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ORIGINAL RESEARCH

Balance functors and relative tilting modules

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Abstract Let $\mathfrak{C}$ and $\mathfrak{D}$ be two classes of left R-modules, $l_{\mathfrak{D}}\mathfrak{C}$ = the class of all left R-modules admitting exact left $\mathfrak{C}$-resolutions which are $\mathrm{Hom}(-, \mathfrak{D})$-exact, $r_{\mathfrak{C}}\mathfrak{D}$ = the class of all left R-modules admitting exact right $\mathfrak{D}$-resolutions which are $\mathrm{Hom}(\mathfrak{C}, -)$-exact. We first study some properties of $l_{\mathfrak{D}}\mathfrak{C}$ and $r_{\mathfrak{C}}\mathfrak{D}$. Then, using the Hom balance functor determined by the above two special classes of modules, we introduce and investigate F-(Wakamatsu) tilting and F-(Wakamatsu) cotilting modules which are possibly infinitely generated over arbitrary rings for an additive subfunctor F of $\mathrm{Ext}^1(-, -)$. Some classical results are extended.

Keywords Balance functor · F-Wakamatsu tilting module · F-Wakamatsu cotilting module · n-F-tilting module · n-F-cotilting module

Mathematics Subject Classification 16D90 · 16E05 · 18G25

1 Introduction

Tilting theory plays an important role in the representation theory of algebras. The classic tilting modules were first introduced in the context of finite dimensional algebras by Brenner and Butler [10] and by Happel and Ringel [16]. Beginning with Miyashita [19] and in turn to Colby and Fuller [11], finitely generated tilting modules over arbitrary rings were studied. Moreover, generalizations to infinitely generated tilting and cotilting modules of finite projective dimension over arbitrary rings were investigated by Colpi and Trlifaj [12] and by Angeleri Hügel and Coelho [1]. A further generalization of tilting modules to modules of possible infinite projective dimension was made by Wakamatsu [23], which are called Wakamatsu tilting modules in [15,18]. On the other hand, in their series of papers [5–7], Auslander and Solberg applied relative homological algebra to the study of the representation theory of Artin algebras in terms of an additive subfunctor of $\mathrm{Ext}^1(-, -)$. Particularly, they introduced finitely generated relative tilting and cotilting modules in Artin algebras using the method of relative homology theory.

Encouraged by these developments, in the present paper, we will extend some results of tilting theory to a more general setting. In [13], Enochs and Jenda introduced the notion of a balanced functor, which plays an important role in relative homological algebra. We find that the classical (Wakamatsu) tilting modules are closely related to balanced pairs. Motivated by this fact, we introduce and investigate relative (Wakamatsu) tilting modules and (Wakamatsu) cotilting modules in the context of arbitrary rings and arbitrary modules from the view point of balanced functors. Let F be an additive subfunctor of $\mathrm{Ext}^1(-, -)$ and $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) be the class of

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F-projective (resp. F-injective) left R-modules. A left R-module (not necessary finitely generated) T is called an *F-Wakamatsu tilting module* if the functor $\mathrm{Hom}(-,-)$ is right balanced on $\mathrm{Add}(T) \times \mathcal{P}(F)$ by $\mathcal{P}(F) \times \mathrm{Add}(T)$ and T is called an *F-Wakamatsu cotilting module* if the functor $\mathrm{Hom}(-,-)$ is right balanced on $\mathcal{I}(F) \times \mathrm{Prod}(T)$ by $\mathrm{Prod}(T) \times \mathcal{I}(F)$, where $\mathrm{Add}(T)$ (resp. $\mathrm{Prod}(T)$) denotes the class of left R-modules isomorphic to direct summands of direct sums (resp. products) of copies of T. Furthermore, n-F-tilting and n-F-cotilting modules in the context of arbitrary rings and arbitrary modules are also introduced and investigated. We obtain many interesting properties of these relative (co)tilting modules, which generalize some known results.

Throughout this paper, all rings are associative rings and all modules are unitary. For a ring R, we write R-Mod for the category of left R-modules. ${}_RM$ stands for a left R-module. All classes of modules are assumed to be closed under isomorphisms and contain 0. For left R-modules M and N, $\mathrm{Hom}(M, N)$ and $\mathrm{Ext}^i(M, N)$ mean $\mathrm{Hom}_R(M, N)$ and $\mathrm{Ext}^i_R(M, N)$ respectively.

2 Closure properties of two special classes of modules

Let $\mathfrak{C}$ and $\mathfrak{D}$ be two classes of left R-modules. Following [13], a complex $\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$ of left R-modules is called *$\mathrm{Hom}(\mathfrak{C}, -)$-exact* if the induced sequence $\cdots \to \mathrm{Hom}(C, A_1) \to \mathrm{Hom}(C, A_0) \to \mathrm{Hom}(C, A^0) \to \mathrm{Hom}(C, A^1) \to \cdots$ is exact for any $C \in \mathfrak{C}$, and it is called *$\mathrm{Hom}(-, \mathfrak{D})$-exact* if the induced sequence $\cdots \to \mathrm{Hom}(A^1, D) \to \mathrm{Hom}(A^0, D) \to \mathrm{Hom}(A_0, D) \to \mathrm{Hom}(A_1, D) \to \cdots$ is exact for any $D \in \mathfrak{D}$. A complex $\cdots \to A_1 \to A_0 \to M \to 0$ of left R-modules with each $A_i \in \mathfrak{C}$ is called a *left $\mathfrak{C}$-resolution* of M if it is $\mathrm{Hom}(\mathfrak{C}, -)$-exact. Dually, a complex $0 \to M \to A^0 \to A^1 \to \cdots$ of left R-modules with each $A^i \in \mathfrak{D}$ is called a *right $\mathfrak{D}$-resolution* of M if it is $\mathrm{Hom}(-, \mathfrak{D})$-exact. In general, the left $\mathfrak{C}$-resolutions and right $\mathfrak{D}$-resolutions need not be exact. However we will concentrate on exact left $\mathfrak{C}$-resolutions and exact right $\mathfrak{D}$-resolutions in this paper. We always write

$l_{\mathfrak{D}}\mathfrak{C}$ (resp. $l_{\mathfrak{D}}\mathfrak{C}_n$) = the class of all left R-modules admitting exact left $\mathfrak{C}$-resolutions (resp. with vanishing $(n+1)$th-kernels) which are $\mathrm{Hom}(-, \mathfrak{D})$-exact,

$r_{\mathfrak{C}}\mathfrak{D}$ (resp. $r_{\mathfrak{C}}\mathfrak{D}_n$) = the class of all left R-modules admitting exact right $\mathfrak{D}$-resolutions (resp. with vanishing $(n+1)$th-cokernels) which are $\mathrm{Hom}(\mathfrak{C}, -)$-exact.

These special classes of modules admit nice closure properties as follows.

Theorem 2.1 *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two classes of left R-modules with $\mathfrak{C}$ closed under finite direct sums, $0 \to A \xrightarrow{\varphi} A' \xrightarrow{\gamma} A'' \to 0$ be an exact sequence of left R-modules which is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact.*

(1) If $A \in l_{\mathfrak{D}}\mathfrak{C}$, then $A' \in l_{\mathfrak{D}}\mathfrak{C}$ if and only if $A'' \in l_{\mathfrak{D}}\mathfrak{C}$.
(2) $l_{\mathfrak{D}}\mathfrak{C}$ is closed under finite direct sums and direct summands.
(3) If A' and $A'' \in l_{\mathfrak{D}}\mathfrak{C}$, then $A \in l_{\mathfrak{D}}\mathfrak{C}$.

Proof (1) Suppose that $\cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A \to 0$ is an exact left $\mathfrak{C}$-resolution of A which is $\mathrm{Hom}(-, \mathfrak{D})$-exact.

"$\Leftarrow$" If A'' has an exact left $\mathfrak{C}$-resolution $\cdots \xrightarrow{f''_2} A''_1 \xrightarrow{f''_1} A''_0 \xrightarrow{f''_0} A'' \to 0$ which is $\mathrm{Hom}(-, \mathfrak{D})$-exact, then there is $\xi : A''_0 \to A'$ such that $\gamma\xi = f''_0$. Define $f'_0 : A_0 \oplus A''_0 \to A'$ by $f'_0(x, y) = \varphi f_0(x) + \xi(y)$ for $x \in A_0$ and $y \in A''_0$. By 3×3 Lemma, we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(f_0) \arrow[r, dotted, "\phi"] \arrow[d, "\iota"] & \ker(f'_0) \arrow[r, dotted, "\rho"] \arrow[d, "\iota'"] & \ker(f''_0) \arrow[r] \arrow[d, "\iota''"] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d, "f_0"] & A_0 \oplus A''_0 \arrow[r] \arrow[d, "f'_0"] & A''_0 \arrow[r] \arrow[d, "f''_0"] & 0 \\
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d] & A' \arrow[r, "\gamma"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By hypothesis, we have $A_0 \oplus A_0'' \in \mathfrak{C}$. It is easy to verify that $0 \to \ker(f_0') \xrightarrow{\iota'} A_0 \oplus A_0'' \xrightarrow{f_0'} A' \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact by 3×3 Lemma. Note that $\ker(f_0)$ and $\ker(f_0'') \in l_{\mathfrak{D}}\mathfrak{C}$, and $0 \to \ker(f_0) \xrightarrow{\phi} \ker(f_0') \xrightarrow{\rho} \ker(f_0'') \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. Repeating the process, we get an exact left $\mathfrak{C}$-resolution of A' which is $\mathrm{Hom}(-, \mathfrak{D})$-exact. Thus $A' \in l_{\mathfrak{D}}\mathfrak{C}$.

" $\Rightarrow$" Assume that A' has an exact left $\mathfrak{C}$-resolution $\cdots \to A_1' \xrightarrow{h_1} A_0' \xrightarrow{h_0} A' \to 0$ which is $\mathrm{Hom}(-, \mathfrak{D})$-exact. Let $X = \ker(h_0)$. We get the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] & & \\
0 \arrow[r] & G \arrow[r] \arrow[d, "\delta"'] & A_0' \arrow[r] \arrow[d, "h_0"'] & A'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d] & A' \arrow[r, "\gamma"] \arrow[d] & A'' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

For any $\alpha : C \to A$ with $C \in \mathfrak{C}$, there is $\beta : C \to A_0'$ such that $h_0\beta = \varphi\alpha$. By the universal property of a pullback, there is $\psi : C \to G$ such that $\delta\psi = \alpha$. So $0 \to X \to G \xrightarrow{\delta} A \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. Since A and $X \in l_{\mathfrak{D}}\mathfrak{C}$, we have $G \in l_{\mathfrak{D}}\mathfrak{C}$ by the previous result. It is easy to check that $0 \to G \to A_0' \to A'' \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. So we get $A'' \in l_{\mathfrak{D}}\mathfrak{C}$.

(2) Let $B \in l_{\mathfrak{D}}\mathfrak{C}$ and M be a direct summand of B. Then there is N such that $B = N \oplus M$. Assume that B has an exact left $\mathfrak{C}$-resolution $\cdots \xrightarrow{g_2} B_1 \xrightarrow{g_1} B_0 \xrightarrow{g_0} B \to 0$ which is $\mathrm{Hom}(-, \mathfrak{D})$-exact. Let $K = \ker(g_0)$, we get the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & Q \arrow[r, "\theta"] \arrow[d] & B_0 \arrow[r] \arrow[d, "g_0"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By the universal property of a pullback, $0 \to K \to Q \to N \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. So the direct sum $0 \to K \to Q \oplus M \to N \oplus M \to 0$ of $0 \to K \to Q \to N \to 0$ and $0 \to 0 \to M \to M \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. Since $K \in l_{\mathfrak{D}}\mathfrak{C}$ and $N \oplus M = B \in l_{\mathfrak{D}}\mathfrak{C}$, we have $Q \oplus M \in l_{\mathfrak{D}}\mathfrak{C}$ by (1). It is easy to verify that $0 \to Q \xrightarrow{\theta} B_0 \to M \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. Repeating the process above to $0 \to M \to Q \oplus M \to Q \to 0$ and so on, we get an exact left $\mathfrak{C}$-resolution of M which is $\mathrm{Hom}(-, \mathfrak{D})$-exact. It follows that $l_{\mathfrak{D}}\mathfrak{C}$ is closed under direct summands.

$l_{\mathfrak{D}}\mathfrak{C}$ is closed under finite direct sums by (1).

(3) There is a $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact exact sequence $0 \to U \to A_0'' \to A'' \to 0$ with $A_0'' \in \mathfrak{C}$ and $U \in l_{\mathfrak{D}}\mathfrak{C}$. So we get the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & U \arrow[r, equal] \arrow[d] & U \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & A_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & A' \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By the universal property of a pullback, $0 \to U \to V \to A' \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and $\mathrm{Hom}(-, \mathfrak{D})$-exact. Since U and $A' \in l_{\mathfrak{D}}\mathfrak{C}$, we have $V \in l_{\mathfrak{D}}\mathfrak{C}$ by (1). It is easy to check that $0 \to A \to V \to A_0'' \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact and so is split since $A_0'' \in \mathfrak{C}$. Then $V \cong A \oplus A_0''$. Hence $A \in l_{\mathfrak{D}}\mathfrak{C}$ by (2). □

Dually, we have

Theorem 2.2 *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two classes of left R-modules with $\mathfrak{D}$ closed under finite direct sums, $0 \to A \to A' \to A'' \to 0$ be an exact sequence which is* $\mathrm{Hom}(\mathfrak{C}, -)$*-exact and* $\mathrm{Hom}(-, \mathfrak{D})$*-exact.*

(1) If $A'' \in r_{\mathfrak{C}}\mathfrak{D}$, then $A \in r_{\mathfrak{C}}\mathfrak{D}$ if and only if $A' \in r_{\mathfrak{C}}\mathfrak{D}$.
(2) $r_{\mathfrak{C}}\mathfrak{D}$ is closed under finite direct sums and direct summands.
(3) If A and $A' \in r_{\mathfrak{C}}\mathfrak{D}$, then $A'' \in r_{\mathfrak{C}}\mathfrak{D}$.

Proposition 2.3 *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two classes of left R-modules.*

(1) If an exact sequence of left R-modules $0 \to A \xrightarrow{f} A' \xrightarrow{g} A'' \to 0$ with $A'' \in l_{\mathfrak{D}}\mathfrak{C}$ is $\mathrm{Hom}(\mathfrak{C}, -)$*-exact, then it is* $\mathrm{Hom}(-, \mathfrak{D})$*-exact.*
(2) If an exact sequence of left R-modules $0 \to A \xrightarrow{f} A' \xrightarrow{g} A'' \to 0$ with $A \in r_{\mathfrak{C}}\mathfrak{D}$ is $\mathrm{Hom}(-, \mathfrak{D})$*-exact, then it is* $\mathrm{Hom}(\mathfrak{C}, -)$*-exact.*

Proof (1) There is an exact sequence $0 \to K \xrightarrow{h} X \xrightarrow{\phi} A'' \to 0$ with $X \in \mathfrak{C}$ which is $\mathrm{Hom}(-, \mathfrak{D})$-exact. So there exists $\tau : X \to A'$ such that $g\tau = \phi$. Thus there is $\theta : K \to A$ such that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "h"] \arrow[d, dotted, "\theta"'] & X \arrow[r, "\phi"] \arrow[d, "\tau"'] & A'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & A' \arrow[r, "g"] & A'' \arrow[r] & 0.
\end{tikzcd}$$

It is clearly a pushout diagram. For any $\alpha : A \to D$ with $D \in \mathfrak{D}$, there is $\beta : X \to D$ such that $\beta h = \alpha\theta$. By the universal property of a pushout, there is $\gamma : A \to D$ such that $\gamma f = \alpha$. Therefore $0 \to A \xrightarrow{f} A' \xrightarrow{g} A'' \to 0$ is $\mathrm{Hom}(-, \mathfrak{D})$-exact.

The proof of (2) is dual. □

3 Relative orthogonality and resolutions of modules

In what follows, F always means an additive subfunctor of $\mathrm{Ext}^1(-, -)$. Following [5], a short exact sequence $0 \to M \xrightarrow{f} G \xrightarrow{g} N \to 0$ of left R-modules is called *F-exact* if it is in $F(N, M)$. In this case, f is called an F-*monomorphism* and g is called an F-*epimorphism*. A long exact sequence is called F-*exact* if it splits into short F-exact sequences. A left R-module P is called F-*projective* if all short F-exact sequences ending in P split. An F-injective module is defined dually. We denote by $\mathcal{P}(F)$ and $\mathcal{I}(F)$ the class of F-projective left R-modules and the class of F-injective left R-modules respectively.

Example 3.1 (1) Let $F = \mathrm{Ext}^1(-,-)$. Then F-exact sequences are usual exact sequences, F-projective and F-injective modules are just classical projective and injective modules respectively.

(2) The rule that associates to a pair of left R-modules (N, M) the subgroup $\mathrm{Pext}^1(N, M) \subseteq \mathrm{Ext}^1(N, M)$ of pure exact sequences $0 \to M \to U \to N \to 0$ constitutes a subfunctor F of $\mathrm{Ext}^1(-,-)$. F-projective and F-injective modules are exactly pure projective and pure injective modules respectively.

(3) Suppose that R is a Gorenstein ring, i.e., R is a left and right Noetherian ring with finite left and right self-injective dimensions, $GProj$ and $GInj$ are the subcategories of R-Mod consisting of Gorenstein projective and Gorenstein injective left R-modules respectively (see [13]). Let F be the subfunctor of $\mathrm{Ext}^1(-,-)$ defined by the condition that $\eta \in F(N, M)$ provided that $\eta : 0 \to M \to U \to N \to 0$ is $\mathrm{Hom}(GProj, -)$-exact, equivalently, η is $\mathrm{Hom}(-, GInj)$-exact by [13, Corollary 12.3.5]. Then F-projective and F-injective modules are exactly Gorenstein projective and Gorenstein injective modules respectively.

We say that an additive subfunctor F of $\mathrm{Ext}^1(-,-)$ has enough projectives if for any left R-module M, there exists an F-exact sequence $0 \to N \to P \to M \to 0$ with $P \in \mathcal{P}(F)$. Having enough injectives is defined dually. We always assume that F has enough projectives and injectives in the sequel. By [5, Proposition 1.5], a short exact sequence is F-exact if and only if it is $\mathrm{Hom}(\mathcal{P}(F), -)$-exact if and only if it is $\mathrm{Hom}(-, \mathcal{I}(F))$-exact. Every pushout and pullback of a short F-exact sequence are again F-exact and the direct sum (product) of short exact sequences is F-exact if and only if each of them is F-exact.

We denote by $\mathrm{Ext}^i_F(-,-)$ the ith right derived functor of $\mathrm{Hom}(-,-)$, which can be computed using a left $\mathcal{P}(F)$-resolution of the first variable or a right $\mathcal{I}(F)$-resolution of the second variable. As usual, we denote by $pd_F(M)$ the F-projective dimension of a left R-module M, i.e., $pd_F(M)$ = the minimum length of left $\mathcal{P}(F)$-resolutions of M. Dually, we denote by $id_F(N)$ the F-injective dimension of a left R-module N. $\mathcal{P}(F)^{\leq n}$ (resp. $\mathcal{I}(F)^{\leq n}$) stands for the subcategory of all left R-modules with F-projective (resp. F-injective) dimension at most n, $\mathcal{P}(F)^{<\infty}$ (resp. $\mathcal{I}(F)^{<\infty}$) stands for the subcategory of all left R-modules with finite F-projective (resp. F-injective) dimension. Note that $pd_F(M) \leq n$ if and only if $\mathrm{Ext}^{n+i}_F(M, N) = 0$ for any left R-module N and $i \geq 1$, $id_F(N) \leq n$ if and only if $\mathrm{Ext}^{n+i}_F(M, N) = 0$ for any left R-module M and $i \geq 1$.

Given a class $\mathfrak{C}$ of left R-modules, to simply notations, we always write $\mathfrak{C}^{\perp_i} = \{M \in R\text{-Mod}: \mathrm{Ext}^i_F(C, M) = 0 \text{ for all } C \in \mathfrak{C}\}$ and ${}^{\perp_i}\mathfrak{C} = \{M \in R\text{-Mod}: \mathrm{Ext}^i_F(M, C) = 0 \text{ for all } C \in \mathfrak{C}\}$ in what follows. We define $\mathfrak{C}^{\perp} = \bigcap\limits_{i\geq 1} \mathfrak{C}^{\perp_i}$ and ${}^{\perp}\mathfrak{C} = \bigcap\limits_{i\geq 1} {}^{\perp_i}\mathfrak{C}$. These are called the *right* and *left F-orthogonal classes* of $\mathfrak{C}$ respectively. We call a class $\mathfrak{C}$ of left R-modules F-*self-orthogonal* if $\mathfrak{C} \subseteq \mathfrak{C}^{\perp}$. In the special case where $\mathfrak{C}$ consists of only one element T, we let $T^{\perp_i} = \{T\}^{\perp_i}$ and ${}^{\perp_i}T = {}^{\perp_i}\{T\}$ for convenience. Similarly, we can define $T^{\perp}$ and ${}^{\perp}T$.

Given a class $\mathfrak{C}$ of left R-modules, we always write $\mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)} = \mathfrak{C}^{\perp} \bigcap l_{\mathcal{I}(F)}\mathfrak{C}$ and $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} = {}^{\perp}\mathfrak{C} \bigcap r_{\mathcal{P}(F)}\mathfrak{C}$ in the sequel.

Proposition 3.2 *Let $\mathfrak{C}$ be a class of left R-modules closed under finite direct sums.*

(1) $\mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$ and $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$ are closed under finite direct sums and direct summands.
(2) If an exact sequence $0 \to A \to A' \to A'' \to 0$ is F-exact with $A \in \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$, then $A' \in \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$ if and only if $A'' \in \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$.
(3) If an exact sequence $0 \to A \to A' \to A'' \to 0$ is F-exact with $A'' \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$, then $A \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$ if and only if $A' \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$.
(4) If an exact sequence $0 \to A \to A' \to A'' \to 0$ is $\mathrm{Hom}(\mathfrak{C}, -)$-exact with $A'' \in \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$, then $A \in \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$ if and only if $A' \in \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$.
(5) If an exact sequence $0 \to A \to A' \to A'' \to 0$ is $\mathrm{Hom}(-, \mathfrak{C})$-exact with $A \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$, then $A' \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$ if and only if $A'' \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$.

Proof It is straightforward by Theorems 2.1, 2.2 and Proposition 2.3. □

Let $\mathfrak{C}$ and $\mathfrak{D}$ be two classes of left R-modules. We call $\mathfrak{C}$ an F-*generator* of $\mathfrak{D}$ if $\mathfrak{C} \subseteq \mathfrak{D}$ and for any $M \in \mathfrak{D}$, there exists an F-exact sequence $0 \to K \to C \to M \to 0$ with $C \in \mathfrak{C}$ and $K \in \mathfrak{D}$. Dually, we call $\mathfrak{C}$ an

F*-cogenerator* of $\mathfrak{D}$ if $\mathfrak{C} \subseteq \mathfrak{D}$ and for any $N \in \mathfrak{D}$, there exists an F-exact sequence $0 \to N \to X \to Y \to 0$ with $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$. Specializing $F = \mathrm{Ext}^1(-,-)$, an F-generator (resp. F-cogenerator) is just the generator (resp. cogenerator) in [22].

Theorem 3.3 *Let $\mathfrak{C}$ be a class of left R-modules closed under direct summands. The following conditions are equivalent:*

(1) $\mathfrak{C}$ is an F-self-orthogonal class.
(2) $\mathfrak{C} \subseteq l_{\mathfrak{C}}\mathcal{P}(F)$.
(3) $\mathfrak{C} \subseteq r_{\mathfrak{C}}\mathcal{I}(F)$.
(4) $\mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)} \bigcap {}^{\perp}\mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)} = \mathfrak{C}$.
(5) $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap (\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)})^{\perp} = \mathfrak{C}$.
(6) $l_{\mathcal{I}(F)}\mathfrak{C}_n = \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)} \bigcap {}^{\perp_{n+1}}(\mathfrak{C}^{\perp})$ for any $n \geq 1$.
(7) $r_{\mathcal{P}(F)}\mathfrak{C}_n = \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap ({}^{\perp}\mathfrak{C})^{\perp_{n+1}}$ for any $n \geq 1$.
(8) $\mathfrak{C}$ is an F-generator of $\mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)}$.
(9) $\mathfrak{C}$ is an F-cogenerator of $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$.

Proof (1) $\Rightarrow$ (2) Any $M \in \mathfrak{C}$ has an exact left $\mathcal{P}(F)$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$. Since $\mathrm{Ext}^i_F(M, N) = 0$ for any $N \in \mathfrak{C}$ and $i \geq 1$, the left $\mathcal{P}(F)$-resolution is $\mathrm{Hom}(-, \mathfrak{C})$-exact. So $\mathfrak{C} \subseteq l_{\mathfrak{C}}\mathcal{P}(F)$.

(2) $\Rightarrow$ (1) For any $M \in \mathfrak{C}$, there is an exact left $\mathcal{P}(F)$-resolution $\cdots \to C_1 \to C_0 \to M \to 0$ which is $\mathrm{Hom}(-, \mathfrak{C})$-exact. Then $\mathrm{Ext}^i_F(M, N) = 0$ for any $N \in \mathfrak{C}$ and $i \geq 1$. So $\mathfrak{C}$ is an F-self-orthogonal class.

(1) $\Rightarrow$ (5) It is clear that $\mathfrak{C} \subseteq \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap (\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)})^{\perp}$.

Conversely, let $X \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap (\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)})^{\perp}$, there is an F-exact sequence $0 \to X \to Y \to Z \to 0$ with $Y \in \mathfrak{C}$ and $Z \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$. Since $\mathrm{Ext}^1_F(Z, X) = 0$, the exact sequence is split. Thus $X \in \mathfrak{C}$ and so $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap (\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)})^{\perp} \subseteq \mathfrak{C}$. Hence $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap (\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)})^{\perp} = \mathfrak{C}$.

(5) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (7) Let $M \in r_{\mathcal{P}(F)}\mathfrak{C}_n$. Then there is an F-exact right $\mathfrak{C}$-resolution $0 \to M \to D^0 \to D^1 \to \cdots \to D^n \to 0$. So $\mathrm{Ext}^i_F(M, N) \cong \mathrm{Ext}^{n+i}_F(D^n, N) = 0$ for any $N \in \mathfrak{C}$ and $i \geq 1$. Thus $M \in {}^{\perp}\mathfrak{C}$. On the other hand, for any $U \in {}^{\perp}\mathfrak{C}$, $\mathrm{Ext}^{n+1}_F(U, M) \cong \mathrm{Ext}^1_F(U, D^n) = 0$. Hence $M \in ({}^{\perp}\mathfrak{C})^{\perp_{n+1}}$. So $r_{\mathcal{P}(F)}\mathfrak{C}_n \subseteq \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap ({}^{\perp}\mathfrak{C})^{\perp_{n+1}}$.

Conversely, let $V \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap ({}^{\perp}\mathfrak{C})^{\perp_{n+1}}$, then there is an F-exact right $\mathfrak{C}$-resolution $0 \to V \xrightarrow{f^0} Q^0 \xrightarrow{f^1} Q^1 \xrightarrow{f^2} \cdots$. It is easy to see that $\mathrm{coker}(f^n) \in {}^{\perp}\mathfrak{C}$. So we have $\mathrm{Ext}^1_F(\mathrm{coker}(f^n), \mathrm{coker}(f^{n-1})) \cong \mathrm{Ext}^{n+1}_F(\mathrm{coker}(f^n), V) = 0$. Hence the exact sequence $0 \to \mathrm{coker}(f^{n-1}) \to Q^n \to \mathrm{coker}(f^n) \to 0$ is split. Thus $\mathrm{coker}(f^{n-1}) \in \mathfrak{C}$. Therefore $V \in r_{\mathcal{P}(F)}\mathfrak{C}_n$ and so $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap ({}^{\perp}\mathfrak{C})^{\perp_{n+1}} \subseteq r_{\mathcal{P}(F)}\mathfrak{C}_n$. It follows that $r_{\mathcal{P}(F)}\mathfrak{C}_n = \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap ({}^{\perp}\mathfrak{C})^{\perp_{n+1}}$.

(7) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (9) It is clear that $\mathfrak{C} \subseteq \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$. Let $A \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$. There is an F-exact sequence $0 \to A \to D \to B \to 0$ with $D \in \mathfrak{C}$ and $B \in r_{\mathcal{P}(F)}\mathfrak{C}$ which is $\mathrm{Hom}(-, \mathfrak{C})$-exact. For any $N \in \mathfrak{C}$, we get the induced exact sequence $\mathrm{Hom}(D, N) \to \mathrm{Hom}(A, N) \to \mathrm{Ext}^1_F(B, N) \to \mathrm{Ext}^1_F(D, N) = 0$. Since $\mathrm{Hom}(D, N) \to \mathrm{Hom}(A, N)$ is an epimorphism, we get $B \in {}^{\perp_1}\mathfrak{C}$. From the induced exact sequence $0 = \mathrm{Ext}^i_F(A, N) \to \mathrm{Ext}^{i+1}_F(B, N) \to \mathrm{Ext}^{i+1}_F(D, N) = 0$, we have $B \in {}^{\perp_i}\mathfrak{C}$ for any $i \geq 1$. Hence $B \in {}^{\perp}\mathfrak{C}$ and so $B \in \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$. Thus $\mathfrak{C}$ is an F-cogenerator of $\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}$.

(9) $\Rightarrow$ (1) is obvious.

The rest are dual. □

Proposition 3.4 *Let $\mathfrak{C}$ be a class of left R-modules closed under direct summands.*

(1) $\mathcal{P}(F) \subseteq r_{\mathcal{P}(F)}\mathfrak{C}$ if and only if ${}^{\perp}\mathfrak{C} \bigcap \mathcal{P}(F)^{<\infty} = \mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)} \bigcap \mathcal{P}(F)^{<\infty}$.
(2) $\mathcal{I}(F) \subseteq l_{\mathcal{I}(F)}\mathfrak{C}$ if and only if $\mathfrak{C}^{\perp} \bigcap \mathcal{I}(F)^{<\infty} = \mathfrak{L}^{\mathfrak{C}}_{\mathcal{I}(F)} \bigcap \mathcal{I}(F)^{<\infty}$.

Proof (1) "$\Rightarrow$" Let $M \in {}^{\perp}\mathfrak{C} \bigcap \mathcal{P}(F)^{<\infty}$. Then M has a left $\mathcal{P}(F)$-resolution of the form $0 \to C_m \to \cdots \to C_1 \to C_0 \to M \to 0$ which is $\mathrm{Hom}(-, \mathfrak{C})$-exact. Since each $C_i \in r_{\mathcal{P}(F)}\mathfrak{C}$, we infer that $M \in r_{\mathcal{P}(F)}\mathfrak{C}$ by

Theorem 2.2(3) and so ${}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}\subseteq\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}\bigcap\mathcal{P}(F)^{<\infty}$. The converse is clear. Hence ${}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}=\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}\bigcap\mathcal{P}(F)^{<\infty}$.

"$\Leftarrow$" Since $\mathcal{P}(F)\subseteq{}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}=\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}\bigcap\mathcal{P}(F)^{<\infty}$, $\mathcal{P}(F)\subseteq r_{\mathcal{P}(F)}\mathfrak{C}$.

The proof of (2) is dual. □

Theorem 3.5 *Let $\mathfrak{C}$ be a class of left R-modules closed under direct summands. The following conditions are equivalent:*

(1) $\mathfrak{C}\subseteq l_{\mathfrak{C}}\mathcal{P}(F)_n$ and $\mathcal{P}(F)\subseteq r_{\mathcal{P}(F)}\mathfrak{C}_n$.
(2) $\mathfrak{C}\subseteq l_{\mathfrak{C}}\mathcal{P}(F)_n$ and every $C\in\mathcal{P}(F)$ has an F-exact right $\mathfrak{C}$-resolution of finite length.
(3) $\mathfrak{C}\subseteq\mathcal{P}(F)^{\leq n}$ and ${}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}=(r_{\mathcal{P}(F)}\mathfrak{C}_n)\bigcap\mathcal{P}(F)^{<\infty}$.

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $M\in\mathcal{P}(F)$. Then M has an F-exact right $\mathfrak{C}$-resolution of the form $0\to M\xrightarrow{g^0}C^0\xrightarrow{g^1}C^1\xrightarrow{g^2}\cdots\xrightarrow{g^r}C^r\to 0$. If $r\leq n$, then it is done. Now suppose that $r>n$. We get the F-exact sequence $0\to M\xrightarrow{g^0}C^0\xrightarrow{g^1}C^1\xrightarrow{g^2}\cdots\xrightarrow{g^{n-1}}C^{n-1}\to\operatorname{coker}(g^{n-1})\to 0$ which is $\operatorname{Hom}(-,\mathfrak{C})$-exact. For any $N\in\mathfrak{C}$, we have $N\in\mathcal{P}(F)^{\leq n}$. By Theorem 3.3, $\operatorname{Ext}^i_F(N,\operatorname{coker}(g^{n-1}))\cong\operatorname{Ext}^{i+n}_F(N,M)=0$ for any $i\geq 1$. Thus $\operatorname{coker}(g^{n-1})\in\mathfrak{C}^{\perp}$. From the F-exact sequence $0\to\operatorname{coker}(g^n)\to C^{n+1}\xrightarrow{g^{n+2}}\cdots\xrightarrow{g^r}C^r\to 0$, we get $\operatorname{Ext}^1_F(\operatorname{coker}(g^n),\operatorname{coker}(g^{n-1}))\cong\operatorname{Ext}^{r-n}_F(C^r,\operatorname{coker}(g^{n-1}))=0$. Hence the exact sequence $0\to\operatorname{coker}(g^{n-1})\to C^n\to\operatorname{coker}(g^n)\to 0$ is split, which means $\operatorname{coker}(g^{n-1})\in\mathfrak{C}$. Thus $M\in r_{\mathcal{P}(F)}\mathfrak{C}_n$ and hence $\mathcal{P}(F)\subseteq r_{\mathcal{P}(F)}\mathfrak{C}_n$.

(1) $\Rightarrow$ (3) It is clear that $\mathfrak{C}\subseteq\mathcal{P}(F)^{\leq n}$ and $(r_{\mathcal{P}(F)}\mathfrak{C}_n)\bigcap\mathcal{P}(F)^{<\infty}\subseteq{}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}$.

Let $A\in{}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}$. Then A has a left $\mathcal{P}(F)$-resolution of the form $0\to N_m\to\cdots\to N_1\to N_0\to A\to 0$ which is $\operatorname{Hom}(-,\mathfrak{C})$-exact. By Theorem 3.3, each $N_i\in\mathcal{P}(F)\subseteq r_{\mathcal{P}(F)}\mathfrak{C}_n=\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}\bigcap({}^{\perp}\mathfrak{C})^{\perp_{n+1}}$. By Proposition 3.2(5), $A\in\mathfrak{R}^{\mathfrak{C}}_{\mathcal{P}(F)}\bigcap({}^{\perp}\mathfrak{C})^{\perp_{n+1}}=r_{\mathcal{P}(F)}\mathfrak{C}_n$. So ${}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}\subseteq(r_{\mathcal{P}(F)}\mathfrak{C}_n)\bigcap\mathcal{P}(F)^{<\infty}$. Hence ${}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}=(r_{\mathcal{P}(F)}\mathfrak{C}_n)\bigcap\mathcal{P}(F)^{<\infty}$.

(3) $\Rightarrow$ (1) Since $\mathfrak{C}\subseteq\mathcal{P}(F)^{\leq n}$, $\mathfrak{C}\subseteq(r_{\mathcal{P}(F)}\mathfrak{C}_n)\bigcap\mathcal{P}(F)^{<\infty}={}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}$. Thus $\mathfrak{C}$ is an F-self-orthogonal class and so $\mathfrak{C}\subseteq l_{\mathfrak{C}}\mathcal{P}(F)_n$. Since $\mathcal{P}(F)\subseteq{}^{\perp}\mathfrak{C}\bigcap\mathcal{P}(F)^{<\infty}=(r_{\mathcal{P}(F)}\mathfrak{C}_n)\bigcap\mathcal{P}(F)^{<\infty}$, $\mathcal{P}(F)\subseteq r_{\mathcal{P}(F)}\mathfrak{C}_n$. □

Dually, we have

Theorem 3.6 *Let $\mathfrak{C}$ be a class of left R-modules closed under direct summands. The following conditions are equivalent:*

(1) $\mathcal{I}(F)\subseteq l_{\mathcal{I}(F)}\mathfrak{C}_n$ and $\mathfrak{C}\subseteq r_{\mathfrak{C}}\mathcal{I}(F)_n$.
(2) $\mathfrak{C}\subseteq r_{\mathfrak{C}}\mathcal{I}(F)_n$ and every $D\in\mathcal{I}(F)$ has an F-exact left $\mathfrak{C}$-resolution of finite length.
(3) $\mathfrak{C}\subseteq\mathcal{I}(F)^{\leq n}$ and $\mathfrak{C}^{\perp}\bigcap\mathcal{I}(F)^{<\infty}=(l_{\mathcal{I}(F)}\mathfrak{C}_n)\bigcap\mathcal{I}(F)^{<\infty}$.

4 Relative Wakamatsu tilting and cotilting modules

Definition 4.1 Let F be an additive subfunctor of $\operatorname{Ext}^1(-,-)$. A left R-module (not necessary finitely generated) T is called *F-Wakamatsu tilting* if the functor $\operatorname{Hom}(-,-)$ is right balanced on $\operatorname{Add}(T)\times\mathcal{P}(F)$ by $\mathcal{P}(F)\times\operatorname{Add}(T)$ in the sense of [13], i.e., $\operatorname{Add}(T)\subseteq l_{\operatorname{Add}(T)}\mathcal{P}(F)$ and $\mathcal{P}(F)\subseteq r_{\mathcal{P}(F)}\operatorname{Add}(T)$.

Dually, T is called an *F-Wakamatsu cotilting module* if the functor $\operatorname{Hom}(-,-)$ is right balanced on $\mathcal{I}(F)\times\operatorname{Prod}(T)$ by $\operatorname{Prod}(T)\times\mathcal{I}(F)$, i.e., $\mathcal{I}(F)\subseteq l_{\mathcal{I}(F)}\operatorname{Prod}(T)$ and $\operatorname{Prod}(T)\subseteq r_{\operatorname{Prod}(T)}\mathcal{I}(F)$.

We first give some characterizations of F-Wakamatsu tilting and F-Wakamatsu cotilting modules.

Let $\mathfrak{C}$ be a class of left R-modules and M be a left R-module. Following [13], we say that a morphism $\phi: C\to M$ is a *$\mathfrak{C}$-precover* of M if $C\in\mathfrak{C}$ and the Abelian group homomorphism $\operatorname{Hom}(C',\phi):\operatorname{Hom}(C',C)\to\operatorname{Hom}(C',M)$ is an epimorphism for any $C'\in\mathfrak{C}$. Dually we have the definition of a $\mathfrak{C}$-preenvelope. The concepts of precovers and preenvelopes (which were also called *approximations* in [4]) play a significant role in tilting theory.

Proposition 4.2 *The following conditions are equivalent for a left R-module T:*

(1) T is an F-Wakamatsu tilting module.

(2) $\mathrm{Add}(T)$ *is an* F*-self-orthogonal class and* $\mathcal{P}(F) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)$.
(3) $\mathrm{Add}(T)$ *is an* F*-self-orthogonal class and* ${}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty} = \mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty}$.
(4) $\mathrm{Add}(T)$ *is an* F*-self-orthogonal class and* ${}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty} = \mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty}$.

In this case, $\mathrm{Add}(T)$ *is an* F*-generator of* $(\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)})^{\perp}$.

Proof (1) $\Leftrightarrow$ (2) follows from Theorem 3.3. (2) $\Leftrightarrow$ (3) holds by Proposition 3.4.

(3) $\Leftrightarrow$ (4) It suffices to show that ${}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty} = {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty}$.

It is obvious that ${}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty} \subseteq {}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty}$.

Let $M \in {}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty}$ and $N \in \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$. Then there exists an F-exact left $\mathrm{Add}(T)$-resolution $\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} N \to 0$. Let $pd_F(M) = m < \infty$. Then $\mathrm{Ext}_F^i(M, N) \cong \mathrm{Ext}_F^{m+i}(M, \ker(f_{m-1})) = 0$ for any $i \geq 1$. Thus $M \in {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$. So ${}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty} \subseteq {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty}$. It follows that ${}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty} = {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} \bigcap \mathcal{P}(F)^{<\infty}$.

Now suppose that T is an F-Wakamatsu tilting module. Let $U \in (\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)})^{\perp}$. There is an F-epimorphism $P \to U$ with $P \in \mathcal{P}(F)$. So there is an F-exact sequence $0 \to P \to B \to V \to 0$ with $B \in \mathrm{Add}(T)$ and $V \in \mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)}$. Thus we have the following pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r] & Q \arrow[r] & V \arrow[r] & 0.
\end{tikzcd}$$

Note that the second row is split and so there is an F-epimorphism $B \to U$. By [21, Corollary 3.7], U has an epic $\mathrm{Add}(T)$-precover $\beta : W \to U$. Then we get an F-exact sequence $0 \to \ker(\beta) \to W \to U \to 0$ with $\ker(\beta) \in T^{\perp}$. Let $G \in \mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)}$. There is an F-exact sequence $0 \to G \to X \to Y \to 0$ with $X \in \mathrm{Add}(T)$ and $Y \in \mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)}$. So $\mathrm{Ext}_F^j(G, \ker(\beta)) \cong \mathrm{Ext}_F^{j+1}(Y, \ker(\beta)) \cong \mathrm{Ext}_F^j(Y, U) = 0$ for any $j \geq 1$. Thus $\ker(\beta) \in (\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)})^{\perp}$. Hence $\mathrm{Add}(T)$ is an F-generator of $(\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)})^{\perp}$. □

Dually, we have

Proposition 4.3 *The following conditions are equivalent for a left* R*-module* T*:*

(1) T *is an* F*-Wakamatsu cotilting module.*
(2) $\mathrm{Prod}(T)$ *is an* F*-self-orthogonal class and* $\mathcal{I}(F) \subseteq l_{\mathcal{I}(F)}\mathrm{Prod}(T)$.
(3) $\mathrm{Prod}(T)$ *is an* F*-self-orthogonal class and* $\mathrm{Prod}(T)^{\perp} \bigcap \mathcal{I}(F)^{<\infty} = \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Prod}(T)} \bigcap \mathcal{I}(F)^{<\infty}$.
(4) $\mathrm{Prod}(T)$ *is an* F*-self-orthogonal class and* $(\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Prod}(T)})^{\perp} \bigcap \mathcal{I}(F)^{<\infty} = \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Prod}(T)} \bigcap \mathcal{I}(F)^{<\infty}$.

In this case, $\mathrm{Prod}(T)$ *is an* F*-cogenerator of* ${}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Prod}(T)}$.

Remark 4.4 Let R be an Artin algebra. Recall that a finitely generated left R-module T is a *Wakamatsu tilting module* [15,18] if $\mathrm{Ext}^i(T, T) = 0$ for any $i \geq 1$, and there exists a $\mathrm{Hom}(-, T)$-exact exact sequence $0 \to {}_RR \to T^0 \to T^1 \to \cdots$ with each $T^i \in \mathrm{add}(T)$ for $i \geq 0$, where $\mathrm{add}(M)$ denotes the class of left R-modules isomorphic to direct summands of finite direct sums of copies of T. A Wakamatsu cotilting module can be defined dually. By Propositions 3.2, 4.2 and 4.3, the concept of an F-Wakamatsu tilting (resp. F-Wakamatsu cotilting) module is a generalization of a Wakamatsu tilting (resp. Wakamatsu cotilting) module in the sense of [15,18].

Recall that a class $\mathfrak{C}$ of left R-modules is *closed under* F*-extensions* if for any F-exact sequence $0 \to A \to A' \to A'' \to 0$ with A and $A'' \in \mathfrak{C}$, $A' \in \mathfrak{C}$. $\mathfrak{C}$ is said to be *closed under cokernels of* F*-monomorphisms* if for any F-exact sequence $0 \to A \to A' \to A'' \to 0$ with A and $A' \in \mathfrak{C}$, $A'' \in \mathfrak{C}$. $\mathfrak{C}$ is said to be *closed under kernels of* F*-epimorphisms* if for any F-exact sequence $0 \to A \to A' \to A'' \to 0$ with A' and $A'' \in \mathfrak{C}$, $A \in \mathfrak{C}$. $\mathfrak{C}$ is called F*-resolving* if it contains $\mathcal{P}(F)$ and is closed under F-extensions and kernels of F-epimorphisms. $\mathfrak{C}$ is called F*-coresolving* if it contains $\mathcal{I}(F)$ and is closed under F-extensions and cokernels of F-monomorphisms. If $\mathfrak{C}$ is F-resolving, then one can easily verify that $\mathfrak{C}^{\perp} = \mathfrak{C}^{\perp_1}$. Dually, if $\mathfrak{C}$ is F-coresolving, then ${}^{\perp}\mathfrak{C} = {}^{\perp_1}\mathfrak{C}$.

According to [17,20], a pair $(\mathfrak{C}, \mathfrak{C}')$ of classes of left R-modules is called an *F-cotorsion pair* if $\mathfrak{C}^{\perp_1} = \mathfrak{C}'$ and $\mathfrak{C} = {}^{\perp_1}\mathfrak{C}'$. An F-cotorsion pair $(\mathfrak{C}, \mathfrak{C}')$ is called *complete* if every left R-module M has an F-special $\mathfrak{C}$-precover, i.e., there exists an F-exact sequence $0 \to K \to B \to M \to 0$ with $B \in \mathfrak{C}$ and $K \in \mathfrak{C}'$, equivalently, every left R-module M has an F-special $\mathfrak{C}'$-preenvelope, i.e., there exists an F-exact sequence $0 \to M \to A \to N \to 0$ with $A \in \mathfrak{C}'$ and $N \in \mathfrak{C}$ by [17, Theorem 3.11]. An F-cotorsion pair $(\mathfrak{C}, \mathfrak{C}')$ is called *hereditary* if $\mathfrak{C}$ is F-resolving, equivalently, $\mathfrak{C}'$ is F-coresolving. It is easy to see that both $({}^{\perp}(\mathfrak{C}^{\perp}), \mathfrak{C}^{\perp})$ and $({}^{\perp}\mathfrak{C}', ({}^{\perp}\mathfrak{C}')^{\perp})$ are hereditary F-cotorsion pairs.

As we see in the next result, under the additional condition that "$\mathcal{P}(F) = \mathrm{Add}(H)$ for a left R-module H", we can describe F-Wakamatsu tilting left R-modules in terms of F-cotorsion pairs. Some proofs are modelled on that of [18, Propositions 2.2 and 2.13], but we give a complete proof here for the reader's convenience.

Theorem 4.5 *Let $\mathcal{P}(F) = \mathrm{Add}(H)$ for a left R-module H. The following conditions are equivalent for a left R-module T:*

(1) T is an F-Wakamatsu tilting left R-module.
(2) $\mathrm{Add}(T)$ is an F-self-orthogonal class and $H \in r_{\mathcal{P}(F)}\mathrm{Add}(T)$.
(3) $\mathrm{Add}(T)$ is an F-self-orthogonal class and there is a left R-module W such that $\mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)} = W^{\perp}$.
(4) $\mathrm{Add}(T)$ is an F-self-orthogonal class and $({}^{\perp}\mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)}, \mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)})$ is a complete and hereditary F-cotorsion pair.
(5) $\mathrm{Add}(T)$ is an F-self-orthogonal class and $\mathrm{Add}(T)$ is an F-cogenerator of ${}^{\perp}\mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)}$.

Proof (1) $\Leftrightarrow$ (2) is clear since $r_{\mathcal{P}(F)}\mathrm{Add}(T)$ is closed under direct sums and direct summands by Proposition 3.2.

(1) $\Rightarrow$ (3) Suppose that H has an F-exact right $\mathrm{Add}(T)$-resolution $0 \to H \overset{g^0}{\to} T^0 \overset{g^1}{\to} T^1 \overset{g^2}{\to} \cdots$. Let $W = \oplus_{i\geq 0}\mathrm{coker}(g^i) \oplus T$. We will show that $\mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)} = W^{\perp}$.

Let $M \in \mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)}$. Then there is an F-exact left $\mathrm{Add}(T)$-resolution $\cdots \overset{f_2}{\to} T_1 \overset{f_1}{\to} T_0 \overset{f_0}{\to} M \to 0$. The F-exact sequence $0 \to H \overset{g^0}{\to} T^0 \to \mathrm{coker}(g^0) \to 0$ induces the exact sequence $\mathrm{Hom}(T^0, M) \overset{(g^0)^*}{\to} \mathrm{Hom}(H, M) \to \mathrm{Ext}^1_F(\mathrm{coker}(g^0), M) \to \mathrm{Ext}^1_F(T^0, M) = 0$. For any $\varphi_0 : H \to M$, there is $\psi_0 : H \to T_0$ such that $\varphi_0 = f_0\psi_0$, whence there is $\xi_0 : T^0 \to T_0$ such that $\psi_0 = \xi_0 g^0$. Hence $\varphi_0 = (f_0\xi_0)g^0 = (g^0)^*(f_0\xi_0)$. Thus $(g^0)^*$ is an epimorphism and so $\mathrm{Ext}^1_F(\mathrm{coker}(g^0), M) = 0$. Also $\mathrm{Ext}^j_F(\mathrm{coker}(g^0), M) \cong \mathrm{Ext}^{j-1}_F(H, M) = 0$ for $j \geq 2$. It is easy to check that $\ker(f_0) \in \mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)}$ and so $\mathrm{Ext}^1_F(\mathrm{coker}(g^0), \ker(f_0)) = 0$ by the proof above. The F-exact sequence $0 \to \ker(f_0) \to T_0 \overset{f_0}{\to} M \to 0$ induces the exact sequence $\mathrm{Hom}(\mathrm{coker}(g^0), T_0) \overset{(f_0)_*}{\to} \mathrm{Hom}(\mathrm{coker}(g^0), M) \to \mathrm{Ext}^1_F(\mathrm{coker}(g^0), \ker(f_0)) = 0$. Thus for any $\varphi_1 : \mathrm{coker}(g^0) \to M$, there is $\psi_1 : \mathrm{coker}(g^0) \to T_0$ such that $\varphi_1 = (f_0)_*(\psi_1) = f_0\psi_1$. Also the F-exact sequence $0 \to \mathrm{coker}(g^0) \overset{g^1}{\to} T^1 \to \mathrm{coker}(g^1) \to 0$ induces the exact sequence $\mathrm{Hom}(T^1, M) \overset{(g^1)^*}{\to} \mathrm{Hom}(\mathrm{coker}(g^0), M) \to \mathrm{Ext}^1_F(\mathrm{coker}(g^1), M) \to \mathrm{Ext}^1_F(T^1, M) = 0$. Since $0 \to \mathrm{coker}(g^0) \overset{g^1}{\to} T^1 \to \mathrm{coker}(g^1) \to 0$ is $\mathrm{Hom}(-, \mathrm{Add}(T))$-exact, there is $\xi_1 : T^1 \to T_0$ such that $\psi_1 = \xi_1 g^1$. Hence $\varphi_1 = (f_0\xi_1)g^1 = (g^1)^*(f_0\xi_1)$. Thus $(g^1)^*$ is an epimorphism and so $\mathrm{Ext}^1_F(\mathrm{coker}(g^1), M) = 0$. Also $\mathrm{Ext}^j_F(\mathrm{coker}(g^1), M) \cong \mathrm{Ext}^{j-1}_F(\mathrm{coker}(g^0), M) = 0$ for $j \geq 2$. By induction, we have $\mathrm{Ext}^j_F(\mathrm{coker}(g^i), M) = 0$ for any $j \geq 1$ and $i \geq 0$. Therefore $\mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)} \subseteq (\oplus_{i\geq 0}\mathrm{coker}(g^i) \oplus T)^{\perp} = W^{\perp}$.

Conversely, for $N \in (\oplus_{i\geq 0}\mathrm{coker}(g^i) \oplus T)^{\perp}$, there is an F-epimorphism $\gamma : \oplus H \to N$. Note that $\oplus H$ has an F-exact right $\mathrm{Add}(T)$-resolution $0 \to \oplus H \overset{\oplus g^0}{\to} \oplus T^0 \overset{\oplus g^1}{\to} \oplus T^1 \to \cdots$. So the F-exact sequence $0 \to \oplus H \overset{\oplus g^0}{\to} \oplus T^0 \to \oplus\mathrm{coker}(g^0) \to 0$ induces the exact sequence $\mathrm{Hom}(\oplus T^0, N) \overset{(\oplus g^0)^*}{\to} \mathrm{Hom}(\oplus H, N) \to \mathrm{Ext}^1_F(\oplus\mathrm{coker}(g^0), N) = 0$. Hence there exists $\alpha_0 : \oplus T^0 \to N$ such that $\gamma = \alpha_0(\oplus g^0)$. By [21, Corollary 3.7], N has an $\mathrm{Add}(T)$-precover $\beta_0 : N_0 \to N$ which is an F-epimorphism and $\mathrm{Ext}^j_F(T, \ker(\beta_0)) = 0$ for any $j \geq 1$. Note that $\mathrm{Ext}^j_F(\mathrm{coker}(g^{i+1}), N_0) = 0$. So $\mathrm{Ext}^j_F(\mathrm{coker}(g^i), \ker(\beta_0)) \cong \mathrm{Ext}^{j+1}_F(\mathrm{coker}(g^{i+1}), \ker(\beta_0)) \cong \mathrm{Ext}^j_F(\mathrm{coker}(g^{i+1}), N) = 0$ for any $j \geq 1$ and $i \geq 0$. Then $\ker(\beta_0) \in (\oplus_{i\geq 0}\mathrm{coker}(g^i) \oplus T)^{\perp}$. By the proof above, $\ker(\beta_0)$ has an $\mathrm{Add}(T)$-precover $\beta_1 : N_1 \to \ker(\beta_0)$ which is an F-epimorphism. Repeating this process, we get an F-exact left $\mathrm{Add}(T)$-resolution $\cdots \overset{\beta_2}{\to} N_1 \overset{\beta_1}{\to} N_0 \overset{\beta_0}{\to} N \to 0$. So $(\oplus_{i\geq 0}\mathrm{coker}(g^i) \oplus T)^{\perp} \subseteq \mathfrak{L}^{\mathrm{Add}(T)}_{\mathcal{I}(F)}$.

It follows that $\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} = (\oplus_{i\geq 0}\mathrm{coker}(g^i) \oplus T)^{\perp} = W^{\perp}$.

(3) $\Rightarrow$ (4) $({}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}, \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)})$ is a complete and hereditary F-cotorsion pair by [20, Theorem 4.11].

(4) $\Rightarrow$ (5) Let $A \in {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$. There is an F-exact sequence $0 \to A \to N \to U \to 0$ with $U \in {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$ and $N \in \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$. So $N \in \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} \bigcap {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)} = \mathrm{Add}(T)$ by Theorem 3.3.

(5) $\Rightarrow$ (1) Since $\mathcal{P}(F) \subseteq {}^{\perp}\mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$, we have $\mathcal{P}(F) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)$. □

Remark 4.6 We give some examples to show that the condition "$\mathcal{P}(F) = \mathrm{Add}(H)$ for a left R-module H" in Theorem 4.5 can be satisfied.

(1) Let $F = \mathrm{Ext}^1(-,-)$ and $H = R$, then the class of projective left modules = $\mathrm{Add}(H)$.

(2) Let $F = \mathrm{Pext}^1(-,-)$ in Example 3.1(2) and H = the direct sum of a representative set of all finitely presented left modules, then the class of pure projective left modules = $\mathrm{Add}(H)$.

(3) If R is a virtually Gorenstein CM-finite artin algebra, then there is an additive subfunctor $\mathrm{Gext}^1(-,-)$ of $\mathrm{Ext}^1(-,-)$ [20]. Let H = the representation generator of the class of finitely generated Gorenstein projective left R-modules, then the class of Gorenstein projective left R-modules = $\mathrm{Add}(H)$ by [9, Theorem 4.10].

Similar to the proof of Theorem 4.5, we have

Proposition 4.7 *Let $\mathcal{I}(F) = \mathrm{Prod}(E)$ for a left R-module E. Then T is an F-Wakamatsu cotilting left R-module if and only if $\mathrm{Prod}(T)$ is an F-self-orthogonal class and $E \in \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Prod}(T)}$. In this case, there is a left R-module W such that $\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Prod}(T)} = {}^{\perp}W$. Consequently, $(\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Prod}(T)}, (\mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Prod}(T)})^{\perp})$ is a hereditary F-cotorsion pair.*

Let T be a left R-module. Denote by $\mathrm{Pres}_F^n(T)$ the class consisting of all left R-modules M for which there exists an F-exact sequence of the form $C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to M \to 0$ with each $C_i \in \mathrm{Add}(T)$. Dually, we denote by $\mathrm{Copres}_F^n(T)$ the class consisting of all left R-modules M for which there exists an F-exact sequence of the form $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n$ with each $C^i \in \mathrm{Prod}(T)$. As usual, we write $\mathrm{Gen}_F(T) = \mathrm{Pres}_F^0(T)$ and $\mathrm{Cogen}_F(T) = \mathrm{Copres}_F^0(T)$. The following theorem characterizes when the kernel $T^{\perp} \bigcap {}^{\perp}(T^{\perp})$ of an F-cotorsion pair $({}^{\perp}(T^{\perp}), T^{\perp})$ is exactly equal to $\mathrm{Add}(T)$.

Theorem 4.8 *The following conditions are equivalent for a left R-module T:*

(1) T is an F-Wakamatsu tilting left R-module and $T^{\perp} = \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$.
(2) $\mathrm{Add}(T) = T^{\perp} \bigcap {}^{\perp}(T^{\perp})$.
(3) $\mathrm{Add}(T)$ is an F-self-orthogonal class and $T^{\perp} \subseteq \mathrm{Gen}_F(T)$.
(4) $\mathrm{Add}(T)$ is an F-generator of $T^{\perp}$.
(5) $\mathrm{Add}(T)$ is an F-self-orthogonal class and $\mathrm{Add}(T)$ is an F-cogenerator of ${}^{\perp}(T^{\perp})$.

Proof (1) $\Rightarrow$ (2) follows from Theorem 3.3.

(2) $\Rightarrow$ (3) $\mathrm{Add}(T)$ is clearly an F-self-orthogonal class. Let $M \in T^{\perp}$. By [20, Corollary 4.13], there is an F-exact sequence $0 \to K \to N \to M \to 0$ with $N \in {}^{\perp}(T^{\perp})$ and $K \in T^{\perp}$. So $N \in T^{\perp} \bigcap {}^{\perp}(T^{\perp}) = \mathrm{Add}(T)$. Thus $T^{\perp} \subseteq \mathrm{Gen}_F(T)$.

(3) $\Rightarrow$ (1) We will prove that ${}^{\perp}(T^{\perp}) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)$. Let $A \in {}^{\perp}(T^{\perp})$. By [20, Corollary 4.13], A has a $T^{\perp}$-preenvelope $\alpha : A \to B$ which is an F-monomorphism. By [21, Corollary 3.7], B has an $\mathrm{Add}(T)$-precover $\beta : C^0 \to B$. Since $B \in T^{\perp} \subseteq \mathrm{Gen}_F(T)$, β is an F-epimorphism. So there is an F-exact $0 \to K \to C^0 \overset{\beta}{\to} B \to 0$. Since $K \in T^{\perp}$, $\mathrm{Ext}_F^1(A, K) = 0$. Thus there is $g^0 : A \to C^0$ such that $\alpha = \beta g^0$. Note that g^0 is an $\mathrm{Add}(T)$-preenvelope which is an F-monomorphism. So there is an F-exact sequence $0 \to A \overset{g^0}{\to} C^0 \to G^1 \to 0$. It is easy to check that $G^1 \in {}^{\perp}(T^{\perp})$. Iterating the above construction, we get an F-exact right $\mathrm{Add}(T)$-resolution $0 \to M \overset{g^0}{\to} C^0 \overset{g^1}{\to} C^1 \overset{g^2}{\to} \cdots$. In particular, any $M \in \mathcal{P}(F)$ has an F-exact right $\mathrm{Add}(T)$-resolution. So T is F-Wakamatsu tilting.

Since $T^{\perp} \subseteq \mathrm{Gen}_F(T)$, any $D \in T^{\perp}$ has an $\mathrm{Add}(T)$-precover $Q \to D$ which is an F-epimorphism. Thus there is an F-exact sequence $0 \to K \to Q \to D \to 0$. Note that $K \in T^{\perp}$. Repeating the process to K and so on, we obtain that $D \in \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$. Thus $T^{\perp} = \mathfrak{L}_{\mathcal{I}(F)}^{\mathrm{Add}(T)}$.

(3) $\Leftrightarrow$ (4) is easy.

(2) $\Rightarrow$ (5) $\mathrm{Add}(T)$ is clearly an F-self-orthogonal class. Let $A \in {}^{\perp}(T^{\perp})$. By [20, Corollary 4.13], there is an F-exact sequence $0 \to A \to N \to Z \to 0$ with $N \in T^{\perp}$ and $Z \in {}^{\perp}(T^{\perp})$. So $N \in T^{\perp} \bigcap {}^{\perp}(T^{\perp}) = \mathrm{Add}(T)$, whence $\mathrm{Add}(T)$ is an F-cogenerator of ${}^{\perp}(T^{\perp})$.

(5) $\Rightarrow$ (2) Let $X \in T^{\perp} \bigcap {}^{\perp}(T^{\perp})$. There is an F-exact sequence $0 \to X \to N \to Y \to 0$ with $N \in \mathrm{Add}(T)$ and $Y \in {}^{\perp}(T^{\perp})$. Thus it is split, and so $X \in \mathrm{Add}(T)$. Therefore $T^{\perp} \bigcap {}^{\perp}(T^{\perp}) \subseteq \mathrm{Add}(T)$. It is clear that $\mathrm{Add}(T) \subseteq T^{\perp} \bigcap {}^{\perp}(T^{\perp})$. Hence $\mathrm{Add}(T) = T^{\perp} \bigcap {}^{\perp}(T^{\perp})$. □

5 Relative tilting and cotilting modules

Definition 5.1 Let F be an additive subfunctor of $\mathrm{Ext}^1(-,-)$ and T be a left R-module (not necessary finitely generated). T is called an *n-F-tilting left R-module* if $\mathrm{Add}(T) \subseteq l_{\mathrm{Add}(T)}\mathcal{P}(F)_n$ and $\mathcal{P}(F) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$. T is called an *n-F-cotilting left R-module* if $\mathcal{I}(F) \subseteq l_{\mathcal{I}(F)}\mathrm{Prod}(T)_n$ and $\mathrm{Prod}(T) \subseteq r_{\mathrm{Prod}(T)}\mathcal{I}(F)_n$.

The following theorem, motivated by [8, Theorem 3.11], [14, Corollary 5.1.10] and [24, Theorem 3.10], characterizes relative tilting modules in a more general setting.

Theorem 5.2 *The following conditions are equivalent for a left R-module T:*

(1) T is an n-F-tilting left R-module.
(2) $pd_F(T) \le n$, $\mathrm{Add}(T)$ is an F-self-orthogonal class and every $P \in \mathcal{P}(F)$ has an F-exact right $\mathrm{Add}(T)$-resolution of finite length.
(3) $pd_F(T) \le n$ and ${}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty} = (r_{\mathcal{P}(F)}\mathrm{Add}(T)_n) \bigcap \mathcal{P}(F)^{<\infty}$.
(4) $pd_F(T) \le n$ and $T^{\perp} = l_{\mathcal{I}(F)}\mathrm{Add}(T)$.
(5) $pd_F(T) \le n$ and ${}^{\perp}(T^{\perp}) = {}^{\perp}\mathrm{Add}(T) \bigcap r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$.
(6) $T^{\perp} = \mathrm{Pres}_F^{n-1}(T)$.

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 3.5.

(2) $\Rightarrow$ (6) Let $M \in \mathrm{Pres}_F^{n-1}(T)$. Then there is an F-exact sequence $0 \to K_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to M \to 0$ with each $T_i \in \mathrm{Add}(T)$. So $\mathrm{Ext}_F^i(T, M) \cong \mathrm{Ext}_F^{i+n}(T, K_n) = 0$ for any $i \ge 1$. Thus $M \in T^{\perp}$ and hence $\mathrm{Pres}_F^{n-1}(T) \subseteq T^{\perp}$.

Conversely, let $X \in T^{\perp}$ and $\alpha : Y \to X$ be a $\mathcal{P}(F)$-precover. By (2), Y has an F-exact right $\mathrm{Add}(T)$-resolution $0 \to Y \overset{g^0}{\to} T^0 \overset{g^1}{\to} T^1 \overset{g^2}{\to} \cdots \overset{g^r}{\to} T^r \to 0$ for some $r \in \mathbb{N}$, which induces the exact sequence $\mathrm{Hom}(T^0, X) \overset{(g^0)^*}{\to} \mathrm{Hom}(Y, X) \to \mathrm{Ext}_F^1(\mathrm{coker}(g^0), X) \cong \mathrm{Ext}_F^r(T^r, X) = 0$. So there is $\varphi : T^0 \to X$ such that $\alpha = \varphi g^0$. Therefore $X \in \mathrm{Gen}_F(T)$. Hence $T^{\perp} \subseteq \mathrm{Gen}_F(T)$. It is easy to see that $T^{\perp} \subseteq \mathrm{Pres}_F^{n-1}(T)$. Thus $T^{\perp} = \mathrm{Pres}_F^{n-1}(T)$.

(6) $\Rightarrow$ (4) Since $\mathrm{Add}(T) \subseteq \mathrm{Pres}_F^{n-1}(T) = T^{\perp}$, $\mathrm{Add}(T)$ is an F-self-orthogonal class. Next we prove that $pd_F(T) \le n$.

For any left R-module M, there is an F-exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to U^n \to 0$ with each $E^i \in \mathcal{I}(F)$. Since $E^{n-1} \in T^{\perp} = \mathrm{Pres}_F^{n-1}(T)$, there is an F-exact $0 \to K^{n-1} \to C^{n-1} \to E^{n-1} \to 0$ with $C^{n-1} \to E^{n-1}$ an $\mathrm{Add}(T)$-precover. So we get the following pullback:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K^{n-1} \arrow[r, equal] \arrow[d] & K^{n-1} \arrow[d] & & \\
0 \arrow[r] & Q^{n-1} \arrow[r, "q^{n-1}"] \arrow[d] & C^{n-1} \arrow[r, "f^n"] \arrow[d] & U^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U^{n-1} \arrow[r] \arrow[d] & E^{n-1} \arrow[r] \arrow[d] & U^n \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

which gives rise to the following pullback:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K^{n-1} \arrow[r, equal] \arrow[d] & K^{n-1} \arrow[d] & \\
0 \arrow[r] & U^{n-2} \arrow[r] \arrow[d, equal] & G^{n-2} \arrow[r] \arrow[d] & Q^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U^{n-2} \arrow[r] & E^{n-2} \arrow[r] \arrow[d] & U^{n-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Notice that $K^{n-1} \in T^{\perp}$ and so $G^{n-2} \in T^{\perp} = \mathrm{Pres}_F^{n-1}(T)$. Thus there is an F-exact sequence $0 \to K^{n-2} \to C^{n-2} \to G^{n-2} \to 0$ with $C^{n-2} \to G^{n-2}$ an $\mathrm{Add}(T)$-precover. So we get the following pullback:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K^{n-2} \arrow[r, equal] \arrow[d] & K^{n-2} \arrow[d] & & \\
0 \arrow[r] & Q^{n-2} \arrow[r, "q^{n-2}"] \arrow[d] & C^{n-2} \arrow[r, "\alpha^{n-1}"] \arrow[d] & Q^{n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U^{n-2} \arrow[r] \arrow[d] & G^{n-2} \arrow[r] \arrow[d] & Q^{n-1} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Continuing this process, we get the F-exact sequence

$$0 \to Q^0 \overset{q^0}{\to} C^0 \overset{q^1\alpha^1}{\to} C^1 \to \cdots \to C^{n-2} \overset{q^{n-1}\alpha^{n-1}}{\to} C^{n-1} \overset{f^n}{\to} U^n \to 0.$$

So $U^n \in \mathrm{Pres}_F^{n-1}(T) = T^{\perp}$. Thus $\mathrm{Ext}_F^{n+i}(T, M) \cong \mathrm{Ext}_F^i(T, U^n) = 0$ for any $i \geq 1$, whence $pd_F(T) \leq n$.

It is clear that $T^{\perp} \subseteq \mathrm{Gen}_F(T)$ and so $T^{\perp} \subseteq l_{\mathcal{I}(F)}\mathrm{Add}(T)$ by Theorem 4.8.

Conversely, let $M \in l_{\mathcal{I}(F)}\mathrm{Add}(T)$, then M has an F-exact left $\mathrm{Add}(T)$-resolution $\cdots \overset{f_2}{\to} T_1 \overset{f_1}{\to} T_0 \overset{f_0}{\to} M \to 0$. So $\mathrm{Ext}_F^i(T, M) \cong \mathrm{Ext}_F^{i+n}(T, \ker(f_{n-1})) = 0$ for any $i \geq 1$. Hence $M \in T^{\perp}$. Thus $l_{\mathcal{I}(F)}\mathrm{Add}(T) \subseteq T^{\perp}$ and so $T^{\perp} = l_{\mathcal{I}(F)}\mathrm{Add}(T)$.

(4) $\Rightarrow$ (5) By Theorem 4.8, $\mathrm{Add}(T)$ is an F-cogenerator of ${}^{\perp}(T^{\perp})$. For any $A \in {}^{\perp}(T^{\perp})$, we get an F-exact right $\mathrm{Add}(T)$-resolution $0 \to A \overset{g^0}{\to} T^0 \overset{g^1}{\to} T^1 \overset{g^2}{\to} \cdots$ with each $\mathrm{coker}(g^i) \in {}^{\perp}(T^{\perp})$. Since $pd_F(T) \leq n$, we have $\mathrm{Ext}_F^i(T, \mathrm{coker}(g^{n-1})) \cong \mathrm{Ext}_F^{i+n}(T, A) = 0$ for any $i \geq 1$. Therefore $\mathrm{coker}(g^{n-1}) \in T^{\perp}$. Consequently $\mathrm{Ext}_F^1(\mathrm{coker}(g^n), \mathrm{coker}(g^{n-1})) = 0$. Thus the F-exact sequence $0 \to \mathrm{coker}(g^{n-1}) \to T^n \to \mathrm{coker}(g^n) \to 0$ is split and so $\mathrm{coker}(g^{n-1}) \in \mathrm{Add}(T)$. Hence ${}^{\perp}(T^{\perp}) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$. Since ${}^{\perp}(T^{\perp}) \subseteq {}^{\perp}\mathrm{Add}(T)$, ${}^{\perp}(T^{\perp}) \subseteq {}^{\perp}\mathrm{Add}(T) \bigcap r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$.

Conversely, let $M \in {}^{\perp}\mathrm{Add}(T) \bigcap r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$, then we get an F-exact right $\mathrm{Add}(T)$-resolution $0 \to M \to T^0 \to T^1 \to \cdots \to T^n \to 0$. For any $N \in T^{\perp}$ and any $i \geq 1$, $\mathrm{Ext}_F^i(M, N) \cong \mathrm{Ext}_F^{i+n}(T^n, N) = 0$. Hence $M \in {}^{\perp}(T^{\perp})$ and so ${}^{\perp}\mathrm{Add}(T) \bigcap r_{\mathcal{P}(F)}\mathrm{Add}(T)_n \subseteq {}^{\perp}(T^{\perp})$. Thus ${}^{\perp}(T^{\perp}) = {}^{\perp}\mathrm{Add}(T) \bigcap r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$.

(5) $\Rightarrow$ (2) is clear since $\mathcal{P}(F) \subseteq {}^{\perp}(T^{\perp})$. $\square$

Dually, we have

Theorem 5.3 *The following conditions are equivalent for a left R-module T:*

(1) T is an n-F-cotilting left R-module.
(2) $id_F(T) \leq n$, $\mathrm{Prod}(T)$ *is an F-self-orthogonal class and every $E \in \mathcal{I}(F)$ has an F-exact left* $\mathrm{Prod}(T)$*-resolution of finite length.*
(3) $id_F(T) \leq n$ and $\mathrm{Prod}(T)^{\perp} \bigcap \mathcal{I}(F)^{<\infty} = (l_{\mathcal{I}(F)}\mathrm{Prod}(T)_n) \bigcap \mathcal{I}(F)^{<\infty}$.
(4) $id_F(T) \leq n$ and ${}^{\perp}T = r_{P(F)}\mathrm{Prod}(T)$.
(5) $id_F(T) \leq n$ and $({}^{\perp}T)^{\perp} = \mathrm{Prod}(T)^{\perp} \bigcap l_{\mathcal{I}(F)}\mathrm{Prod}(T)_n$.
(6) ${}^{\perp}T = \mathrm{Copres}_F^{n-1}(T)$.

Remark 5.4 (1) If we specilaze $F = \mathrm{Ext}^1(-,-)$, then an n-F-tilting (resp. n-F-cotilting) module for some n is exactly a tilting (resp. cotilting) module defined by Angeleri Hügel and Coelho in [1] by Theorems 5.2 and 5.3.

(2) Finitely generated F-tilting and F-cotilting modules over an Artin algebra have been defined and investigated by Auslander and Solberg [6] and by Wei [24]. We here relax the defining conditions of F-tilting and F-cotilting modules in the sense of [6] to arbitrary rings and possibly infinitely generated modules.

Proposition 5.5 *Let T be a left R-module.*

(1) Suppose that $\mathcal{P}(F) = \mathrm{Add}(H)$ for a left R-module H. Then T is an n-F-tilting left R-module if and only if $pd_F(T) \leq n$, $\mathrm{Add}(T)$ is an F-self-orthogonal class and $H \in r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$.
(2) Suppose that $\mathcal{I}(F) = \mathrm{Prod}(E)$ for a left R-module E. Then T is an n-F-cotilting left R-module if and only if $id_F(T) \leq n$, $\mathrm{Prod}(T)$ is an F-self-orthogonal class and $E \in l_{\mathcal{I}(F)}\mathrm{Prod}(T)_n$.

Proof (1) "$\Rightarrow$" It is clear by Theorem 5.2.

"$\Leftarrow$" It is obvious that $r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$ is closed under direct sums. By Proposition 3.2 and Theorem 3.3, $r_{\mathcal{P}(F)}\mathrm{Add}(T)_n = \mathfrak{R}_{\mathcal{P}(F)}^{\mathrm{Add}(T)} \bigcap ({}^{\perp}\mathrm{Add}(T))^{\perp_{n+1}}$ is closed under direct summands. So $\mathcal{P}(F) = \mathrm{Add}(H) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)_n$. Then Theorem 5.2 implies that T is an n-F-tilting left R-module.

The proof of (2) is dual. □

We will call a class $\mathfrak{W}$ of left R-modules an *n-F-tilting class* if $\mathfrak{W} = T^{\perp}$ for some n-F-tilting left R-module T. Similarly, $\mathfrak{W}$ is called an *n-F-cotilting class* if $\mathfrak{W} = {}^{\perp}T$ for some n-F-cotilting left R-module T. If we specialize $F = \mathrm{Ext}^1(-,-)$, then an n-F-tilting (resp. n-F-cotilting) class for some n is just a tilting (resp. cotilting) class in the sense of [14]. Inspired by [1, Theorems 4.1 and 4.2], [2, Theorem 2.1] and [3, Theorem 5.5], we have the following results.

Theorem 5.6 *Let $\mathcal{P}(F) = \mathrm{Add}(H)$ for a left R-module H. The following conditions are equivalent for a class $\mathfrak{W}$ of left R-modules:*

(1) $\mathfrak{W}$ is an n-F-tilting class.
(2) $\mathfrak{W}$ has an F-generator $\mathrm{Add}(T)$ with $T \in {}^{\perp}\mathfrak{W}$, $\mathfrak{W}$ is F-coresolving and is closed under direct summands and direct sums, and for every left R-module M, there is an F-exact sequence of the form $0 \to M \to W^0 \to W^1 \to \cdots \to W^{n-1} \to W^n \to 0$ with each $W^i \in \mathfrak{W}$.
(3) Every left R-module has an F-special $\mathfrak{W}$-preenvelope, $\mathfrak{W}$ is F-coresolving and is closed under direct summands and direct sums, and ${}^{\perp}\mathfrak{W} \subseteq \mathcal{P}(F)^{\leq n}$.
(4) $({}^{\perp}\mathfrak{W}, \mathfrak{W})$ is a complete and hereditary F-cotorsion pair, $\mathfrak{W}$ is closed under direct sums and ${}^{\perp}\mathfrak{W} \subseteq \mathcal{P}(F)^{\leq n}$.

Proof (1) $\Rightarrow$ (2) There is an n-F-tilting left R-module T such that $\mathfrak{W} = T^{\perp}$. By Theorem 5.2, $T^{\perp} = l_{\mathcal{I}(F)}\mathrm{Add}(T)$. So $\mathrm{Add}(T)$ is an F-generator of $\mathfrak{W}$. It is obvious that $T \in {}^{\perp}\mathfrak{W}$ and $\mathfrak{W}$ is F-coresolving and is closed under direct summands. By Theorem 5.2, $T^{\perp} = \mathrm{Pres}_F^{n-1}(T)$, which implies that $\mathfrak{W}$ is closed under direct sums. For every left R-module M, there is an F-exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to V^n \to 0$ with each $E^i \in \mathcal{I}(F) \subseteq \mathfrak{W}$. For any $i \geq 1$, we have $\mathrm{Ext}_F^i(T, V^n) \cong \mathrm{Ext}_F^{n+i}(T, M) = 0$. So $V^n \in \mathfrak{W}$.

(2) $\Rightarrow$ (3) For every left R-module M, there is an F-exact sequence $0 \to M \xrightarrow{\omega^0} W^0 \xrightarrow{\omega^1} W^1 \xrightarrow{\omega^2} \cdots \xrightarrow{\omega^{n-1}} W^{n-1} \xrightarrow{\omega^n} W^n \to 0$ with each $W^i \in \mathfrak{W}$. Write $V^i = \mathrm{coker}(\omega^{i-1})$ for $i \geq 1$ and $V^0 = M$. By (2), there is the F-exact sequence $0 \to K^n \to U^n \to W^n \to 0$ with $U^n \in \mathrm{Add}(T)$ and $K^n \in \mathfrak{W}$. Then we get the following

pullback:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K^n \arrow[r, equal] \arrow[d] & K^n \arrow[d] & \\
0 \arrow[r] & V^{n-1} \arrow[r] \arrow[d, equal] & G^{n-1} \arrow[r] \arrow[d] & U^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V^{n-1} \arrow[r] & W^{n-1} \arrow[r] \arrow[d] & W^n \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathfrak{W}$ is closed under F-extensions, $G^{n-1} \in \mathfrak{W}$. So there is the F-exact sequence $0 \to K^{n-1} \to U^{n-1} \to G^{n-1} \to 0$ with $U^{n-1} \in \mathrm{Add}(T)$ and $K^{n-1} \in \mathfrak{W}$. Then we get the following pullback:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K^{n-1} \arrow[r, equal] \arrow[d] & K^{n-1} \arrow[d] & & \\
0 \arrow[r] & Q^{n-1} \arrow[r] \arrow[d] & U^{n-1} \arrow[r] \arrow[d] & U^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V^{n-1} \arrow[r] \arrow[d] & G^{n-1} \arrow[r] \arrow[d] & U^n \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

which gives rise to the following pullback:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K^{n-1} \arrow[r, equal] \arrow[d] & K^{n-1} \arrow[d] & \\
0 \arrow[r] & V^{n-2} \arrow[r] \arrow[d, equal] & G^{n-2} \arrow[r] \arrow[d] & Q^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V^{n-2} \arrow[r] & W^{n-2} \arrow[r] \arrow[d] & V^{n-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Repeating this process above, we get an F-exact sequence $0 \to M \to G^0 \to Q^1 \to 0$ with $G^0 \in \mathfrak{W}$ and an F-exact sequence $0 \to Q^1 \to U^1 \to U^2 \to \cdots \to U^n \to 0$ with each $U^i \in \mathrm{Add}(T)$. For any $B \in \mathfrak{W}$, $\mathrm{Ext}^i_F(Q^1, B) \cong \mathrm{Ext}^{n+i-1}_F(U^n, B) = 0$ for any $i \geq 1$. So $Q^1 \in {}^{\perp}\mathfrak{W}$. Since $G^0 \in \mathfrak{W}$, $M \to G^0$ is an F-special $\mathfrak{W}$-preenvelope.

Let $A \in {}^{\perp}\mathfrak{W}$. Then $\mathrm{Ext}_F^{n+i}(A, M) \cong \mathrm{Ext}_F^{i}(A, W^n) = 0$ for any $i \geq 1$. So $A \in \mathcal{P}(F)^{\leq n}$. Thus ${}^{\perp}\mathfrak{W} \subseteq \mathcal{P}(F)^{\leq n}$.

(3) $\Rightarrow$ (1) By (3), H has an F-exact right $\mathfrak{W}$-resolution $0 \to H \xrightarrow{g^0} T^0 \xrightarrow{g^1} T^1 \xrightarrow{g^2} \cdots \xrightarrow{g^n} T^n \to \mathrm{coker}(g^n) \to 0$ with each $\mathrm{coker}(g^i) \in {}^{\perp}\mathfrak{W}$. Then $T^i \in {}^{\perp}\mathfrak{W} \bigcap \mathfrak{W}$. Since $pd_F(\mathrm{coker}(g^n)) \leq n$, $\mathrm{Ext}_F^1(\mathrm{coker}(g^n), \mathrm{coker}(g^{n-1})) \cong \mathrm{Ext}_F^{n+1}(\mathrm{coker}(g^n), H) = 0$. Hence $\mathrm{coker}(g^{n-1})$ is isomorphic to a direct summand of T^n and so $\mathrm{coker}(g^{n-1}) \in {}^{\perp}\mathfrak{W} \bigcap \mathfrak{W}$. Let $T = T^0 \oplus T^1 \oplus \cdots \oplus T^{n-1} \oplus \mathrm{coker}(g^{n-1})$. Then $T \in {}^{\perp}\mathfrak{W} \subseteq \mathcal{P}(F)^{\leq n}$ and $\mathrm{Add}(T)$ is an F-self-orthogonal class. By Proposition 5.5, T is n-F-tilting.

We next prove that ${}^{\perp}\mathfrak{W} \bigcap \mathfrak{W} = \mathrm{Add}(T)$. It is clear that $\mathrm{Add}(T) \subseteq {}^{\perp}\mathfrak{W} \bigcap \mathfrak{W}$. For any $W \in {}^{\perp}\mathfrak{W} \bigcap \mathfrak{W}$, we have $W \in T^{\perp}$ since ${}^{\perp}\mathfrak{W} \bigcap \mathfrak{W} \subseteq T^{\perp}$. By Theorem 5.2, $T^{\perp} = l_{\mathcal{I}(F)}\mathrm{Add}(T)$. So we get an F-exact left $\mathrm{Add}(T)$-resolution $\cdots \xrightarrow{\beta_2} W_1 \xrightarrow{\beta_1} W_0 \xrightarrow{\beta_0} W \to 0$. Then $\mathrm{Ext}_F^1(W, \ker(\beta_0)) \cong \mathrm{Ext}_F^{n+1}(W, \ker(\beta_n)) = 0$ since $pd_F(W) \leq n$. Thus W is isomorphic to a direct summand of W_0 and so $W \in \mathrm{Add}(T)$. Hence ${}^{\perp}\mathfrak{W} \bigcap \mathfrak{W} \subseteq \mathrm{Add}(T)$. Therefore ${}^{\perp}\mathfrak{W} \bigcap \mathfrak{W} = \mathrm{Add}(T)$.

Finally, let $M \in T^{\perp}$, there exists an F-exact right $\mathfrak{W}$-resolution $0 \to M \xrightarrow{\alpha^0} C^0 \xrightarrow{\alpha^1} C^1 \xrightarrow{\alpha^2} \cdots$ with each $\mathrm{coker}(\alpha^i) \in {}^{\perp}\mathfrak{W}$ and $C^i \in {}^{\perp}\mathfrak{W} \bigcap \mathfrak{W} = \mathrm{Add}(T)$ for any $i \geq 1$. So $\mathrm{Ext}_F^1(\mathrm{coker}(\alpha^0), M) \cong \mathrm{Ext}_F^{n+1}(\mathrm{coker}(\alpha^n), M) = 0$ since $pd_F(\mathrm{coker}(\alpha^n)) \leq n$. Hence M is isomorphic to a direct summand of C^0. Thus $M \in \mathfrak{W}$ and so $T^{\perp} \subseteq \mathfrak{W}$. It is clear that $\mathfrak{W} \subseteq T^{\perp}$. Hence $\mathfrak{W} = T^{\perp}$.

(1) $\Rightarrow$ (4) There is an n-F-tilting left R-module T such that $\mathfrak{W} = T^{\perp}$. Since (1) is equivalent to (3), every left R-module has an F-special $\mathfrak{W}$-preenvelope, $\mathfrak{W}$ is closed under direct sums and ${}^{\perp}\mathfrak{W} \subseteq \mathcal{P}(F)^{\leq n}$. It follows that $({}^{\perp}\mathfrak{W}, \mathfrak{W})$ is a complete and hereditary F-cotorsion pair.

(4) $\Rightarrow$ (3) is trivial. □

Dually, we have

Theorem 5.7 *Let $\mathcal{I}(F) = \mathrm{Prod}(E)$ for a left R-module E. The following conditions are equivalent for a class $\mathfrak{W}$ of left R-modules:*

(1) $\mathfrak{W}$ is an n-F-cotilting class.
(2) $\mathfrak{W}$ has an F-cogenerator $\mathrm{Prod}(T)$ with $T \in \mathfrak{W}^{\perp}$, $\mathfrak{W}$ is F-resolving and is closed under direct summands and direct products, and for every left R-module M, there is an F-exact sequence of the form $0 \to W_n \to W_{n-1} \to \cdots \to W_1 \to W_0 \to M \to 0$ with each $W_i \in \mathfrak{W}$.
(3) Every left R-module has an F-special $\mathfrak{W}$-precover, $\mathfrak{W}$ is F-resolving and is closed under direct summands and direct products, and $\mathfrak{W}^{\perp} \subseteq \mathcal{I}(F)^{\leq n}$.
(4) $(\mathfrak{W}, \mathfrak{W}^{\perp})$ is a complete and hereditary F-cotorsion pair, $\mathfrak{W}$ is closed under direct products and $\mathfrak{W}^{\perp} \subseteq \mathcal{I}(F)^{\leq n}$.

Finally, we consider the relationship between F-Wakamatsu tilting (resp. F-Wakamatsu cotilting) modules and F-tilting (resp. F-cotilting) modules. Obviously, any F-tilting (resp. F-cotilting) module is an F-Wakamatsu tilting (resp. F-Wakamatsu cotilting) module with finite F-projective (resp. F-injective) dimension. However there exists an F-Wakamatsu tilting R-module with infinite F-projective dimension [18]. Motivated by [18, Proposition 4.4], we have

Proposition 5.8 *Let T be an F-Wakamatsu tilting left R-module with $pd_F(T) \leq n$. If one of the following conditions is satisfied, then T is an n-F-tilting module.*

(1) ${}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty} \subseteq \mathcal{P}(F)^{\leq m}$ for some $m \in \mathbb{N}$.
(2) $\mathcal{P}(F) \subseteq \mathcal{I}(F)^{\leq r}$ for some $r \in \mathbb{N}$.

Proof (1) By Proposition 4.2, $\mathrm{Add}(T)$ is an F-self-orthogonal class. Also any $C \in \mathcal{P}(F)$ has an F-exact right $\mathrm{Add}(T)$-resolution $0 \to C \xrightarrow{g^0} D^0 \xrightarrow{g^1} D^1 \xrightarrow{g^2} \cdots$. Thus $\mathrm{coker}(g^i) \in {}^{\perp}\mathrm{Add}(T)$ for any $i \geq 0$. Since $\mathrm{Add}(T) \subseteq \mathcal{P}(F)^{\leq n}$, $\mathrm{coker}(g^m) \in {}^{\perp}\mathrm{Add}(T) \bigcap \mathcal{P}(F)^{<\infty}$. Hence $\mathrm{coker}(g^m) \in \mathcal{P}(F)^{\leq m}$ by hypothesis. So

$$\mathrm{Ext}_F^1(\mathrm{coker}(g^m), \mathrm{coker}(g^{m-1})) \cong \mathrm{Ext}_F^{m+1}(\mathrm{coker}(g^m), C) = 0.$$

Consequently, the exact sequence $0 \to \mathrm{coker}(g^{m-1}) \to D^m \to \mathrm{coker}(g^m) \to 0$ is split and so $\mathrm{coker}(g^{m-1}) \in \mathrm{Add}(T)$. Thus $C \in r_{\mathcal{P}(F)}\mathrm{Add}(T)_m$ and so $\mathcal{P}(F) \subseteq r_{\mathcal{P}(F)}\mathrm{Add}(T)_m$. It is easy to see that $\mathrm{Add}(T) \subseteq l_{\mathrm{Add}(T)}\mathcal{P}(F)_n$. Therefore T is an n-F-tilting module by Theorem 3.5.

(2) For any $P \in \mathcal{P}(F)$, there exists an F-exact right Add(T)-resolution $0 \to P \xrightarrow{g^0} T^0 \xrightarrow{g^1} T^1 \xrightarrow{g^2} \cdots$. Then $\mathrm{coker}(g^i) \in {}^{\perp}\mathrm{Add}(T)$ for any $i \geq 0$. Thus we have $\mathrm{Ext}^1_F(\mathrm{coker}(g^r), \mathrm{coker}(g^{r-1})) \cong \mathrm{Ext}^{r+1}_F(\mathrm{coker}(g^r), P) = 0$. So $\mathrm{coker}(g^{r-1}) \in \mathrm{Add}(T)$. By Theorem 5.2, T is an n-F-tilting module. □

Dually, we have

Proposition 5.9 *Let T be an F-Wakamatsu cotilting left R-module with $id_F(T) \leq n$. If one of the following conditions is satisfied, then T is an n-F-cotilting module.*

(1) $\mathrm{Prod}(T)^{\perp} \bigcap \mathcal{I}(F)^{<\infty} \subseteq \mathcal{I}(F)^{\leq m}$ *for some* $m \in \mathbb{N}$.
(2) $\mathcal{I}(F) \subseteq \mathcal{P}(F)^{\leq r}$ *for some* $r \in \mathbb{N}$.

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