Indian J Pure Appl Math (2023) 54:1033–1039
https://doi.org/10.1007/s13226-022-00318-6

ORIGINAL RESEARCH

Linear preservers of Hadamard circulant majorization

G. Sankara Raju Kosuru · Subhajit Saha

Received: 21 May 2021 / Accepted: 10 August 2022 / Published online: 6 December 2022

Abstract In this paper, we study the Hadamard circulant majorization. Also, we derive the structure of linear preservers and strong linear preservers of this Hadamard circulant majorization.

Keywords Linear preserver · Hadamard circulant majorization · Circulant doubly stochastic matrix

Mathematics Subject Classification 15A86 · 15A04 · 15B51

1 Introduction and preliminaries

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ be a vector in $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the operator defined by $T(\alpha_1, \alpha_2, \ldots, \alpha_n)^t = (\alpha_n, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})^t$. A circulant matrix associated with the vector α is the $n \times n$ matrix whose k^{th} column is given by $T^{k-1}(\alpha)$, $k = 1, 2, \ldots, n$, here T^0 is the identity operator on $\mathbb{R}^n$. The circulant matrix associated with e_i, the i^{th} element of the standard ordered basis of $\mathbb{R}^n$, is called a circulant permutation matrix. Let C_1 be the circulant permutation matrix associated with the vector e_n and let $C_i = C_1^i$ for $1 \leq i \leq n$. Then each C_i is a circulant permutation matrix, for $1 \leq i \leq n$. We denote the collection of all such circulant permutation matrices by $\mathscr{C}_n$. A square matrix is said to be a permutation matrix, if it has exactly one non-zero entry of 1 in each row and each column. It is well known that each permutation matrix of order n represents a unique permutation on n symbols, say $\{1, 2, \ldots, n\}$. The permutation represented by a circulant permutation matrix is known as a circulant permutation. We denote the set of all such circulant permutations, $\{\sigma_i : 1 \leq i \leq n\}$ by $\mathscr{C}$. Here σ_i is the circulent permutation represented by the circulant permutations matrix C_i, for $1 \leq i \leq n$. A square matrix D is said to be a circulant doubly stochastic matrix if it is a convex combination of circulant permutation matrices. Let x, y be two vectors in $\mathbb{R}^n$. Then x is said to be majorized by y, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \text{ for all } 1 \leq k \leq n-1 \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Here $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ is the non-increasing rearrangement of the components of x. It is well known that, for $x, y \in \mathbb{R}^n$, x is majorized by y if and only if $x = Dy$ for some doubly stochastic matrix $D \in M_n$ ([10]).

Communicated by Rahul Roy.

G. Sankara Raju Kosuru (✉)
Department of Mathematics, Indian Institute of Technology Ropar, Rupnagar, Punjab 140 001, India
E-mail: raju@iitrpr.ac.in

Subhajit Saha
Department of Mathematics, C.V. Raman Global University, Bhubaneswar, Odisha 752 054, India
E-mail: subhajitsahamath@gmail.com

A square matrix is said to be doubly stochastic, if all of its entries are non-negative and each row sum is equal to 1, as does each column sum. As observed by Birkoff ([7]), every doubly stochastic matrix can be written as a convex combination of permutation matrices. Also, for $A, B \in M_n$, A is said to be multivariate majorized by B if $A = DB$ for some doubly stochastic matrix D ([10]). For more details of majorization and multivariate majorization the reader can refer to [1,4–6,10].

Let $M_{m,n}$ be the vector space of all $m \times n$ real matrices and let $\mathscr{R}$ be a relation on $M_{m,n}$. A linear operator T on $M_{m,n}$ is said to be a preserver (respectively a strong preserver) of $\mathscr{R}$ if $\mathscr{R}\big(T(X), T(Y)\big)$ whenever $\mathscr{R}(X, Y)$ (respectively $\mathscr{R}\big(T(X), T(Y)\big)$ if and only if $\mathscr{R}(X, Y)$), for X, Y in $M_{m,n}$. One of the interesting problems in the matrix theory is the linear preserver problem, which analyses all linear operators that preserve (and/or strongly preserve) the relation $\mathscr{R}$. For more details we refer to [3]. Earlier in [2], Ando formulated the structure of linear operators which preserve the majorization. Multiple studies generalized the notion of majorization using different settings and obtained the structure of linear preservers and strong linear preservers [8,9,11,13]. Recently, Motlaghian *et al.* in [12], introduced a notion called Hadamard majorization on $\mathbb{M}_n$ and derived the structure of linear preservers of the same.

Definition 1 [12] Let X and Y be two square matrices of order n. Then X is said to be Hadamard majorized by Y, if there exists a doubly stochastic matrix D of order n such that X is the entry-wise product of D and Y.

In this article, we study a novel notion, Hadamard circulant majorization. Further, we derive the structure of linear preservers and strong linear preservers of this Hadamard circulant majorization

2 Hadamard circulant majorization in M_n

Let $M_{m,n}$ be the vector space of all matrices of order $m \times n$ and $\mathbb{N}_n := \{1, 2, \ldots, n\}$, $\mathbb{N}_{m,n} = \mathbb{N}_m \times \mathbb{N}_n$. We denote the power set, the set of all subsets of $\mathbb{N}_{m,n}$, by $P(\mathbb{N}_{m,n})$. Define a function $\mathcal{A} : M_{m,n} \to P(\mathbb{N}_{m,n})$ by

$$\mathcal{A}(X) = \{(i, j) \in \mathbb{N}_{m,n} : x_{ij} \neq 0\}, \text{ for } X = [x_{ij}] \in M_{m,n}.$$

We denote $\mathcal{A}(X)$ by $\mathcal{A}_X$. A matrix A in $M_{m,n}$ is said be to an elementary matrix, if each non-zero entry of A is 1. Furthermore, if A has only one non-zero entry at the position (i, j), then it is denoted by E_{ij}. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of order $m \times n$. The Hadamard product of A and B is defined as the entry-wise product of A and B. We denote this by $A \odot B \ \big(:= [a_{ij}b_{ij}]\big)$.

Also the matrix A is said to be dominated by a matrix C in $M_{m,n}$ if $A \odot C = A$. It is evident from the definition that if a matrix A is dominated by a matrix C, then $\mathcal{A}_A \subseteq \mathcal{A}_C$. Conversely if $\mathcal{A}_A \subseteq \mathcal{A}_C$ and C is an elementary matrix, then the matrix A is dominated by the matrix C. Let X and Y be two non-zero matrices of order n. If X and Y are dominated by circulant permutation matrix C_i and C_j respectively with $C_i \neq C_j$, then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$ and $X + Y$ can not be dominated by any circulant permutation matrix.

Definition 2 Let X and Y be two real square matrices of order n. We say that X is Hadamard circulant majorized by Y, denoted by $X \prec_{HC} Y$, if there exists a circulant doubly stochastic matrix C of order n such that $X = C \odot Y$.

It is evident from the above definition that, $A \prec_{HC} B$ if

$$A = \begin{bmatrix} \frac{1}{3} & \frac{8}{3} & 0 \\ 0 & 1 & 4 \\ \frac{10}{3} & 0 & \frac{8}{3} \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \\ 5 & 4 & 8 \end{bmatrix}.$$

Let X and Y be two square matrices of order n. If $X \prec_{HC} Y$, then $\mathcal{A}_X \subseteq \mathcal{A}_Y$. Thus if $X \prec_{HC} \mathbf{0}$, then $X = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix. Also we have $X \prec_{HC} X$ if and only if X is dominated by a circulant permutation matrix. Furthermore, if $X \prec_{HC} Y$, $Y \prec_{HC} X$, then $\mathcal{A}_X = \mathcal{A}_Y$ and $X = D \odot C \odot X$, for some circulant doubly stochastic matrices $C = [c_{ij}]$ and $D = [d_{ij}]$. Hence $c_{ij} = 1 = d_{ij}$, whenever $x_{ij} \neq 0$. Thus we have the following:

Fact 1 *Let X and Y be two square matrices of order n. Then $X \prec_{HC} Y$ and $Y \prec_{HC} X$ if and only if $Y = X$ and X is dominated by a circulant permutation matrix.*

Let J be the elementary matrix of order n with $\mathcal{A}_J = \mathbb{N}_{n,n}$. By setting $P = \frac{1}{n^2}J$ and $Q = \frac{1}{n}J$, we have $P \prec_{HC} Q$ and $Q \prec_{HC} J$ but $P \not\prec_{HC} J$. Thus the relation "$\prec_{HC}$" is non-transitive. Furthermore, by setting $R = 2I_n$, where I_n is the identity matrix of order n, we have $R(J \odot J)J \neq (RJJ) \odot (RJJ)$. Thus, for square matrices P, Q, R and S of order n, $P(Q \odot R)S = (PQS) \odot (PRS)$ is not always true. However, for circulant permutation matrices we have the following.

Fact 2 *Let X and Y be two square matrices of order n. Then $C_i(X \odot Y)C_j = (C_iXC_j) \odot (C_iYC_j)$, for any C_i and C_j in $\mathscr{C}_n$.*

3 Linear preservers of the Hadamard circulant majorization

Let M_n be the space of all real square matrices of order n and let $T : M_n \to M_n$ be a linear operator. We say that T preserves the Hadamard circulant majorization if $T(X) \prec_{HC} T(Y)$ whenever $X \prec_{HC} Y$, for $X,\ Y$ in M_n. For a fixed $A \in M_n$, define a linear operator T on M_n by $T(X) = X \odot A$, for all $X \in M_n$. Then T preserves the Hadamard circulant majorization. Suppose T is a linear operator on M_n that preserves the Hadamard circulant majorization. If a matrix X is dominated by a matrix C_i, for some $C_i \in \mathscr{C}_n$, then $T(X)$ is dominated by a matrix C_j for some $C_j \in \mathscr{C}_n$. Furthermore, the following lemma ensures that $\mathcal{A}_{T(C_i)} \cap \mathcal{A}_{T(C_j)} = \emptyset$, for $C_i \neq C_j$ in $\mathscr{C}_n$.

Lemma 1 *Let $T : M_n \to M_n$ be a linear operator. If T preserves the Hadamard circulant majorization, then $T(E_{i\sigma_j(i)}) \odot T(E_{r\sigma_k(r)}) = \mathbf{0}$ for $1 \leq i, r \leq n$ with $\sigma_j, \sigma_k \in \mathscr{C}$ and $j \neq k$. Here $\mathbf{0}$ is the zero matrix.*

Proof Suppose that there exist j and k in $\mathbb{N}_n$ with $j \neq k$ such that $T(E_{i\sigma_j(i)}) \odot T(E_{r\sigma_k(r)}) \neq \mathbf{0}$ for some $i, r \in \mathbb{N}_n$. Without loss of generality, assume that $\langle T(E_{i\sigma_j(i)}), E_{11}\rangle = \alpha$ and $\langle T(E_{r\sigma_k(r)}), E_{11}\rangle = \beta$ for some non-zero scalars $\alpha, \beta \in \mathbb{R}$. Here $\langle, \rangle$ is the standard inner product on M_n. Let $C \in M_n$ be the circulant doubly stochastic matrix such that $\big(i, \sigma_j(i)\big)^{th}$ and $\big(r, \sigma_k(r)\big)^{th}$ entries of C are $\frac{1}{3}$ and $\frac{2}{3}$ respectively. By setting $Y = \frac{1}{\alpha}E_{i\sigma_j(i)} - \frac{1}{\beta}E_{r\sigma_k(r)}$ and $X = C \odot Y$, we have $\langle T(X), E_{11}\rangle \neq 0$ and $\langle T(Y), E_{11}\rangle = 0$. Thus $X \prec_{HC} Y$ but $T(X) \not\prec_{HC} T(Y)$. This leads to a contradiction. □

The following lemma will be useful in the sequel.

Lemma 2 *Let $T : M_n \to M_n$ be a linear operator and let $\sigma_j \in \mathscr{C}$, for some $j \in \mathbb{N}_n$. Suppose T preserves the Hadamard circulant majorization. Then for each $i \in \mathbb{N}_n$, the matrix $T(E_{i\sigma_j(i)})$ is dominated by a matrix C_{k_j}, for some $C_{k_j} \in \mathscr{C}_n$.*

Proof Since for $i \in \mathbb{N}_n$, $T(E_{i\sigma_j(i)})$ is Hadamard circulant majorized by itself, it is dominated by a circulant permutation matrix. Suppose there exist i and k with $i \neq k$ such that the matrices $T(E_{i\sigma_j(i)})$ and $T(E_{k\sigma_j(k)})$ are dominated by two circulant permutation matrices, say C_{P_j} and C_{P_k} respectively with $C_{P_j} \neq C_{P_k}$. Then $\mathcal{A}_{T(E_{i\sigma_j(i)})} \cap \mathcal{A}_{T(E_{k\sigma_j(k)})} = \emptyset$. Set $K = E_{i\sigma_j(i)} + E_{k\sigma_j(k)}$ and $P = T(E_{i\sigma_j(i)})$ and $Q = T(E_{k\sigma_j(k)})$. Without loss of generality, we assume that both the matrices P and Q are non-zero matrices. Since K is dominated by a circulant permutation matrix, $K \prec_{HC} K$. So $T(K) \prec_{HC} T(K)$. Hence $T(K) = P + Q$ is dominated by a circulant permutation matrix, which is a contradiction. This completes the proof. □

It is evident from the above lemma that, for all $1 \leq i, j, p, q \leq n$, both the matrices $T(E_{i\sigma_p(i)})$ and $T(E_{r\sigma_q(r)})$ are dominated by C_{k_p} and C_{k_q} respectively for some C_{k_p} and C_{k_q} in $\mathscr{C}_n$. Suppose $C_{k_p} = C_{k_q}$ with $p \neq q$. Then by Lemma 1, $T(E_{i\sigma_p(i)}) \odot T(E_{r\sigma_q(r)}) = \mathbf{0}$. Without loss of generality, assume $T(E_{i\sigma_p(i)})$ and $T(E_{r\sigma_q(r)})$ are non-zero matrices. Let C be the circulant doubly stochastic matrix in M_n such that the $\big(i, \sigma_p(i)\big)^{th}$ and $\big(r, \sigma_q(r)\big)^{th}$ entries of C are $\frac{1}{3}$ and $\frac{2}{3}$ respectively. By setting $Y = E_{i\sigma_p(i)} + E_{r\sigma_q(r)}$ and $X = C \odot Y$, we have $X \prec_{HC} Y$ but $T(X) \not\prec_{HC} T(Y)$. Thus we have the following:

Lemma 3 *Let $T : M_n \to M_n$ be a linear operator that preserves the Hadamard circulant majorization. Then for $p \neq q$ and i, j in $\mathbb{N}_n$, the matrices $T(E_{i\sigma_p(i)})$ and $T(E_{j\sigma_q(j)})$ are dominated by C_{k_p} and C_{k_q} respectively for some C_{k_p} and C_{k_q} in $\mathscr{C}_n$ with $C_{k_p} \neq C_{k_q}$*

Let T be the linear operator on M_3 defined by

$$T(X) = \begin{bmatrix} x_{11} & 0 & x_{13} \\ x_{21} & x_{12} + x_{23} + x_{31} & 0 \\ 0 & x_{32} & x_{33} + x_{22} \end{bmatrix}, \quad \text{for } X = [x_{ij}] \in M_3.$$

Then $T(E_{i\sigma_j(i)}) \odot T(E_{r\sigma_k(r)}) = \mathbf{0}$ for $1 \leq i, r \leq 3$ with $\sigma_j, \sigma_k \in \mathscr{C}$ and $j \neq k$. Also for each $i \in \mathbb{N}_3$, the matrix $T(E_{i\sigma_j(i)})$ is dominated by a matrix C_{k_j}, for some $C_{k_j} \in \mathscr{C}_3$ $(1 \leq j \leq 3)$. However, both the matrices $T(E_{1\sigma_1(1)})$ and $T(E_{1\sigma_3(1)})$ are dominated by the circulant permutation matrix C_3.

Let T be a linear operator on M_n that preserves the Hadamard circulant majorization. By Lemma 2, for any $i \in \mathbb{N}_n$, $T(E_{i\sigma_j(i)})$ is dominated by a circulant permutation matrix, say C_{k_j}, for $1 \leq j \leq n$. As $C_j = \sum_{i=1}^{n} E_{i\sigma_j(i)}$, the matrix $T(C_j)$ is dominated by the circulant permutation matrix C_{k_j}. Now define a function $P : \mathbb{N}_n \to \mathbb{N}_n$ by $P(j) = k_j$, for $1 \leq j \leq n$. Then by Lemma 3, we have the following:

Proposition 1 *Let $T : M_n \to M_n$ be a linear operator. Suppose T preserves the Hadamard circulant majorization. Then there exists a permutation P on $\mathbb{N}_n$ such that $T(C_j)$ is dominated by $C_{P(j)}$, for $1 \leq j \leq n$.*

The following theorem provides the structure of linear operators that preserve the Hadamard circulant majorization.

Theorem 3 *Let $T : M_n \to M_n$ be a linear operator. Suppose T preserves the Hadamard circulant majorization. Then there exists a permutation P on $\mathbb{N}_n$ and scalars $a^i_{k\sigma_j(k)} \in \mathbb{R}$, $1 \leq i, j, k \leq n$ such that for any $X = [x_{ij}]$ in M_n, we have*

$$\left\langle T(X), E_{i\sigma_{P(j)}(i)} \right\rangle = \sum_{k=1}^{n} a^i_{k\sigma_j(k)} x_{k\sigma_j(k)}$$

for $\sigma_{P(j)} \in \mathscr{C}$ and $1 \leq i, j \leq n$. Here $\langle \cdot, \cdot \rangle$ is the standard inner product on M_n.

Proof For $X = [x_{ij}] = \sum_{j=1}^{n} \sum_{i=1}^{n} x_{i\sigma_j(i)} E_{i\sigma_j(i)}$ in M_n, we have $T(X) = \sum_{j=1}^{n} \sum_{i=1}^{n} x_{i\sigma_j(i)} T(E_{i\sigma_j(i)})$. Also by Lemma 2 and Lemma 3, there exists a permutation P on $\mathbb{N}_n$ such that for each i, $(1 \leq i \leq n)$ the matrix $T(E_{i\sigma_j(i)})$ is dominated by $C_{P(j)}$, for $1 \leq j \leq n$. Let

$$a^i_{j\sigma_k(j)} = \left\langle T(E_{j\sigma_k(j)}, E_{i,g_k(i)} \right\rangle, \tag{1}$$

for all $1 \leq i, j, k \leq n$, where g_k is the function on $\mathbb{N}_n$ defined by

$$g_k(s) = \begin{cases} P(k) + s & \text{if } 1 \leq P(k) + s \leq n \\ (P(k) + s) \pmod n & \text{if } (n+1) < P(k) + s < 2n \\ n & \text{otherwise} \end{cases}$$

for all $1 \leq s \leq n$. Now

$$\begin{aligned} \langle T(X), E_{i\sigma_{P(j)}(i)} \rangle &= \left\langle \sum_{l=1}^{n} \sum_{k=1}^{n} x_{k\sigma_l(k)} T(E_{k\sigma_l(k)}), E_{i\sigma_{P(j)}(i)} \right\rangle \\ &= \sum_{l=1}^{n} \sum_{k=1}^{n} x_{k\sigma_l(k)} \left\langle T(E_{k\sigma_l(k)}), E_{i\sigma_{P(j)}(i)} \right\rangle \\ &= \sum_{k=1}^{n} x_{k\sigma_j(k)} \left\langle T(E_{k\sigma_j(k)}), E_{i\sigma_{P(j)}(i)} \right\rangle \\ &= \sum_{k=1}^{n} x_{k\sigma_j(k)} \left\langle T(E_{k\sigma_j(k)}), E_{ig_j(i)} \right\rangle \\ &= \sum_{k=1}^{n} x_{k\sigma_j(k)} a^i_{k\sigma_j(k)}. \end{aligned}$$

□

Remark 1 In the above theorem, if the permutation P is a circulant permutation, then

$$T(X) = \begin{bmatrix} \sum_{k=1}^{n} a^1_{k\sigma_n(k)} x_{k\sigma_n(k)} & \sum_{k=1}^{n} a^1_{k\sigma_1(k)} x_{k\sigma_1(k)} & \cdots & \sum_{k=1}^{n} a^1_{k\sigma_{n-1}(k)} x_{k\sigma_{n-1}(k)} \\ \sum_{k=1}^{n} a^2_{k\sigma_{n-1}(k)} x_{k\sigma_{n-1}(k)} & \sum_{k=1}^{n} a^2_{k\sigma_n(k)} x_{k\sigma_n(k)} & \cdots & \sum_{k=1}^{n} a^2_{k\sigma_{n-2}(k)} x_{k\sigma_{n-2}(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a^n_{k\sigma_1(k)} x_{k\sigma_1(k)} & \sum_{k=1}^{n} a^n_{k\sigma_2(k)} x_{k\sigma_2(k)} & \cdots & \sum_{k=1}^{n} a^n_{k\sigma_n(k)} x_{k\sigma_n(k)} \end{bmatrix} P_1,$$

where P_1 is a permutation matrix of order n.

The following example illustrates Theorem 3.

Example 1 Let $T : M_4 \to M_4$ be a linear operator, defined by

$$T(X) = \begin{bmatrix} x_{12} & x_{14} & x_{11} & x_{13} \\ x_{24} & x_{23} & x_{21} & x_{22} \\ x_{33} & x_{31} & x_{34} & x_{32} \\ x_{43} & x_{44} & x_{42} & x_{41} \end{bmatrix}, \quad \text{for } X = [x_{ij}] \in M_n.$$

The reader can verify that T preserves the Hadamard circulant majorization.

Let T be a linear operator on M_n. Suppose there exists a permutation P on $\mathbb{N}_n$ and scalars $a^i_{k\sigma_j(k)} \in \mathbb{R}$, $1 \leq i, j, k \leq n$ such that

$$\langle T(X), E_{i\sigma_{P(j)}(i)} \rangle = \sum_{k=1}^{n} a^i_{k\sigma_j(k)} x_{k\sigma_j(k)}$$

for $X = [x_{ij}] \in M_n,\ \sigma_{P(j)} \in \mathscr{C}$ and $1 \leq i, j \leq n$. Then for any $C_k \in \mathscr{C}_n$ and $B \in M_n$, we have $T(B_k) = C_{P(k)} \odot T(B)$, where $C_{P(k)} \in \mathscr{C}_n$ and $B_k = B \odot C_k$. The following theorem ensures that the converse of Theorem 3 holds as well.

Theorem 4 *Let $T : M_n \to M_n$ be a linear operator. Then the following are equivalent:*

(i) T preserves the Hadamard circulant majorization.
(ii) For any $C_k \in \mathscr{C}_n$ and $B \in M_n$, we have $T(B_k) = C_{B_k} \odot T(B)$, where C_{B_k} is a circulant doubly stochastic matrix in M_n and $B_k = B \odot C_k$.

Proof It is sufficient to prove $(ii) \Rightarrow (i)$. Assume (ii). Suppose X and Y are two matrices in M_n such that $X \prec_{HC} Y$. Then $X = C \odot Y$, where $C = \sum_{i=1}^{n} c_i C_i$ is a circulant doubly stochastic matrix. Then we have $c_i \geq 0$ for $1 \leq i \leq n$, $\sum_{i=1}^{n} c_i = 1$ and $X = \sum_{j=1}^{n} c_j Y_j$. Thus

$$\begin{aligned} T(X) &= \sum_{j=1}^{n} c_j T(Y_j) \\ &= \sum_{j=1}^{n} c_j C_{Y_j} \odot T(Y), \text{ by the assumption} \\ &= \Big(\sum_{j=1}^{n} c_j C_{Y_j} \Big) \odot T(Y). \end{aligned}$$

As $\sum_{j=1}^{n} c_j C_{Y_j}$ is a circulant doubly stochastic matrix, T preserves the Hadamard circulant majorization. □

A matrix A in M_n is said to be a non-negative matrix if all its entries are non-negative. We denote it by $A \geq 0$. Also a linear operator T on M_n is said to be a non-negative operator if $T(A) \geq 0$, whenever $A \geq 0$. The following theorem ensures that the converse of Proposition 1 holds as well, if T is a non-negative operator.

Theorem 5 *Let $T : M_n \to M_n$ be a non-negative linear operator. Then T preserves the Hadamard circulant majorization if and only if there exists a permutation P on $\mathbb{N}_n$ such that $T(C_i)$ is dominated by $C_{P(i)}$, for $1 \leq i \leq n$.*

Proof In view of Proposition 1, it is sufficient to prove the "only if" part. Assume that there exists a permutation P on $\mathbb{N}_n$ such that $T(C_i)$ is dominated by $C_{P(i)}$ for $1 \leq i \leq n$. Then $T(E_{j\sigma_i(j)})$ is dominated by $C_{P(i)}$ for $1 \leq j \leq n$. Therefore $T(E_{i\sigma_j(i)}) \odot T(E_{r\sigma_k(r)}) = \mathbf{0}$ for $1 \leq i, r \leq n$ with $j \neq k$. Now let $A, B \in M_n$ be such that $A \prec_{HC} B$. Therefore $A = C \odot B$, where $C = \sum_{i=1}^{n} c_i C_i$ is a circulant doubly stochastic matrix. Then we have $c_i \geq 0$ for $1 \leq i \leq n$, $\sum_{i=1}^{n} c_i = 1$ and $A = \sum_{j=1}^{n} c_j B_j$, where $B_j = B \odot C_j$, $1 \leq j \leq n$. Now $T(A) = \sum_{j=1}^{n} c_j T(B_j)$. As $T(B_i)$ is also dominated by $C_{P(i)}$ for $1 \leq i \leq n$, we get

$$\begin{aligned} T(A) &= \sum_{j=1}^{n} c_j \Big(C_{P(j)} \odot T(B) \Big) \\ &= \Big(\sum_{j=1}^{n} c_j C_{P(j)} \Big) \odot T(B). \end{aligned}$$

As $\sum_{j=1}^{n} c_j C_{P(j)}$ is a circulant doubly stochastic matrix, we have $T(A) \prec_{HC} T(B)$. This completes the proof. □

4 Strong linear preservers of the Hadamard circulant majorization

Let $T : M_n \to M_n$ be a linear operator. We say that T strongly preserves the Hadamard circulant majorization, if the following holds

$$T(X) \prec_{HC} T(Y) \text{ if and only if } X \prec_{HC} Y.$$

Let T be a linear operator M_n defined by $T(X) = X \odot A$, for $X \in M_n$. Here $A = [a_{ij}] \in M_n$ with $a_{ij} \neq 0$ for all $1 \leq i, j \leq n$. Then T strongly preserves the Hadamard circulant majorization. Let T be a linear operator on M_n that strongly preserves the Hadamard circulant majorization. Suppose $T(X) = \mathbf{0}$, for some X in M_n. Then $T(X) \prec_{HC} T(\mathbf{0})$ and so $X \prec_{HC} \mathbf{0}$. Thus we have the following:

Lemma 4 *Let $T : M_n \to M_n$ be a linear operator. If T strongly preserves the Hadamard circulant majorization, then T is invertible.*

Theorem 6 *Let $T : M_n \to M_n$ be an invertible linear operator. If T preserves the Hadamard circulant majorization, then so does T^{-1}.*

Proof Suppose that T preserves the Hadamard circulant majorization. Then for each $i \in \mathbb{N}_n$, we have that $T(E_{i\sigma_j(i)})$ is dominated by $C_{P(j)}$ $(1 \leq j \leq n)$, where P is a permutation on $\mathbb{N}_n$. Therefore for each $i \in \mathbb{N}_n$, $T^{-1}(E_{i\sigma_j(i)})$ is dominated by $C_{P^{-1}(j)}$ for $1 \leq j \leq n$. Thus for any $B \in M_n$, we have $T^{-1}(B_j) = C_{P^{-1}(j)} \odot T^{-1}(B)$ for $1 \leq j \leq n$, where $B_j = B \odot C_j$. Hence by Theorem 4, T^{-1} preserves the Hadamard circulant majorization. □

As a consequences of Lemma 4 and Theorem 6, we get the following characterization.

Theorem 7 *Let $T : M_n \to M_n$ be a linear operator. Then T strongly preserves the Hadamard circulant majorization if and only if T is invertible and preserves the Hadamard circulant majorization.*

By using Fact 2, we give an example that illustrates the above theorem.

Example 2 For $C_i, C_j \in \mathscr{C}_n$, define a linear operator T on M_n by $T(X) = C_i X C_j$, for $X \in M_n$. Then T strongly preserves the Hadamard circulant majorization. Also T^{-1} exists and preserves the Hadamard circulant majorization.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Eaton, Morris L., On group induced orderings, monotone functions, and convolution theorems, Inequalities in statistics and probability, IMS Lecture Notes Monogr. Ser., **5**, 13–25 (1984).
2. Ando,T., Majorization, doubly stochastic matrices, and comparison of eigenvalues, Linear Algebra and its Applications, **118**, 163–248 (1989).
3. Piere, S., A survey of linear preserver problems, Linear and Multilinear Algebra **33**, no. 1-2 (1992).
4. Ando, T., Majorizations and inequalities in matrix theory, Linear Algebra and its Applications, **199**, 17–67 (1994).
5. Bondar, James V., Comments on and complements to: *Inequalities: theory of majorization and its applications* by A. W. Marshall and I. Olkin, Linear Algebra and its Applications, **199**, 115–130 (1994).
6. Cheon, Gi-Sang and Lee, You-Ho., The doubly stochastic matrices of a multivariate majorization, Journal of the Korean Mathematical Society, **32**, 857–867 (1995).
7. Bhatia, Rajendra., Matrix analysis, Springer-Verlag, New York (1997).
8. Li, Chi-Kwong and Poon, Edward., Linear operators preserving directional majorization, Linear Algebra and its Applications, **325**, 141-146 (2001).
9. Hasani, Ahmad M. and Radjabalipour., Mehdi, The structure of linear operators strongly preserving majorizations of matrices, Electronic Journal of Linear Algebra, **15**, 260–268 (2006).
10. Marshall, Albert W and Olkin, Ingram and Arnold, Barry C., Inequalities: theory of majorization and its applications, Springer, 2dn Edition (2011).
11. Soleymani, M. and Armandnejad, A., Linear preservers of circulant majorization on $\mathbb{R}^n$, Linear Algebra and its Applications, **440**, 286–292 (2014).
12. Motlaghian, Sara M. and Armandnejad, Ali and Hall, Frank J., Linear preservers of Hadamard majorization, Electronic Journal of Linear Algebra, **31**, 593–609 (2016).
13. Mohtashami, Shiva and Salemi, Abbas and Soleymani, Mohammad., Linear preservers of circulant majorization on rectangular matrices, Linear and Multilinear Algebra, **67**, 1554-1560 (2019).