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ORIGINAL RESEARCH

Proofs of some conjectures of Sun on the relations between $t(a, b, c; n)$ and $N(a, b, c; n)$

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Abstract Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a, b, c, n \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{Z}$, let $N(a, b, c; n)$ be the number of representations of n as $ax^2 + by^2 + cz^2$ and $t(a, b, c; n)$ be the number of representations of n as $a\frac{x(x+1)}{2} + b\frac{y(y+1)}{2} + c\frac{z(z+1)}{2}$. Recently, Sun established many relations between $t(a, b, c; n)$ and $N(a, b, c; 8n + a + b + c)$ and listed 43 relations need to be confirmed. More recently, Xia and Zhang proved 19 relations conjectured by Sun. In this paper, by employing Ramanujan's theta function identities, we prove the remaining 24 relations.

Keywords Ramanujan's theta function identities · Sum of squares · Sum of triangular numbers

Mathematics Subject Classification 11D85 · 11E25

1 Introduction

Let $\mathbb{Z}^+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers, the set of nonnegative integers and the set of integers, respectively. Define

$$N(a, b, c; n) = \sharp\left\{(x, y, z) \in \mathbb{Z}^3 \middle| ax^2 + by^2 + cz^2 = n\right\},$$
$$t(a, b, c; n) = \sharp\left\{(x, y, z) \in \mathbb{Z}^3 \middle| a\frac{x(x+1)}{2} + b\frac{y(y+1)}{2} + c\frac{z(z+1)}{2} = n\right\}$$

and

$$T(a, b, c; n) = \sharp\left\{(x, y, z) \in \mathbb{N}^3 \middle| a\frac{x(x+1)}{2} + b\frac{y(y+1)}{2} + c\frac{z(z+1)}{2} = n\right\},$$

where $a, b, c \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. The numbers $\frac{x(x+1)}{2}$ $(x \in \mathbb{Z})$ are called triangular numbers. Since $\frac{x(x+1)}{2} = \frac{(-x)(-x-1)}{2}$ we have

$$t(a, b, c; n) = 8T(a, b, c; n). \tag{1.1}$$

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In 2005, Adiga, Cooper and Han [1] established that if $a+b+c \le 7$, then for $n \in \mathbb{N}$,

$$t(a,b,c;n) = \frac{2}{2+C(a,b,c)} N(a,b,c;8n+a+b+c),$$

where

$$C(a,b,c) = \frac{i_1(i_1-1)(i_1-2)(i_1-3)}{24} + \frac{i_1(i_1-1)i_2}{2} + i_1 i_3$$

and i_j denotes the number of elements in $\{a,b,c\}$ which are equal to j. In 2008, Baruah, Cooper and Hirschhorn [2] proved that for $n \in \mathbb{N}$,

$$t(a,b,c;n) = \frac{2}{2+C(a,b,c)} (N(a,b,c;8n+8) - N(a,b,c;2n+2)),$$

where $a+b+c=8$. Recently, Sun [7] further discovered some general relations between $t(a,b,c;n)$ and $N(a,b,c;8n+a+b+c)$ under certain conditions. In particular, in [7], Sun posed four conjectures (Conjecture 6.1–Conjecture 6.4) including 43 relations between $t(a,b,c;n)$ and $N(a,b,c;8n+a+b+c)$ for some special values of (a,b,c). Later, Xia and Zhang [9] confirmed 19 relations of Sun [7] by employing theta function identities.

In this paper, we use elementary techniques and theta function identities to prove the following seven relations between $t(a,b,c;n)$ and $N(a,b,c;8n+a+b+c)$ conjectured by Sun [7].

Theorem 1.1 *Let $n \in \mathbb{N}^+$. Then*

$$t(1,2,5;n) = \frac{1}{2}\Big(N(1,2,5;4(8n+8)) - N(1,2,5;8n+8)\Big). \tag{1.2}$$

Theorem 1.2 *Let $n \in \mathbb{N}^+$. Then*

$$t(1,6,9;n) = \frac{1}{2}\Big(N(1,6,9;4(8n+16)) - N(1,6,9;8n+16)\Big). \tag{1.3}$$

Theorem 1.3 *Let $n \in \mathbb{N}^+$. Then*

$$t(1,5,10;n) = \frac{1}{2}\Big(N(1,5,10;4(8n+16)) - N(1,5,10;8n+16)\Big). \tag{1.4}$$

Theorem 1.4 *Let $n \in \mathbb{N}^+$ with $n \equiv 1 \pmod 2$. Then*

$$t(5,21,35;n) = \frac{1}{2}\Big(N(5,21,35;4(8n+61)) - N(5,21,35;8n+61)\Big). \tag{1.5}$$

Theorem 1.5 *Let $n \in \mathbb{N}^+$ with $n \equiv 0 \pmod 2$. Then*

$$t(1,6,15;n) = \frac{1}{2}\Big(3N(1,6,15;8n+22) - N(1,6,15;4(8n+22))\Big). \tag{1.6}$$

Theorem 1.6 *Let $n \in \mathbb{N}^+$ with $n \equiv 1 \pmod 2$. Then*

$$t(1,2,7;n) = \frac{1}{2}\Big(3N(1,2,7;8n+10) - N(1,2,7;4(8n+10))\Big). \tag{1.7}$$

Theorem 1.7 *Let $n \in \mathbb{N}^+$ with $n \equiv 1 \pmod 2$. Then*

$$t(3,5,30;n) = \frac{1}{2}\Big(3N(3,5,30;8n+38) - N(3,5,30;4(8n+38))\Big). \tag{1.8}$$

REMARK. The remaining three relations in Sun [7, Conjecture 6.4] can be shown by using similar arguments as in Theorems 1.5–1.7. The proofs of the remaining 14 relations proposed by Sun [7, Conjectures 6.1–6.2] are similar to that of Xia and Zhang [9, Conjectures 1.1–1.3], which can be shown directly by using several theta function identities.

The rest of the paper is organized as follows. To prove the results of this paper, we present some definitions and lemmas in the next section. In Sections 3– 9, we prove Theorems 1.1–1.7, respectively.

2 Preliminaries

Ramanujan's general theta functions $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2},\quad |ab|<1.$$

By the well-known Jacobi triple product identity in [4, p. 35, Entry 19], the function $f(a,b)$ satisfies

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{2.1}$$

where

$$(a;b)_\infty:=\prod_{n=1}^{\infty}(1-ab^{n-1}).$$

Two special cases of $f(a,b)$ are

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{2.2}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{2.3}$$

Define

$$f_k:=f_k(q):=\prod_{n=1}^{\infty}(1-q^{kn})=(q^k;q^k)_\infty.$$

It is easy to deduce the following relations:

$$f_{2k}(-q)=f_{2k},\quad f_{2k+1}(-q)=\frac{f_{4k+2}^3}{f_{2k+1}f_{8k+4}}. \tag{2.4}$$

From (2.2), (2.3) and (2.4), we have

$$\varphi(-q)=\frac{f_1^2}{f_2},\quad \psi(-q)=\frac{f_1f_4}{f_2}. \tag{2.5}$$

It is evident from the definitions of $\varphi(q)$ and $\psi(q)$ that

$$\sum_{n=0}^{\infty}N(a,b,c;n)q^n=\varphi(q^a)\varphi(q^b)\varphi(q^c) \tag{2.6}$$

and

$$\sum_{n=0}^{\infty}T(a,b,c;n)q^n=\psi(q^a)\psi(q^b)\psi(q^c). \tag{2.7}$$

Next, we recall some theta function identities which will be used later.

Lemma 2.1

$$\psi(q)^2=\varphi(q)\psi(q^2), \tag{2.8}$$

$$\varphi(q)=\varphi(q^4)+2q\psi(q^8), \tag{2.9}$$

$$\psi(q)\psi(q^3)=\varphi(q^6)\psi(q^4)+q\varphi(q^2)\psi(q^{12}), \tag{2.10}$$

$$\psi(q)\psi(q^7)=\psi(q^8)\varphi(q^{28})+q\psi(q^2)\psi(q^{14})+q^6\varphi(q^4)\psi(q^{56}), \tag{2.11}$$

$$\psi(q^3)\psi(q^5)=\psi(q^8)\varphi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\varphi(q^4)\psi(q^{120}), \tag{2.12}$$

$$\psi(q)\psi(q^{15})=\psi(q^6)\psi(q^{10})+q\varphi(q^{20})\psi(q^{24})+q^3\varphi(q^{12})\psi(q^{40}), \tag{2.13}$$

$$\psi(q)\psi(q^5)=\psi(q^6)\varphi(q^{15})+qf(q^2,q^4)f(q^{40},q^{80})+q^6f(q^2,q^4)f(q^{20},q^{100}). \tag{2.14}$$

Proof Identity (2.8) is given in [4, p. 40, Entry 25 (iv)]. Identity (2.9) follows from [4, p. 40, Entry 25 (i),(ii)]. In [4, p. 69, eq. (36.8)], setting $(\mu, \nu) = (2, 1)$, $(\mu, \nu) = (4, 3)$ and $(\mu, \nu) = (4, 1)$, respectively, we obtain identities (2.10), (2.11) and (2.12). Moreover, Identity (2.13) follows from [3, eq. (2.18)]. Adding (ii) to (iii) in [4, p. 46, Entry 30] yields

$$f(a,b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \tag{2.15}$$

If we put $(\mu, \nu) = (3, 2)$ in [4, p. 69, eq. (36.7)], we have

$$\begin{aligned}\psi(q)\psi(q^5) &= \psi(q^6)\varphi(q^{15}) + qf(q^2, q^4)f(q^5, q^{25})\\ &= \psi(q^6)\varphi(q^{15}) + qf(q^2, q^4)f(q^{40}, q^{80}) + q^6 f(q^2, q^4)f(q^{20}, q^{100}),\end{aligned}$$

where the second equality is due to (2.15). □

Lemma 2.2

$$\varphi(-q^6)\varphi(-q^{10}) + 2q\psi(q^3)\psi(q^5) = \varphi(q)\varphi(q^{15}), \tag{2.16}$$
$$\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\psi(q^6)\psi(q^{10}), \tag{2.17}$$
$$\varphi(-q^2)\varphi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}) = \varphi(q^3)\varphi(q^5), \tag{2.18}$$
$$\varphi(-q^3)\varphi(-q^5) - \varphi(q^3)\varphi(q^5) = 2q^2\psi(-q)\psi(-q^{15}) - 2q^2\psi(q)\psi(q^{15}), \tag{2.19}$$
$$\varphi(q)\varphi(q^7) - 2q\psi(q)\psi(q^7) = \varphi(-q^2)\varphi(-q^{14}), \tag{2.20}$$
$$\varphi(q)\varphi(q^7) - \varphi(-q)\varphi(-q^7) = 4q\psi(q^8)\varphi(q^{28}) + 4q^7\varphi(q^4)\psi(q^{56}), \tag{2.21}$$
$$\varphi(q)\varphi(q^{15}) - \varphi(-q)\varphi(-q^{15}) = 4q\psi(q^3)\psi(q^5) - 4q^4\psi(q^2)\psi(q^{30}). \tag{2.22}$$

Proof Identities (2.16), (2.17) and (2.18) are in [4, p. 377, Entry 9 (ii),(iv) and (iii)].

By changing q to $-q$ in (2.18), we obtain

$$\varphi(-q^3)\varphi(-q^5) = \varphi(-q^2)\varphi(-q^{30}) + 2q^2\psi(-q)\psi(-q^{15}).$$

This together with (2.18) yields (2.19). Identity (2.20) appears in [4, p. 304].

Applying (2.9), we easily deduce that

$$\varphi(q)\varphi(q^7) = \varphi(q^4)\varphi(q^{28}) + 2q\psi(q^8)\varphi(q^{28}) + 2q^7\varphi(q^4)\psi(q^{56}) + 4q^8\psi(q^8)\psi(q^{56}),$$

and so

$$\varphi(-q)\varphi(-q^7) = \varphi(q^4)\varphi(q^{28}) - 2q\psi(q^8)\varphi(q^{28}) - 2q^7\varphi(q^4)\psi(q^{56}) + 4q^8\psi(q^8)\psi(q^{56}),$$

from which we immediately get (2.21).

Replace q by $-q$ in (2.16) and subtract the resulting identity from (2.16) to deduce that

$$\varphi(q)\varphi(q^{15}) - \varphi(-q)\varphi(-q^{15}) = 2q\psi(q^3)\psi(q^5) + 2q\psi(-q^3)\psi(-q^5).$$

Recalling [4, p. 377, Entry 9 (i)],

$$\psi(q^3)\psi(q^5) - \psi(-q^3)\psi(-q^5) = 2q^3\psi(q^2)\psi(q^{30}).$$

Combine the above two identities together to deduce (2.22). □

Lemma 2.3 *The following 2-dissection formulas hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.23}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.24}$$

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}, \tag{2.25}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.26}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{2.27}$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q \frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{2.28}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}. \tag{2.29}$$

Proof Identity (2.23) follows from (2.9). A complete proof of identity (2.25) is given in [8, Lemma 3.2]. Identities (2.24), (2.26)–(2.29) follow from (1.10.1), (30.10.3), (30.10.4), (34.1.17) and (34.1.18) of [6], respectively. □

3 Proof of Theorem 1.1

We first need the following lemma.

Lemma 3.1 *Let*

$$K(q) := 4\frac{f_4^5}{f_{20}} - 2\frac{f_2^5}{f_{10}} - \frac{f_1^5}{f_5} - \frac{f_2^{15} f_5 f_{20}}{f_1^5 f_4^5 f_{10}^3}.$$

Then

$$K(q) = 0.$$

Proof Appealing to (2.4), we conclude that

$$\frac{f_2^{15} f_5 f_{20}}{f_1^5 f_4^5 f_{10}^3} = \frac{f_1^5(-q)}{f_5(-q)},$$

which implies that

$$K(q) = 4\frac{f_4^5}{f_{20}} - 2\frac{f_2^5}{f_{10}} - \left(\frac{f_1^5}{f_5} + \frac{f_1^5(-q)}{f_5(-q)}\right).$$

From [6, Eq. (34.1.23)],

$$\psi(q)^2 - 5q\psi(q^5)^2 = \frac{f_1^3 f_{10}}{f_2 f_5}.$$

Applying (2.3) and (2.25), the above identity can be rewritten as follows,

$$\frac{f_1^5}{f_5} = \frac{f_2^5}{f_{10}} - 5q \frac{f_1^2 f_2 f_{10}^3}{f_5^2} \tag{3.1}$$

$$= \frac{f_2^5}{f_{10}} - 5q f_2 f_{10}^3 \cdot \left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_{40} f_4^2}{f_8 f_{10}^2}\right)^2. \tag{3.2}$$

By (3.2), we have

$$\frac{f_1^5}{f_5}+\frac{f_1^5(-q)}{f_5(-q)}=2\left(\frac{f_2^5}{f_{10}}-5qf_2f_{10}^3\cdot\left(-2q\frac{f_2f_4f_{20}^3}{f_{10}^5}\right)\right),$$

and therefore,

$$K(q)=4\left(\frac{f_4^5}{f_{20}}-\frac{f_2^5}{f_{10}}-5q^2\frac{f_2^2f_4f_{20}^3}{f_{10}^2}\right).$$

If we replace q by q^2 in (3.1), we immediately find that $K(q)=0$. □

REMARK. In fact, Lemma 3.1 was established in [5]. For the sake of completeness, we present a proof here.

To start our proofs, we present the following fact which will be used frequently without being explicitly mentioned in the sequel. Let $\{a(n)\}_0^\infty$ be the sequence defined by

$$\sum_{n=0}^{\infty}a(n)q^n=A_0(q^m)+qA_1(q^m)+\cdots+q^{m-1}A_{m-1}(q^m), \tag{3.3}$$

where $A_i(q)$ is an arbitrary infinite series in q and $m\geq 2$. For $0\leq i<m$, collecting those terms on each side of (3.3) that involve only the powers q^{mn+i}, dividing by q^i and replacing q^m by q, we find that

$$\sum_{n=0}^{\infty}a(mn+i)q^n=A_i(q). \tag{3.4}$$

Proof of Theorem 1.1 In view of (2.6) and (2.9),

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,2,5;n)q^n&=\varphi(q)\varphi(q^2)\varphi(q^5) &(3.5)\\ &=\left(\varphi(q^4)+2q\psi(q^8)\right)\left(\varphi(q^8)+2q^2\psi(q^{16})\right)\left(\varphi(q^{20})+2q^5\psi(q^{40})\right). &(3.6)\end{aligned}$$

If we extract those terms in which the power of q is divisible by 4 in (3.6), and replace q^4 by q, then

$$\sum_{n=0}^{\infty}N(1,2,5;4n)q^n=\varphi(q)\varphi(q^2)\varphi(q^5)+8q^2\psi(q^2)\psi(q^4)\psi(q^{10}). \tag{3.7}$$

Define

$$r_1(n)=N(1,2,5;4n)-N(1,2,5;n). \tag{3.8}$$

By (3.5), (3.7) and (3.8),

$$\sum_{n=0}^{\infty}r_1(n)q^n=8q^2\psi(q^2)\psi(q^4)\psi(q^{10}). \tag{3.9}$$

Picking out the terms involving q^{2n} in (3.9), then replacing q^2 by q and using (2.3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}r_1(2n)q^n&=8q\psi(q)\psi(q^2)\psi(q^5)\\ &=8qf_2f_4^2f_{10}^2\cdot\frac{f_1}{f_5}\cdot\frac{1}{f_1^2}. &(3.10)\end{aligned}$$

Substitute (2.23) and (2.25) in (3.10),

$$\sum_{n=0}^{\infty} r_1(2n)q^n = 8q f_2 f_4^2 f_{10}^2 \cdot \left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2} \right) \cdot \left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \right).$$

It follows that

$$\sum_{n=0}^{\infty} r_1(4n)q^n = 16q\frac{f_2^3 f_8^2 f_{10}^3}{f_{20}} \cdot \frac{f_1}{f_5} \cdot \frac{1}{f_1^4} - 8q\frac{f_2^4 f_4^4 f_{20}}{f_8^2} \cdot \frac{1}{f_1^4}. \tag{3.11}$$

Invoke (2.24) and (2.25) to arrive at

$$\sum_{n=0}^{\infty} r_1(4n)q^n = \left(16q\frac{f_2^3 f_8^2 f_{10}^3}{f_{20}} \left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2} \right) - 8q\frac{f_2^4 f_4^4 f_{20}}{f_8^2} \right) \cdot \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \right),$$

and

$$\sum_{n=0}^{\infty} r_1(8n)q^n = 64q\frac{f_2 f_4^7 f_{10}^2}{f_1^6 f_{20}} - 16q\frac{f_2^{16} f_5 f_{20}}{f_1^{11} f_4^3 f_{10}} - 32q\frac{f_2^6 f_4^2 f_{10}}{f_1^6}.$$

By (2.7) and (2.3), we deduce that

$$\sum_{n=0}^{\infty} T(1,2,5;n)q^n = \frac{f_2 f_4^2 f_{10}^2}{f_1 f_5}.$$

Combining the above two identities together, we obtain

$$\sum_{n=0}^{\infty} r_1(8n)q^n - 16\sum_{n=0}^{\infty} T(1,2,5;n)q^{n+1} = 16q\frac{f_2 f_4^2 f_{10}^2}{f_1^6} K(q).$$

Applying Lemma 3.1, we see that

$$\sum_{n=0}^{\infty} r_1(8n)q^n = 16\sum_{n=0}^{\infty} T(1,2,5;n)q^{n+1},$$

from which it follows that

$$T(1,2,5;n) = \frac{1}{16} r_1(8n+8).$$

From (1.1), (3.8) and the above relation, we immediately deduce (1.2). □

4 Proof of Theorem 1.2

Due to (2.6) and (2.9),

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,6,9;n)q^n &= \varphi(q)\varphi(q^6)\varphi(q^9) \\ &= \left(\varphi(q^4) + 2q\psi(q^8)\right)\left(\varphi(q^{24}) + 2q^6\psi(q^{48})\right)\left(\varphi(q^{36}) + 2q^9\psi(q^{72})\right).\end{aligned} \tag{4.1}$$

If we pick out the terms involving q^{4n} in (4.1) and then replace q^4 by q, we obtain

$$\sum_{n=0}^{\infty} N(1,6,9;4n)q^n = \varphi(q)\varphi(q^6)\varphi(q^9) + 8q^4\psi(q^2)\psi(q^{12})\psi(q^{18}).$$

Define

$$r_2(n) = N(1,6,9;4n) - N(1,6,9;n). \tag{4.2}$$

It follows that

$$\sum_{n=0}^{\infty} r_2(n)q^n = 8q^4\psi(q^2)\psi(q^{12})\psi(q^{18}).$$

From [4, p. 49, Corollary (ii)],

$$\psi(q) = \frac{f_6 f_9^2}{f_3 f_{18}} + q\psi(q^9). \tag{4.3}$$

Thus,

$$\sum_{n=0}^{\infty} r_2(n)q^n = 8q^4 \left(\frac{f_{12} f_{18}^2}{f_6 f_{36}} + q^2\psi(q^{18}) \right) \psi(q^{12})\psi(q^{18}). \tag{4.4}$$

Extracting those terms in which the power of q is congruent to 0 modulo 3 in (4.4), then replacing q^3 by q, we achieve

$$\sum_{n=0}^{\infty} r_2(3n)q^n = 8q^2\psi(q^4)\psi(q^6)^2,$$

and

$$\sum_{n=0}^{\infty} r_2(6n)q^n = 8q\psi(q^2)\psi(q^3)^2.$$

Invoke (2.8) and (2.9),

$$\begin{aligned}\sum_{n=0}^{\infty} r_2(6n)q^n &= 8q\psi(q^2)\varphi(q^3)\psi(q^6)\\ &= 8q\psi(q^2)\left(\varphi(q^{12}) + 2q^3\psi(q^{24})\right)\psi(q^6).\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} r_2(12n)q^n = 16q^2\psi(q)\psi(q^3)\psi(q^{12}).$$

By (2.10), we have

$$\sum_{n=0}^{\infty} r_2(24n)q^n = 16q\psi(q^2)\varphi(q^3)\psi(q^6).$$

Applying (2.8) and dividing by q,

$$\sum_{n=0}^{\infty} r_2(24n+24)q^n = 16\psi(q^2)\psi(q^3)^2. \tag{4.5}$$

If we collect those terms in which the power of q is congruent to 1 modulo 3 in (4.4), then divide by q and replace q^3 by q, we obtain

$$\sum_{n=0}^{\infty} r_2(3n+1)q^n = 8q\psi(q^4)\psi(q^6)\frac{f_4 f_6^2}{f_2 f_{12}},$$

and

$$\sum_{n=0}^{\infty} r_2(6n+4)q^n = 8\psi(q^2)\psi(q^3)\frac{f_2 f_3^2}{f_1 f_6}.$$

Employing (2.3), we find that

$$\sum_{n=0}^{\infty} r_2(6n+4)q^n = 8 f_4^2 f_6 \cdot \frac{f_3}{f_1}.$$

By (2.26),

$$\sum_{n=0}^{\infty} r_2(6n+4)q^n = 8 f_4^2 f_6 \cdot \left(\frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}} \right),$$

from which we extract

$$\sum_{n=0}^{\infty} r_2(12n+4)q^n = 8 \frac{f_2^3 f_8 f_{12}^2}{f_4 f_6 f_{24}} \cdot \left(\frac{f_3}{f_1} \right)^2.$$

From (2.26), we can easily deduce that

$$\sum_{n=0}^{\infty} r_2(24n+16)q^n = 16 \frac{f_3 f_4^2 f_6}{f_1}. \tag{4.6}$$

It is easy to see that there are no terms of q^{3n+2} in (4.4), and $r_2(3n+2) = 0$. Hence, $r_2(24n+32) = 0$. From (2.3), (4.5), and (4.6),

$$\begin{aligned}\sum_{n=0}^{\infty} r_2(8n+16)q^n &= \sum_{n=0}^{\infty} r_2(24n+16)q^{3n} + \sum_{n=0}^{\infty} r_2(24n+24)q^{3n+1} + \sum_{n=0}^{\infty} r_2(24n+32)q^{3n+2} \\ &= 16 \frac{f_9 f_{12}^2 f_{18}}{f_3} + 16 q \psi(q^6)\psi(q^9)^2 \\ &= 16\psi(q^6)\psi(q^9) \left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\psi(q^9) \right).\end{aligned}$$

From (4.3),

$$\sum_{n=0}^{\infty} r_2(8n+16)q^n = 16 \sum_{n=0}^{\infty} T(1,6,9;n)q^n,$$

and hence,

$$T(1,6,9;n) = \frac{1}{16} r_2(8n+16).$$

Combining (1.1), (4.2) and the above relation, we complete the proof (1.3). □

5 Proof of Theorem 1.3

To prove Theorem 1.3, we first state and prove a lemma.

Lemma 5.1 *Let $n \in \mathbb{N}$. Then*

$$T(1,5,10;4n+6)=2T(1,5,10;n). \tag{5.1}$$

Proof From (2.7), (2.9) and (2.14), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T(1,5,10;n)q^n &= \psi(q)\psi(q^5)\psi(q^{10})\\ &= \psi(q^{10})\left(\psi(q^6)\varphi(q^{15})+qf(q^2,q^4)f(q^{40},q^{80})+q^6f(q^2,q^4)f(q^{20},q^{100})\right)\\ &= \psi(q^6)\psi(q^{10})\left(\varphi(q^{60})+2q^{15}\psi(q^{120})\right)\\ &\quad +q\psi(q^{10})f(q^2,q^4)f(q^{40},q^{80})+q^6\psi(q^{10})f(q^2,q^4)f(q^{20},q^{100}),\end{aligned}$$

and

$$\sum_{n=0}^{\infty}T(1,5,10;2n)q^n=\psi(q^3)\psi(q^5)\varphi(q^{30})+q^3\psi(q^5)f(q,q^2)f(q^{10},q^{50}).$$

Letting $a=q$ and $b=q^2$ in (2.1), it is not difficult to obtain

$$f(q,q^2)=\frac{f_2f_3^2}{f_1f_6}. \tag{5.2}$$

Furthermore, with the use of (2.3), (2.12) and (5.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T(1,5,10;2n)q^n &= \left(\psi(q^8)\varphi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\varphi(q^4)\psi(q^{120})\right)\varphi(q^{30})\\ &\quad +q^3f(q^{10},q^{50})\frac{f_2f_{10}^2}{f_6}\cdot\frac{f_3^2}{f_1^2}\cdot\frac{f_1}{f_5}.\end{aligned} \tag{5.3}$$

Put (2.25) and (2.27) in (5.3) to find that

$$\begin{aligned}\sum_{n=0}^{\infty}T(1,5,10;2n)q^n &= \left(\psi(q^8)\varphi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\varphi(q^4)\psi(q^{120})\right)\varphi(q^{30})\\ &\quad +q^3f(q^{10},q^{50})\frac{f_2f_{10}^2}{f_6}\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right)\left(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}T(1,5,10;4n+2)q^n &= q\psi(q)\varphi(q^{15})\psi(q^{15})\\ &\quad +qf(q^5,q^{25})\frac{f_1f_5^2}{f_3}\left(\frac{f_2^3f_3f_6^2f_{10}^3}{f_1^4f_5^3f_{12}f_{20}}-2q\frac{f_2^3f_3^2f_{12}f_{20}}{f_1^4f_5^2f_6}\right).\end{aligned} \tag{5.4}$$

In (2.16), replace q by $-q$ to see that

$$\varphi(-q^6)\varphi(-q^{10})-2q\psi(-q^3)\psi(-q^5)=\varphi(-q)\varphi(-q^{15}).$$

Multiplying by $\frac{f_2^3 f_3 f_{10}}{f_1^4 f_5^3}$ on both sides and using (2.5), we arrive at

$$\frac{f_2^3 f_3 f_6^2 f_{10}^3}{f_1^4 f_5^3 f_{12} f_{20}} - 2q \frac{f_2^3 f_3^2 f_{12} f_{20}}{f_1^4 f_5^2 f_6} = \frac{f_2^2 f_3 f_{10} f_{15}^2}{f_1^2 f_5^3 f_{30}}. \tag{5.5}$$

Besides, we apply (2.1) with $a = q$ and $b = q^5$ to deduce that

$$f(q, q^5) = \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6},$$

and

$$f(q^5, q^{25}) = \frac{f_{10}^2 f_{15} f_{60}}{f_5 f_{20} f_{30}}. \tag{5.6}$$

By (2.3), (5.4), (5.5) and (5.6),

$$\begin{aligned}\sum_{n=0}^{\infty} T(1, 5, 10; 4n+2)q^n &= q\psi(q)\varphi(q^{15})\psi(q^{15}) + q\frac{f_2^2 f_{10}^3 f_{15}^3 f_{60}}{f_1 f_5^2 f_{20} f_{30}^2} \\ &= q\psi(q)\left(\varphi(q^{15})\psi(q^{15}) + \frac{f_{10}^3 f_{15}^3 f_{60}}{f_5^2 f_{20} f_{30}^2}\right).\end{aligned}$$

Now, we need to establish the following identity,

$$\varphi(q^3)\psi(q^3) + \frac{f_2^3 f_3^3 f_{12}}{f_1^2 f_4 f_6^2} = 2\psi(q)\psi(q^2). \tag{5.7}$$

From (2.28) and (2.29),

$$\begin{aligned}\frac{f_1}{f_3^3} \cdot \frac{f_6^7}{f_{12}^2} + \frac{f_3^3}{f_1} \cdot \frac{f_2^3 f_{12}}{f_4 f_6^2} &= \left(\frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}\right) \cdot \frac{f_6^7}{f_{12}^2} + \left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \cdot \frac{f_2^3 f_{12}}{f_4 f_6^2} \\ &= 2 f_2 f_4^2.\end{aligned}$$

Multiplying both sides of above identity by $\frac{1}{f_1}$, we obtain

$$\frac{f_6^5}{f_3^2 f_{12}^2} \cdot \frac{f_6^2}{f_3} + \frac{f_2^3 f_3^3 f_{12}}{f_1^2 f_4 f_6^2} = 2\frac{f_2^2}{f_1} \cdot \frac{f_4^2}{f_2}.$$

We next use (2.2) and (2.3) to deduce (5.7).

Substituting q for q^5 in (5.7), we immediately see that

$$\sum_{n=0}^{\infty} T(1, 5, 10; 4n+2)q^n = 2q\psi(q)\psi(q^5)\psi(q^{10}).$$

Equating the coefficients of q^{n+1} in the above equation, we obtain the required result. □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3 We note that

$$\begin{aligned}\sum_{n=0}^{\infty} N(1, 5, 10; n)q^n &= \varphi(q)\varphi(q^5)\varphi(q^{10}) \\ &= \left(\varphi(q^4) + 2q\psi(q^8)\right)\left(\varphi(q^{20}) + 2q^5\psi(q^{40})\right)\left(\varphi(q^{40}) + 2q^{10}\psi(q^{80})\right),\end{aligned}$$

and

$$\sum_{n=0}^{\infty} N(1,5,10;4n)q^{4n} = \varphi(q^4)\varphi(q^{20})\varphi(q^{40}) + 8q^{16}\psi(q^8)\psi(q^{40})\psi(q^{80}).$$

Replacing q^4 with q, we obtain

$$\sum_{n=0}^{\infty} N(1,5,10;4n)q^{n} = \varphi(q)\varphi(q^{5})\varphi(q^{10}) + 8q^{4}\psi(q^2)\psi(q^{10})\psi(q^{20}).$$

Define

$$r_3(n) = N(1,5,10;4n) - N(1,5,10;n), \tag{5.8}$$

we have

$$\sum_{n=0}^{\infty} r_3(n)q^n = 8q^4\psi(q^2)\psi(q^{10})\psi(q^{20}).$$

It then follows that

$$\sum_{n=0}^{\infty} r_3(n)q^n = 8\sum_{n=0}^{\infty} T(1,5,10;n)q^{2n+4}.$$

By comparing coefficients of q^{2n+4}, we derive

$$r_3(2n+4) = 8T(1,5,10;n),$$

and

$$r_3(8n+16) = 8T(1,5,10;4n+6).$$

Applying Lemma 5.1, we arrive at

$$r_3(8n+16) = 16T(1,5,10;n). \tag{5.9}$$

Utilizing (1.1), (5.8) in (5.9), we complete the proof of (1.4). □

6 Proof of Theorem 1.4

Using (2.6) and (2.9), we achieve

$$\begin{aligned}\sum_{n=0}^{\infty} N(5,21,35;n)q^n &= \varphi(q^5)\varphi(q^{21})\varphi(q^{35})\\ &= \left(\varphi(q^{20}) + 2q^5\psi(q^{40})\right)\left(\varphi(q^{84}) + 2q^{21}\psi(q^{168})\right)\left(\varphi(q^{140}) + 2q^{35}\psi(q^{280})\right),\end{aligned}$$

and

$$\sum_{n=0}^{\infty} N(5,21,35;4n)q^n = \varphi(q^5)\varphi(q^{21})\varphi(q^{35}) + 4q^{14}\varphi(q^5)\psi(q^{42})\psi(q^{70}) + 4q^{10}\psi(q^{10})\varphi(q^{21})\psi(q^{70}).$$

Define

$$r_4(n) = N(5,21,35;4n) - N(5,21,35;n). \tag{6.1}$$

From the generating functions for $N(5,21,35;n)$ and $N(5,21,35;4n)$ above,

$$\begin{aligned}\sum_{n=0}^{\infty} r_4(n)q^n &= 4q^{14}\varphi(q^5)\psi(q^{42})\psi(q^{70}) + 4q^{10}\psi(q^{10})\varphi(q^{21})\psi(q^{70}) \\ &= 4q^{14}\left(\varphi(q^{20}) + 2q^5\psi(q^{40})\right)\psi(q^{42})\psi(q^{70}) \\ &\quad + 4q^{10}\psi(q^{10})\left(\varphi(q^{84}) + 2q^{21}\psi(q^{168})\right)\psi(q^{70}),\end{aligned} \tag{6.2}$$

where the second equality is due to (2.9).

From (6.2), we obtain

$$\sum_{n=0}^{\infty} r_4(2n+1)q^n = 8q^9\psi(q^{20})\psi(q^{21})\psi(q^{35}) + 8q^{15}\psi(q^5)\psi(q^{35})\psi(q^{84}). \tag{6.3}$$

By (2.11), (2.12) and (6.3),

$$\begin{aligned}\sum_{n=0}^{\infty} r_4(2n+1)q^n &= 8q^9\psi(q^{20})\left(\psi(q^{56})\varphi(q^{420}) + q^{21}\psi(q^{14})\psi(q^{210}) + q^{98}\varphi(q^{28})\psi(q^{840})\right) \\ &\quad + 8q^{15}\left(\psi(q^{40})\varphi(q^{140}) + q^5\psi(q^{10})\psi(q^{70}) + q^{30}\varphi(q^{20})\psi(q^{280})\right)\psi(q^{84}),\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} r_4(4n+1)q^n = 8q^{15}\psi(q^{10})\psi(q^7)\psi(q^{105}) + 8q^{10}\psi(q^5)\psi(q^{35})\psi(q^{42}).$$

Invoking (2.11) and (2.13), we deduce from above that

$$\sum_{n=0}^{\infty} r_4(8n+5)q^n = 16q^7\psi(q^5)\psi(q^{21})\psi(q^{35}). \tag{6.4}$$

Equating the coefficients of q^{2n+8} in (6.4), we find that

$$r_4(16n+69) = 16T(5,21,35;2n+1).$$

Theorem 1.4 follows from this relation, (1.1) and (6.1), and we finish the proof. □

7 Proof of Theorem 1.5

From (2.6) and (2.9),

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,6,15;n)q^n &= \varphi(q)\varphi(q^6)\varphi(q^{15}) \\ &= \left(\varphi(q^4) + 2q\psi(q^8)\right)\left(\varphi(q^{24}) + 2q^6\psi(q^{48})\right)\left(\varphi(q^{60}) + 2q^{15}\psi(q^{120})\right).\end{aligned} \tag{7.1}$$

It follows that

$$\sum_{n=0}^{\infty} N(1,6,15;4n)q^n = \varphi(q)\varphi(q^6)\varphi(q^{15}) + 4q^4\psi(q^2)\varphi(q^6)\psi(q^{30}). \tag{7.2}$$

Define

$$r_5(n) = 3N(1,6,15;n) - N(1,6,15;4n). \tag{7.3}$$

In light of (7.1), (7.2) and (7.3),

$$\sum_{n=0}^{\infty} r_5(n)q^n = 2\varphi(q)\varphi(q^6)\varphi(q^{15}) - 4q^4\psi(q^2)\varphi(q^6)\psi(q^{30}).$$

This together with (2.9) gives

$$\begin{aligned}\sum_{n=0}^{\infty} r_5(n)q^n &= 2\left(\varphi(q^4)+2q\psi(q^8)\right)\varphi(q^6)\left(\varphi(q^{60})+2q^{15}\psi(q^{120})\right)\\&\quad - 4q^4\psi(q^2)\varphi(q^6)\psi(q^{30}),\end{aligned}$$

and

$$\sum_{n=0}^{\infty} r_5(2n)q^n = 2\varphi(q^2)\varphi(q^3)\varphi(q^{30}) + 8q^8\varphi(q^3)\psi(q^4)\psi(q^{60}) - 4q^2\psi(q)\varphi(q^3)\psi(q^{15}). \tag{7.4}$$

With the use of (2.9), (2.13) and (7.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} r_5(2n)q^n &= 2\varphi(q^2)\left(\varphi(q^{12})+2q^3\psi(q^{24})\right)\varphi(q^{30}) + 8q^8\left(\varphi(q^{12})+2q^3\psi(q^{24})\right)\psi(q^4)\psi(q^{60})\\&\quad - 4q^2\left(\varphi(q^{12})+2q^3\psi(q^{24})\right)\left(\psi(q^6)\psi(q^{10})+q\varphi(q^{20})\psi(q^{24})+q^3\varphi(q^{12})\psi(q^{40})\right).\end{aligned}$$

If we extract those terms in which the power of q is congruent to 1 modulo 2, then divide by q and replace q^2 by q and employ (2.16), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} r_5(4n+2)q^n &= 4q\varphi(q)\psi(q^{12})\varphi(q^{15}) + 16q^5\psi(q^2)\psi(q^{12})\psi(q^{30})\\&\quad - 4q\varphi(q^6)\varphi(q^{10})\psi(q^{12}) - 4q^2\varphi(q^6)^2\psi(q^{20}) - 8q^2\psi(q^3)\psi(q^5)\psi(q^{12})\\&= 4q\psi(q^{12})\left(\varphi(q)\varphi(q^{15})-2q\psi(q^3)\psi(q^5)\right)\\&\quad + 16q^5\psi(q^2)\psi(q^{12})\psi(q^{30}) - 4q\varphi(q^6)\varphi(q^{10})\psi(q^{12}) - 4q^2\varphi(q^6)^2\psi(q^{20})\\&= 4q\varphi(-q^6)\varphi(-q^{10})\psi(q^{12}) + 16q^5\psi(q^2)\psi(q^{12})\psi(q^{30})\\&\quad - 4q\varphi(q^6)\varphi(q^{10})\psi(q^{12}) - 4q^2\varphi(q^6)^2\psi(q^{20}).\end{aligned} \tag{7.5}$$

Due to (2.17), (2.19) and (7.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} r_5(8n+6)q^n &= 4\psi(q^6)\left(\varphi(-q^3)\varphi(-q^5)-\varphi(q^3)\varphi(q^5)\right) + 16q^2\psi(q)\psi(q^6)\psi(q^{15})\\&= 4\psi(q^6)\left(2q^2\psi(-q)\psi(-q^{15})-2q^2\psi(q)\psi(q^{15})\right) + 16q^2\psi(q)\psi(q^6)\psi(q^{15})\\&= 8q^2\psi(q^6)\left(\psi(-q)\psi(-q^{15})+\psi(q)\psi(q^{15})\right)\\&= 16q^2\psi(q^6)^2\psi(q^{10}).\end{aligned}$$

It then follows that

$$\sum_{n=0}^{\infty} r_5(16n+6)q^n = 16q\psi(q^3)^2\psi(q^5). \tag{7.6}$$

By (2.7) and (2.13),

$$\begin{aligned}\sum_{n=0}^{\infty} T(1,6,15;n)q^n &= \psi(q)\psi(q^6)\psi(q^{15})\\ &= \psi(q^6)\left(\psi(q^6)\psi(q^{10}) + q\varphi(q^{20})\psi(q^{24}) + q^3\varphi(q^{12})\psi(q^{40})\right),\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} T(1,6,15;2n)q^n = \psi(q^3)^2\psi(q^5). \tag{7.7}$$

Equating the coefficients of q^{n+1} in (7.6) and (7.7), we find that

$$T(1,6,15;2n) = \frac{1}{16} r_5(16n+22).$$

Combining this relation, (1.1) and (7.3), we complete the proof of Theorem 1.5. □

8 Proof of Theorem 1.6

We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,2,7;n)q^n &= \varphi(q)\varphi(q^2)\varphi(q^7)\\ &= \left(\varphi(q^4) + 2q\psi(q^8)\right)\left(\varphi(q^8) + 2q^2\psi(q^{16})\right)\left(\varphi(q^{28}) + 2q^7\psi(q^{56})\right),\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{\infty} N(1,2,7;4n)q^n = \varphi(q)\varphi(q^2)\varphi(q^7) + 4q^2\varphi(q^2)\psi(q^2)\psi(q^{14}).$$

Define

$$r_6(n) = 3N(1,2,7;n) - N(1,2,7;4n), \tag{8.1}$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} r_6(n)q^n &= 2\varphi(q)\varphi(q^2)\varphi(q^7) - 4q^2\varphi(q^2)\psi(q^2)\psi(q^{14})\\ &= 2\left(\varphi(q^4) + 2q\psi(q^8)\right)\varphi(q^2)\left(\varphi(q^{28}) + 2q^7\psi(q^{56})\right)\\ &\quad - 4q^2\varphi(q^2)\psi(q^2)\psi(q^{14}).\end{aligned}$$

It is easy to see that

$$\sum_{n=0}^{\infty} r_6(2n)q^n = 2\varphi(q)\varphi(q^2)\varphi(q^{14}) + 8q^4\varphi(q)\psi(q^4)\psi(q^{28}) - 4q\varphi(q)\psi(q)\psi(q^7). \tag{8.2}$$

Using (2.9) and (2.11) in (8.2), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} r_6(2n)q^n &= 2\left(\varphi(q^4) + 2q\psi(q^8)\right)\varphi(q^2)\varphi(q^{14}) + 8q^4\left(\varphi(q^4) + 2q\psi(q^8)\right)\psi(q^4)\psi(q^{28})\\ &\quad - 4q\left(\varphi(q^4) + 2q\psi(q^8)\right)\left(\psi(q^8)\varphi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\varphi(q^4)\psi(q^{56})\right).\end{aligned}$$

Extracting those terms in which the power of q is congruent to 1 modulo 2, then dividing by q and replacing q^2 by q and using (2.20), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} r_6(4n+2)q^n &= 4\varphi(q)\psi(q^4)\varphi(q^7) + 16q^2\psi(q^2)\psi(q^4)\psi(q^{14}) \\
&\quad - 4\varphi(q^2)\psi(q^4)\varphi(q^{14}) - 4q^3\varphi(q^2)^2\psi(q^{28}) - 8q\psi(q)\psi(q^4)\psi(q^7) \\
&= 4\psi(q^4)\left(\varphi(q)\varphi(q^7) - 2q\psi(q)\psi(q^7)\right) + 16q^2\psi(q^2)\psi(q^4)\psi(q^{14}) \\
&\quad - 4\varphi(q^2)\psi(q^4)\varphi(q^{14}) - 4q^3\varphi(q^2)^2\psi(q^{28}) \\
&= 4\psi(q^4)\varphi(-q^2)\varphi(-q^{14}) + 16q^2\psi(q^2)\psi(q^4)\psi(q^{14}) \\
&\quad - 4\varphi(q^2)\psi(q^4)\varphi(q^{14}) - 4q^3\varphi(q^2)^2\psi(q^{28}).
\end{aligned} \tag{8.3}$$

With the use of (2.11), (2.21) and (8.3), we arrive at

$$\begin{aligned}
\sum_{n=0}^{\infty} r_6(8n+2)q^n &= 4\psi(q^2)\left(\varphi(-q)\varphi(-q^7) - \varphi(q)\varphi(q^7)\right) + 16q\psi(q^2)\psi(q)\psi(q^7) \\
&= 4\psi(q^2)\left(-4q\psi(q^8)\varphi(q^{28}) - 4q^7\varphi(q^4)\psi(q^{56})\right) \\
&\quad + 16q\psi(q^2)\left(\psi(q^8)\varphi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\varphi(q^4)\psi(q^{56})\right) \\
&= 16q^2\psi(q^2)^2\psi(q^{14}).
\end{aligned}$$

It then follows that

$$\sum_{n=0}^{\infty} r_6(16n+2)q^n = 16q\psi(q)^2\psi(q^7). \tag{8.4}$$

From (2.7) and (2.11),

$$\sum_{n=0}^{\infty} T(1,2,7;n)q^n = \psi(q^2)\left(\psi(q^8)\varphi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\varphi(q^4)\psi(q^{56})\right).$$

Hence,

$$\sum_{n=0}^{\infty} T(1,2,7;2n+1)q^n = \psi(q)^2\psi(q^7). \tag{8.5}$$

From (8.4) and (8.5),

$$\sum_{n=0}^{\infty} r_6(16n+2)q^n = 16\sum_{n=0}^{\infty} T(1,2,7;2n+1)q^{n+1}.$$

This immediately implies that

$$T(1,2,7;2n+1) = \frac{1}{16} r_6(16n+18). \tag{8.6}$$

Combining (1.1), (8.1) and (8.6) together, we complete the proof. □

9 Proof of Theorem 1.7

Observe that

$$\sum_{n=0}^{\infty} N(3,5,30;n)q^n = \varphi(q^3)\varphi(q^5)\varphi(q^{30})$$
$$= \left(\varphi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\varphi(q^{20}) + 2q^5\psi(q^{40})\right)\left(\varphi(q^{120}) + 2q^{30}\psi(q^{240})\right),$$

and

$$\sum_{n=0}^{\infty} N(3,5,30;4n)q^n = \varphi(q^3)\varphi(q^5)\varphi(q^{30}) + 4q^2\psi(q^6)\psi(q^{10})\varphi(q^{30}).$$

Define

$$r_7(n) = 3N(3,5,30;n) - N(3,5,30;4n). \tag{9.1}$$

Then, we obtain

$$\sum_{n=0}^{\infty} r_7(n)q^n = 2\varphi(q^3)\varphi(q^5)\varphi(q^{30}) - 4q^2\psi(q^6)\psi(q^{10})\varphi(q^{30})$$
$$= 2\left(\varphi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\varphi(q^{20}) + 2q^5\psi(q^{40})\right)\varphi(q^{30})$$
$$- 4q^2\psi(q^6)\psi(q^{10})\varphi(q^{30}),$$

from which we extract

$$\sum_{n=0}^{\infty} r_7(2n)q^n = (2\varphi(q^6)\varphi(q^{10}) + 8q^4\psi(q^{12})\psi(q^{20}))\varphi(q^{15}) - 4q\psi(q^3)\psi(q^5)\varphi(q^{15}).$$

By (2.9) and (2.12),

$$\sum_{n=0}^{\infty} r_7(2n)q^n = \left(2\varphi(q^6)\varphi(q^{10}) + 8q^4\psi(q^{12})\psi(q^{20})\right)\left(\varphi(q^{60}) + 2q^{15}\psi(q^{120})\right)$$
$$- 4q\left(\psi(q^8)\varphi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\varphi(q^4)\psi(q^{120})\right)$$
$$\times\left(\varphi(q^{60}) + 2q^{15}\psi(q^{120})\right).$$

Collecting those terms in which the power of q is odd, then dividing by q and replacing q^2 by q and utilizing (2.18), we obtain

$$\sum_{n=0}^{\infty} r_7(4n+2)q^n = 4q^7\varphi(q^3)\varphi(q^5)\psi(q^{60}) + 16q^9\psi(q^6)\psi(q^{10})\psi(q^{60}) - 4\psi(q^4)\varphi(q^{30})^2$$
$$- 4q^7\varphi(q^2)\varphi(q^{30})\psi(q^{60}) - 8q^9\psi(q)\psi(q^{15})\psi(q^{60})$$
$$= 4q^7\left(\varphi(q^3)\varphi(q^5) - 2q^2\psi(q)\psi(q^{15})\right)\psi(q^{60}) + 16q^9\psi(q^6)\psi(q^{10})\psi(q^{60})$$
$$- 4\psi(q^4)\varphi(q^{30})^2 - 4q^7\varphi(q^2)\varphi(q^{30})\psi(q^{60})$$
$$= 4q^7\varphi(-q^2)\varphi(-q^{30})\psi(q^{60}) + 16q^9\psi(q^6)\psi(q^{10})\psi(q^{60})$$
$$- 4\psi(q^4)\varphi(q^{30})^2 - 4q^7\varphi(q^2)\varphi(q^{30})\psi(q^{60}),$$

and

$$\sum_{n=0}^{\infty} r_7(8n+6)q^n = 4q^3\left(\varphi(-q)\varphi(-q^{15}) - \varphi(q)\varphi(q^{15})\right)\psi(q^{30}) + 16q^4\psi(q^3)\psi(q^5)\psi(q^{30}).$$

From (2.22),

$$\sum_{n=0}^{\infty} r_7(8n+6)q^n = 4q^3\left(4q^4\psi(q^2)\psi(q^{30}) - 4q\psi(q^3)\psi(q^5)\right)\psi(q^{30}) + 16q^4\psi(q^3)\psi(q^5)\psi(q^{30})$$
$$= 16q^7\psi(q^2)\psi(q^{30})^2. \tag{9.2}$$

By (2.7) and (2.12),

$$\sum_{n=0}^{\infty} T(3,5,30;n)q^n = \left(\psi(q^8)\varphi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\varphi(q^4)\psi(q^{120})\right)\psi(q^{30}),$$

and

$$\sum_{n=0}^{\infty} T(3,5,30;2n+1)q^{2n+1} = q^3\psi(q^2)\psi(q^{30})^2. \tag{9.3}$$

In view of (9.2) and (9.3), we conclude that

$$\sum_{n=0}^{\infty} r_7(8n+6)q^n = 16\sum_{n=0}^{\infty} T(3,5,30;2n+1)q^{2n+5}.$$

Equating the coefficients of q^{2n+5} in the above equation, we see that

$$T(3,5,30;2n+1) = \frac{1}{16}r_7(16n+46). \tag{9.4}$$

Combining (1.1), (9.1) and (9.4), we complete the proof of Theorem 1.7. □

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