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ORIGINAL RESEARCH

K-uniform convexity in Orlicz-Lorentz function space equipped with the Orlicz norm

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Abstract The necessary and sufficient conditions for k-uniform convexity in Orlicz-Lorentz function spaces equipped with the Orlicz norm and generated by N-functions as well as any non-increasing weight sequences are given. Moreover, Some tools useful in the proofs of the main results are also provided. Besides, in the proof process, we use the reachability of the infimum in Orlicz norm formula to obtain a function working like supporting functional, which makes us complete the proof even though the characterization of dual space is still unclear.

Keywords Uniform convexity · Orlicz-Lorentz function space · K-uniform convexity

Mathematics Subject Classification 46E30 · 46B20

1 Introduction

The research on Orlicz-Lorentz space began in the 1990s, and the research on geometric properties of this space has attracted the attention of a large number of scholars. J. Cerdà, H. Hudzik, A. Kamińska and M. Mastyło gave some conclusion about the basic convexity properties as well as Kadec-Klee property for pointwise converge in [1] in 1998. Besides, H. Hudzik, A. kamińska and M. Matyło characterized the Köthe dual of Orlicz-Lorentz space in [2] in 2002. It is worth mentioning that W. Gong and D. Zhang gave the description of monotonicity in Orlicz-Lorentz spaces in [3] in 2016. Moreover, B. Chen and W. Gong gave the sufficient and necessary conditions for normal structure and uniform non-squareness in Orlicz-Lorentz space in [4] in 2021. Among these geometric properties, uniform convexity has been studied intensively [5].

A Banach space $(X, \|\cdot\|)$ is uniformly convex ([6]) if whenever $f_n \in S(X)$, $g_n \in S(X)$, $\|f_n + g_n\| \to 2$, then $\|f_n - g_n\| \to 0$ where $S(X)$ denotes the unite sphere of a Banach space $X = (X, \|\cdot\|)$. We can also give the equivalent definition of uniform convexity, that is, for every $f, g \in S(X)$ and arbitrary $\varepsilon > 0$, if $\|f - g\| > \varepsilon$, then there exists $\delta > 0$ such that $\|\frac{f+g}{2}\| < 1 - \delta$. X is called k-uniformly convex ($k \geq 1$) ([7], pg.31) provided that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0, x_1, \ldots, x_k \in B(X)$ and $\|x_0 + x_1 + \ldots + x_k\| \geq k + 1 - \delta$

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imply

$$\Delta\left(x_0, x_1, \ldots, x_k\right) = \sup_{f_i \in B(X^*)} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ f_1\left(x_0\right) & f_1\left(x_1\right) & \ldots & f_1\left(x_k\right) \\ \ldots & \ldots & \ldots & \ldots \\ f_k\left(x_0\right) & f_k\left(x_1\right) & \ldots & f_k\left(x_k\right) \end{vmatrix} < \varepsilon.$$

Clearly, uniform convexity, which can be regarded as uniform 1-convexity, implies uniform k-convexity for $k \geq 1$. Uniform convexity has attracted much attention since Clarkson put forward this concept in 1936. Uniform convexity is related to fixed point property, Kadec-Klee property as well as normal structure theory. For more information about the geometric properties mentioned above please refer to [4] [8] [9].

By L_0 we denote the space of all equivalence classes of Lebesgue measurable functions $f : (0, \infty) \to R$. By m we denote the Lebesgue measure on $M = M(R_+) :=$ the lebesgue-measurable subsets of R_+. An element $f \in X$ is said to process absolutely continuous norm if $\| f_n\|_X \to 0 (n \to \infty)$ for every sequence $\{f_n\}_{n\in N}$ with $|f_n| \leq |f|$ and $|f_n| \to 0 (n \to \infty)$ on R_+. A Banach function space X is said to be order continuous if every element of X has absolutely continuous norm.

A Banach space $E = (E, \| \cdot \|_E)$ is said to be a symmetric space if $f \in L^0$, $g \in E$ and $m_f = m_g$, then $f \in E$ and $\| f\|_E = \|g\|_E$, where $m_f(t)$ is the distribution function of f and defined by

$$m_f(t) := m(\{s \in R_+ : |f(s)| > t\}), \quad t > 0.$$

For $f \in L_0$, we define its non-increasing rearrangement f^* which is defined by

$$f^*(t) := \inf\{\lambda > 0 : m_f(\lambda) \leq t\} \quad t > 0.$$

Obviously, $\| f^*(t)\|_E = \| f\|_E$ whenever E is a symmetric Banach space.

$\sigma : (0, r) \to (0, r)$ $(r \leq \infty)$ is called a measure preserving transformation [10] if for every measurable set $E \subset (0, r)$, $\sigma^{-1}(E)$ is measurable and $m(\sigma^{-1}(E)) = m(E)$.

A function $\varphi : R_+ \to R_+$ is said to be an Orlicz function if φ is convex, $\varphi = 0$ and $\varphi > 0$ for all $u > 0$. For any Orlicz function φ we define its complementary function ψ in the sense of Young by

$$\psi(v) = \sup_{u>0}\{|uv| - \varphi(u)\}.$$

Let $p(u)$, $q(v)$ be the left-derivative of φ and ψ, respectively. $p(u)$, $q(v)$ is non-decreasing and satisfy:

$$\varphi(u) = \int_0^{|u|} p(t)dt, \quad u \geq 0$$

and

$$\psi(v) = \int_0^{|v|} q(s)ds, \quad v \geq 0.$$

Besides, φ and ψ satisfy Young Inequality:

$$uv \leq \varphi(u) + \psi(v), \quad u \geq 0, v \geq 0.$$

Furthermore, $uv = \varphi(u) + \psi(v)$ whenever $v = p(u)$ or $u = q(v)$. We recall that an Orlicz function φ satisfies the Δ_2-condition($\varphi \in \Delta_2$) ([11]) if there exist $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t > 0$. We say $\varphi \in \nabla_2$ when its complementary function ψ satisfies the Δ_2-condition. We can further define N-function if Orlicz function f satisfys

$$\lim_{t\to 0} \frac{\varphi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

A function $\omega : (0, \infty) \to (0, \infty)$ is said to be a weight function if it is non-increasing, locally integrable with respect to the Lebesgue measure m and

$$\int_0^\infty \omega(t)dt = \infty.$$

Also we define the function $W : R_+ \to R_+$ by

$$W(t) := \int_0^t \omega(s)ds \quad t \geq 0.$$

Given any Orlicz function φ and a weight function ω we define the modular $\rho_{\varphi,\omega}$ by

$$\rho_{\varphi,\omega}(f) = \int_0^\infty \varphi(f^*(t))\omega(t)dt$$

and the Orlicz-Lorentz function space

$$\lambda_{\varphi,\omega} = \{f \in L_0 : \rho_{\varphi,\omega}(\lambda f) = \int_0^\infty \varphi(\lambda f^*(t))\omega(t)dt < \infty \text{ for some } \lambda > 0\}$$

which becomes a Banach space under the Luxemburg norm

$$\|f\| = \|f\|_{\varphi,\omega} = \inf\{\lambda > 0 : \rho_{\varphi,\omega}(\frac{f}{\lambda}) \leq 1\}.$$

For any $f \in \lambda_{\varphi,\omega}$ its Orlicz norm is defined by:

$$\|f\|^o = \|f\|^o_{\varphi,\omega} = \sup_{\rho_{\psi,\omega}(g)\leq 1} \int_0^\infty f^*(t)g^*(t)\omega(t)dt.$$

Orlicz-Lorentz space is a classical Banach space. For more information please refer to [12] [13] [14] [15].

From now on, we let $\lambda_{\varphi,\omega}$ be the Orlicz-Lorentz function space equipped with the Luxemburg norm and $\lambda^o_{\varphi,\omega}$ be the Orlicz-Lorentz function space equipped with the Orlicz norm. It has been proven in [16] that for arbitrary Orlicz function φ and any non-increasing weight function ω, the equality

$$\|f\|^o_{\varphi,\omega} = \inf_{k>0} \frac{1}{k}\{1 + \rho_{\varphi,\omega}(kf)\}$$

holds for every $f \in \lambda^o_{\Phi,\omega}$. Now we consider the attainability of the infimum in the formula arises. Define

$$k^* = k^*(f) = \inf\{k > 0 : \rho_{\psi,\omega}(p(kf)) \geq 1\}$$

and

$$k^{**} = k^{**}(f) = \sup\{k > 0 : \rho_{\psi,\omega}(p(kf)) \leq 1\}.$$

Obviously, $0 \leq k^* \leq k^{**} \leq \infty$. If φ is N-function, then $k^{**} < \infty$ for arbitrary $f \in \lambda^o_{\varphi,\omega}$. Indeed if $k^{**} = \infty$, then there exists a non-negative sequence $\{k_n\}$ such that $k_n \uparrow \infty$ and $\int_0^\infty \psi(p(k_n f)^*)\omega(t)dt \leq 1$. Assume $t_0 = m\{t : f^*(t) > 1\} < \infty$, we obtain

$$\begin{aligned}
\psi\left(p\left(k_n\right)\right) W\left(t_0\right) &= \int_0^{t_0} \psi\left(p\left(k_n\right)\right) \omega(t)dt \\
&= \int_0^{m\{t:f^*>1\}} \psi\left(p\left(k_n\right)\right) \omega(t)dt \\
&\leq \int_0^\infty \psi(p(k_n f^*))\omega(t)dt \\
&\leq 1,
\end{aligned}$$

then $\frac{\psi(p(k_n))}{p(k_n)} \leq \frac{1}{W(t_0)p(k_n)}$, the left side of the inequality approaches to ∞ while the right side tends to zero, a contradiction.

2 Auxiliary Result

Theorem 2.1 *(see theorem 2.4, 2.5, 2.6 in [16])*
Set $K(f) := [k^*, k^{**}]$, there is:

(1) *If there exists $k > 0$ such that $\rho_{\psi,\omega}(p(kf)) = 1$, then $\|f\|^o_{\phi,\omega} = \frac{1}{k}\{1 + \rho_{\varphi,\omega}(kf)\}$,*
(2) *$k \in K(f)$ if and only if $\|f\|^o_{\varphi,\omega} = \frac{1}{k}\{1 + \rho_{\varphi,\omega}(kf)\}$.*

Theorem 2.2 *([11,17])*

(1) *$\inf\{k : k \in K(x), \|x\|^o_{\varphi,\omega} = 1\} > 1$ if and only if $\Phi \in \Delta_2$,*
(2) *The set $Q = \cup\{K(x) : a \leq \|x\|^o_{\varphi,\omega} \leq b\}$ is bounded for each $b \geq a > 0$ if and only if $\Phi \in \nabla_2$.*

Proof This theorem is a natural generalization of the relevant conclusions in classical Orlicz spaces.

Lemma 2.3 *([11,17]) For any Orlicz-Lorentz function space $\lambda_{\varphi,\omega}$, the following condition are equivalent:*

(1) *$\varphi \in \Delta_2$,*
(2) *$\rho_{\varphi,\omega}(f) = 1$ if and only if $\|f\|_{\varphi,\omega} = 1$,*
(3) *For arbitrary $\varepsilon > 0$, there exists $\delta > 0$, such that $\rho_{\varphi,\omega}(f) \geq 1 - \varepsilon$ whenever $\|f\|_{\varphi,\omega} \geq 1 - \delta$,*
(4) *Converge in norm and converge in modular are equivalent. That is, $\rho_{\varphi,\omega}(f_n) \to 0$ if and only if $\rho_{\varphi,\omega}(\lambda f_n) \to 0$ for $\lambda > 0$.*

Lemma 2.4 *Assume $\varphi \in \Delta_2$. For arbitrary $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$ such that if $f, g \in S(X)$, $\|f - g\|^o_{\varphi,\omega} > \varepsilon_1$, then $\rho_{\varphi,\omega}(kf - hg) > \varepsilon_2$ where $k \in K(f)$ and $h \in K(g)$.*

Proof If not, then there exist $f_n, g_n \in S(\lambda_{\varphi,\omega})$ such that $\|f_n - g_n\|^o_{\varphi,\omega} > \varepsilon_1$ while $\lim_{n\to\infty} \rho_{\varphi,\omega}(k_n f_n - h_n g_n) = 0$ where $k_n \in K(f_n)$ and $h_n \in K(g_n)$. Since $\varphi \in \Delta_2$, we obtain $\lim_{n\to\infty} \|k_n f_n - h_n g_n\|^o_{\varphi,\omega} = 0$, therefore

$$|k_n - h_n| = |k_n\|f_n\|^o_{\varphi,\omega} - h_n\|g_n\|^o_{\varphi,\omega}| \leq \|k_n f_n - h_n g_n\|^o_{\varphi,\omega} \to 0.$$

It still follows from $\varphi \in \Delta_2$ that there exists $K > 0$ such that

$$\varphi(2u) \leq K\varphi(u), \quad u \in [0, \infty).$$

Since

$$\begin{aligned}
\rho_{\varphi,\omega}(h_n(f_n - g_n)) &= \rho_{\varphi,\omega}(k_n f_n - h_n g_n - (k_n - h_n) f_n) \\
&\leq \rho_{\varphi,\omega}(|k_n f_n - h_n g_n| + |k_n - h_n| f_n) \\
&\leq \frac{1}{2}\rho_{\varphi,\omega}(2|k_n f_n - h_n g_n|) + \frac{1}{2}\rho_{\varphi,\omega}(2|k_n - h_n||f_n|) \\
&\leq \frac{K}{2}\rho_{\varphi,\omega}(k_n f_n - h_n g_n) + |k_n - h_n|\rho_{\varphi,\omega}(f_n),
\end{aligned}$$

assume $\rho_{\varphi,\omega}(f_n - g_n) > \varepsilon_3$ for $\|f_n - g_n\|^o_{\varphi,\omega} > \varepsilon_1$, then

$$\begin{aligned}
\rho_{\varphi,\omega}(k_n f_n - h_n g_n) &\geq \frac{2}{K}[\rho_{\varphi,\omega}(h_n(f_n - g_n)) - |k_n - h_n|\rho_{\varphi,\omega}(f_n)] \\
&\geq \frac{2}{K}[\rho_{\varphi,\omega}(f_n - g_n) - \frac{\varepsilon_3}{2}] \\
&\geq \frac{\varepsilon_3}{K},
\end{aligned}$$

which lead to a contradiction.

3 Main Result

Theorem 3.1 *For any N-function φ, the following conditions are equivalent:*

(1) $\lambda^o_{\varphi,\omega}$ is uniformly convex,
(2) $\lambda^o_{\varphi,\omega}$ is k-uniformly convex,
(3) $\varphi \in \Delta_2$ and φ is uniformly convex.

Proof (1) $\Rightarrow$ (2) is obviously. Now we tend to (2) $\Rightarrow$ (3). Since uniformly k-convex Banach space is reflexive, we have $\varphi \in \Delta_2 \cap \nabla_2$. Now we tend to prove that φ is uniform convex. If not, there exists $0 < \varepsilon_0 < 1$ and $u_n \uparrow \infty$ such that

$$p((1+\varepsilon_0)u_n) < (1+\frac{1}{n})p(u_n) \qquad (n \in N). \tag{1}$$

Assume $u_1 > 2$, pick $G_n = [0, a_n]$ such that

$$[k\psi(p(u_n)) + \psi(p(1+\varepsilon_0)u_n)]\int_{G_n} \omega(t)dt = k+1 \tag{2}$$

and set

$$k_n = \frac{1}{k+1}[ku_n p(u_n) + (1+\varepsilon_0)u_n p((1+\varepsilon_0)u_n)]\int_{G_n} \omega(t)dt$$

then we partition G_n into $G_n^0, G_n^1, G_n^2 ... G_n^k$ such that $\int_{G_n^i} \omega(t)dt = \frac{1}{k+1}\int_{G_n} \omega(t)dt$. and define

$$x_n^0 = \frac{1}{k_n}[u_n \sum_{j=0}^{k-1} \chi_{G_n^j} + (1+\varepsilon_0)u_n \chi_{G_n^k}],$$
$$x_n^i = \frac{1}{k_n}[u_n \sum_{j\neq i} \chi_{G_n^j} + (1+\varepsilon_0)u_n \chi_{G_n^i}]$$

For $i = 1, 2...k$ and $n \in N$. It is easy to verify $\rho_{\psi,\omega}(p(k_n x_n^i)) = 1$ and thus $\|x_n^i\|^o_{\varphi,\omega} =< p(k_n x_n^i), x_n^i >= 1$. Define $v_n = p(u_n)\chi_{G_n}$, then $\rho_{\psi,\omega}(v_n) \le 1$, so

$$\begin{aligned}
\frac{1}{k+1}\|\sum_{i=0}^{k} x_n^i\|^o_{\varphi,\omega} &\ge \frac{1}{k+1} < v_n, \sum_{i=0}^{k} x_n^i > \\
&= \frac{1}{(k+1)k_n}[ku_n p(u_n)W(mG_n) + (1+\varepsilon_0)u_n p(u_n W(mG_n))] \\
&\ge (1+\frac{1}{n})^{-1}\frac{1}{(1+k)k_n}[ku_n p(u_n) + (1+\varepsilon_0)u_n p((1+\varepsilon_0)u_n)]W(mG_n) \\
&= (1+\frac{1}{n})^{-1} \to 1.
\end{aligned}$$

Let $c_n = (k+1)W(mG_n)$, $\alpha_n = \psi^{-1}(\frac{1}{c_n})$ and $b = [(k+1)(1+\varepsilon_0)]^{-1}$. Since $\psi(p(u_n))W(mG_n) \le 1$, thus $p(u_n) \le \psi^{-1}(\frac{1}{W(mG_n)}) \le \alpha_n$. Therefore, formula(1) and formula(2) imply

$$\begin{aligned}
1 &\le (1+\varepsilon_0)u_n p((1+\varepsilon_0)u_n)W(mG_n) \\
&\le 2(1+\varepsilon_0)u_n p(u_n)W(mG_n) \\
&\le 2(1+\varepsilon_0)(k+1)c_n\alpha_n u_n.
\end{aligned}$$

Define $g_n^i=\alpha_n\chi_{G_n^i}$, $\|g_n^i\|_{\psi,\omega}=1$, $i=1,2...k$, $n\in N$. Let

$$\Delta_n=\begin{vmatrix}1 & 1 & \ldots & 1\\ g_n^1\left(x_n^0\right) & g_n^1\left(x_n^1\right) & \ldots & g_n^1\left(x_n^k\right)\\ \ldots & \ldots & \ldots & \ldots\\ g_n^k\left(x_n^0\right) & g_n^k\left(x_n^1\right) & \ldots & g_n^k\left(x_n^k\right)\end{vmatrix}.$$

Subtract the first column from all other columns in the determinant and expanding it by the first row.

$$\begin{aligned}\Delta_n&=(\frac{\varepsilon_0\alpha_n u_n W(mG_n)}{k_n(k+1)})^k\\ &\geq(\frac{\varepsilon_0}{2(1+\varepsilon_0)k_n(k+1)^3})^k\\ &\geq(\frac{\varepsilon_0}{2(1+\varepsilon_0)d(k+1)^3})^k\end{aligned}$$

where d is define in formula(3). It implies $\lambda_{\varphi,\omega}^o$ is not k-uniformly convex.

(3) $\Rightarrow$ (1): Let $f,g\in S(\lambda_{\varphi,\omega}^o)$ and $\|f-g\|_{\varphi,\omega}^o>\varepsilon_1$. Thus, from lemma 2.4, there exists φ_2 such that $\rho_{\varphi,\omega}(kf-hy)>\varepsilon_2$ where $k\in K(f)$ and $h\in K(g)$. Since φ is uniformly convex, then $\varphi\in\nabla_2$. It follows from lemma 2.3 that there exists $d>1$ such that

$$d=\sup\{l:l\in K(f),\|f\|_{\varphi,\omega}^o=1\}. \tag{3}$$

Assume that

$$\begin{aligned}a&=\min\{\inf\frac{k}{k+h},\inf\frac{h}{k+h}\},\\ b&=\max\{\sup\frac{k}{k+h},\sup\frac{h}{k+h}\}.\end{aligned}$$

Obviously, $[a,b]\subset(0,1)$. Since $\varphi\in\Delta_2$, thus there exists $K>0$ such that

$$\varphi(2du)\leq K\varphi(u).$$

Since φ is uniformly convex, there exists $\delta>0$ such that

$$\varphi(\lambda u+(1-\lambda)v)\leq(1-\delta)[\lambda\varphi(u)+(1-\lambda)\varphi(v)]$$

for any $\lambda\in[a,b]$ and all $u,v\in R$ satisfying $|u-v|\geq\frac{\varepsilon_2}{4}\max\{|u|,|v|\}$. For the given f,g there exists a measure preserving transformation([?][?]) $\sigma:\operatorname{supp}(f+g)^*\to\operatorname{supp}(f+g)$ such that

$$(f+g)^*(t)=f(\sigma(t))+g(\sigma(t)).$$

Let

$$T=\{t\in[0,\infty):|kf(\sigma(t))-hg(\sigma(t))|\geq\frac{\varepsilon_2}{4(d-1)}\max\{|kf(\sigma(t))|,|hg(\sigma(t))|\},$$

thus

$$\begin{aligned}\rho_{\varphi,\omega}[((kf(\sigma(t))-hg(\sigma(t)))\chi_{T^c})^*]&=\int_0^\infty\varphi[((kf(\sigma(t))-hg(\sigma(t)))\chi_{T^c})^*]\omega(t)dt\\ &\leq\frac{\varepsilon_2}{4(d-1)}\int_0^\infty\varphi[(kf(\sigma(t))+hg(\sigma(t))\chi_{T^c})^*]\\ &\leq\frac{\varepsilon_2}{4(d-1)}[\rho_{\varphi,\omega}(kf)+\rho_{\varphi,\omega}(hg)]\\ &\leq\frac{\varepsilon_2}{4(d-1)}(k+h-2)\\ &\leq\frac{\varepsilon_2}{4(d-1)}(2d-2)\\ &\leq\frac{\varepsilon_2}{2}\end{aligned}$$

and

$$
\begin{aligned}
2-\|f+g\|^o_{\varphi,\omega} &= \|f\|^o_{\varphi,\omega}+\|g\|^o_{\varphi,\omega}-\|f+g\|^o_{\varphi,\omega}\\
&\geq \frac{1}{k}\{1+\rho_{\varphi,\omega}(kf)\}+\frac{1}{h}\{1+\rho_{\varphi,\omega}(hg)\}-\frac{k+h}{kh}[1+\rho_{\varphi,\omega}(\frac{kh}{k+h}(f+g))]\\
&= \frac{k+h}{kh}\int_0^\infty[\frac{h}{k+h}\varphi(kf(t))^*+\frac{k}{k+h}\varphi(hg(t))^*\\
&-\varphi(\frac{kh}{k+h}(f(t)+g(t)))^*]\omega(t)dt\\
&\geq \frac{k+h}{kh}\int_T[\frac{h}{k+h}\varphi(kf(\sigma(t)))+\frac{k}{k+h}\varphi(hg(\sigma(t)))\\
&-\varphi(\frac{kh}{k+h}(f(\sigma(t))+g(\sigma(t))))]\omega(t)dt\\
&+\frac{k+h}{kh}\int_{T^c}^\infty[\frac{h}{k+h}\varphi(kf(\sigma(t)))+\frac{k}{k+h}\varphi(hg(\sigma(t)))^*\\
&-\varphi(\frac{kh}{k+h}(f(\sigma(t))+g(\sigma(t))))]\omega(t)dt\\
&\geq \frac{k+h}{kh}\int_T[\frac{h}{k+h}\varphi(kf(\sigma(t)))\\
&+\frac{k}{k+h}\varphi(hg(\sigma(t)))-\varphi(\frac{h}{k+h}kf(\sigma(t))+\frac{k}{k+h}hg(\sigma(t)))]\omega(t)dt\\
&\geq \delta\int_T[\frac{1}{k}\varphi(kf(\sigma(t)))+\frac{1}{h}\varphi(hg(\sigma(t)))]\\
&\geq \frac{2\delta}{Kd^2}\int_T\varphi((f+g)^*(t))\omega(t)dt.
\end{aligned}
\tag{4}
$$

Since $\rho_{\varphi,\omega}[(kf(\sigma(t))-hg(\sigma(t)))\chi_{T^c}]\leq\frac{\varepsilon_2}{2}$ and $\rho_{\varphi,\omega}(kf-hg)>\varepsilon_2$, then

$$\rho_{\varphi,\omega}[(kf(\sigma(t))-hg(\sigma(t)))\chi_T]\geq\frac{\varepsilon_2}{2}.$$

Therefore

$$
\begin{aligned}
\frac{\varepsilon_2}{2} &\leq \rho_{\varphi,\omega}[(kf(\sigma(t))+hg(\sigma(t)))\chi_T]\\
&= \rho_{\varphi,\omega}[\frac{kh}{k+h}(\frac{k+h}{h}f(\sigma(t))+\frac{k+h}{k}g(\sigma(t)))\chi_T]\\
&\leq K^2\rho_{\varphi,\omega}[(k+h)(f(\sigma(t))-g(\sigma(t)))\chi_T]\\
&\leq K^2\rho_{\varphi,\omega}(f(\sigma(t))+g(\sigma(t)))\chi_T\\
&\leq K^2\rho_{\varphi,\omega}[(f(t)+g(t))^*\chi_T],
\end{aligned}
$$

that is

$$\rho_{\varphi,\omega}[(f(t)+g(t))^*\chi_T]\geq\frac{\varepsilon_2}{2K^2}.$$

It is obviously to prove that there exists $\varepsilon_3>0$ such that

$$\int_T\varphi[(f(t)+g(t))^*]\omega(t)dt\geq\varepsilon_3.$$

It follows from formula(4) that

$$\|\frac{f+g}{2}\|^o_{\varphi,\omega}\leq 1-\frac{\delta\varepsilon_3}{Kd^2}.$$

Then we finish this part of proof. □

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