=1}^{n}\partial_i^p(K_k\vee(K_t\cup K_{n-k-t})) =&(k-1)(\alpha n-1)^p+(t-1)(\alpha(2n-k-t)-1)^p\\ &+(n-k-t-1)(\alpha n+\alpha t)^p+z_1^p+z_2^p+z_3^p.\end{aligned}$$

For $\alpha=p=\frac{1}{2}$, we get $\sum\limits_{i=1}^{n}\sqrt{\partial_i}$, which is known as the distance signless Laplacian-energy-like invariant [24] and is denoted by $DEL(G)$. This energy-like invariant is a molecular descriptor and describes the properties [29] like entropy, molar volume, molar refraction, boiling point, melting point and partition coefficient of hydrocarbons. Motivated by this the D_α energy-like invariant is defined by

$$D_\alpha EL(G) = \sum_{i=1}^{n}\sqrt{\partial_i}.$$

For $p=\frac{1}{2}$, various bounds for $D_\alpha EL(G)$ are given by Theorems 3, 4, 6, 7 and 8 of Section 2.

The present articles studies the Schatten norms of the D_α matrix. Several interesting bounds are presented along with the characterization of extremal graphs. However, in general the characterization of maximal and minimal graphs for these norms remains open at large. Also, strong relations can be obtained between the trace norm of the matrix $D_\alpha(G)-\frac{2\alpha W(G)}{n}I_n$ and the Ky Fan k-norm M_k of D_α matrix, which in turn can help us in identifying the extremal graphs for the spectral invariant $E^{D_\alpha}(G)=\sum_{i=1}^{n}$. The classification of graphs with particular values of σ are very well studied. Lastly, the D_α energy like invariant is yet to be investigated and its corresponding applications in theoretical chemistry.

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Data Availability There is no data associated with this article.

References

1. A. Alhevaz, M. Baghipur, H. A. Ganie and Y. Shang, The generalized distance spectrum of the join of graphs, Symmetry, 12(1) (2020) 169.
2. M. Aouchiche and P. Hansen, Distance spectra of graphs: a survey, Linear Algebra Appl., 458 (2014) 301–386.
3. M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra Appl., 439 (2013) 21–33.
4. M. Aouchiche and P. Hansen, Some properties of distance Laplacian spectra of a graph, Czech. Math. J., 64 (2014) 751–761.
5. M. J. Cloud, B. C. Drachman and L. P. Lebedev, Inequalities with Applications to Engineering, second edition, Springer, New York, 2014.
6. J. N. Cooper, Constraints on Brouwer's Laplacian spectrum conjecture, Linear Algebra Appl., 615 (2021) 11–27.
7. D. M. Cvetković, P. Rowlison and S. Simić, An Introduction to the Theory of Graph Spectra. London Mathematical Society Student Texts 75, Cambridge University Press, Cambridge, 2010.
8. S. Y. Cui, J. X. He and G. X. Tian, The generalized distance matrix, Linear Algebra Appl., 563 (2019) 1–23.
9. R. C. Diaz, German Pasten and O. Rojo, On the minimal D_α-spectral radius of graphs subject to fixed connectivity, Linear Algebra Appl., 584 (2020) 353–370.
10. R. C. Diaz, German Pasten and O. Rojo, New results on the D_α-matrix of connected graphs, Linear Algebra Appl., 577 (2019) 168–185.
11. J. B. Diaz and F. T. Metcalf, Stronger forms of a class of inequalities of G. Pólya, G. Szegö and L. K. Kantorovich, Bull. Am. Math. Soc., 69 (1963) 415–418.
12. H. A. Ganie, S. Pirzada, B. A. Rather and V. Trevisan, Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl., 588 (2020) 1–18.
13. H. A. Ganie, S. Pirzada, B. A. Rather and R. U. Shaban, On Laplacian eigenvalues of graphs and Brouwer's conjecture, J. Ramanujan Math. Soc., 36(1) (2021) 1–9.
14. R. Horn and C. Johnson, *Matrix Analysis*, Second Edition, Cambridge University Press, New York, 2013.
15. C. H. Huang, J. S. Chen and C. C. Hu, The Schatten p-norm on $\mathbb{R}^n$, J. Nonlinear Conver Anal., 21(1) (2020) 21–29.
16. X. Li, Y. Shi and I. Gutman, Graph Energy, Springer, New York, 2012.
17. H. Lin, On the sum of k largest distance eigenvalues of graphs, Discrete Appl. Math., 259 (2019) 153–159.
18. H. Lin, J. Xue and J. Shu, On the D_α spectra of graphs, Linear Multilinear Algebra, 69(6) (2021) 997–1019 .
19. Huiqiu Liu and K. C. Das, Characterization of extremal graphs from distance signless Laplacian eigenvalues, Linear Algebra Appl., 500 (2016) 77–87.
20. M. S. Moslehian, Ky Fan inequalities, Linear multilinear Algebra, 60 (2011) 1–13.
21. V. Nikiforov, Beyond graph energy: norms of graphs and matrices, Linear Algebra Appl., 506 (2016) 82–138.
22. N. Ozeki, On the estimation of the inequality by the maximum and minimum values, J. College Arts Sci. Chiba Univ., 5 (1968) 199–203.
23. S. Pirzada, H. A. Ganie, B. A. Rather and R. U. Shaban, On generalized distance energy of graphs, Linear Algebra Appl., 603 (2020) 1–19.
24. S. Pirzada, H. A. Ganie, A. Alhevaz and M. Baghipur, On the sum of the powers of distance signless Laplacian eigenvalues of graphs, Indian J. Pure Appl. Math., 51(3) (2020) 1143–1163.
25. S. Pirzada, B. A. Rather, H. A. Ganie and Rezwan ul Shaban, On generalized distance spectral radius of a bipartite graph, Mate. Vesnik, 72(4) (2020) 327–336.
26. S. Pirzada, B. A. Rather, R. U. Shaban and T. Chishti, On the sum of the powers of A_α eigenvalues of graphs and A_α-energy like invariant, Boletim da Soc. Paran. de Mate., 40 (2022) 1–12, https://doi.org/10.5269/bspm.52469.
27. S. Pirzada, B. A. Rather, R. U. Shaban and Merajuddin, On graphs with minimal distance signless Laplacian energy, Acta Univ. Sap. Math., 13(2) (2021) 450–467.
28. G. Pólya and G. Szego, Problems and theorems in analysis, Vol. 1, New York, Springer 1992.
29. D. Stevanovíc, A. Ilíc, C. Onisor and M. Diudea, LEL a newly designed molecular descriptor, Acta Chim. Slov., 56 (2009) 410–417.
30. B. S.Thomson, B. B. Judith and A. M. Bruckner, Elementary Real Analysis, Prentice-Hall, New York, 2008.