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ORIGINAL RESEARCH

On two theta function identities of Ramanujan

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Abstract On page 310 of his second notebook, Ramanujan recorded two theta function identities. B. C. Berndt proved them by the method of parameterization. The purpose of this article is to give an elementary proof for those two identities and also we have obtained two theta function identities which are analogues to the identities of Ramanujan. Further, we establish two distinct formula for 3-core partitions in terms of divisor function.

Keywords Theta functions · Divisor functions

Mathematics Subject Classification 11F20 · 11A25

1 Introduction

On page 310 of his second notebook [10], Ramanujan recorded the following two theta function identities:
For $|q| < 1$,

$$\frac{\varphi^3(q^{\frac{1}{3}})}{\varphi(q)} = \frac{\varphi^3(q)}{\varphi(q^3)} + 6q^{\frac{1}{3}}\frac{f^3(q^3)}{f(q)} + 12q^{\frac{2}{3}}\frac{f^3(-q^6)}{f(-q^2)} \tag{1.1}$$

and

$$\frac{\psi^3(q^{\frac{1}{3}})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{\frac{1}{3}}\frac{f^3(-q^3)}{f(-q)} + 3q^{\frac{2}{3}}\frac{f^3(-q^6)}{f(-q^2)}, \tag{1.2}$$

where $\varphi(q)$, $\psi(q)$ and $f(-q)$ are as defined below in (1.12),(1.13) and (1.14) respectively. B. C. Berndt [2, pp. 185-186] proved these identities by the method of parameterization. The equivalent form of (1.1) and (1.2) can be found in [6, eq. (vii), eq. (xv)]. One of the objective of this article is to give proof of these identities by making use of Ramanujan's ${}_1\psi_1$ summation formula [2, p. 34] and Lambert series sum found in [5, p. 71]. Further we prove the following two identities which are analogues to (1.1) and (1.2):

$$\frac{\varphi^3(q)}{\varphi(q^{\frac{1}{3}})} = \frac{\varphi^3(q^3)}{\varphi(q)} - 2q^{\frac{1}{3}}\frac{f^3(q^3)}{f(q)} + 4q^{\frac{2}{3}}\frac{f^3(-q^6)}{f(-q^2)} \tag{1.3}$$

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and

$$q^{\frac{1}{3}}\frac{\psi^3(q)}{\psi(q^{\frac{1}{3}})}=q\frac{\psi^3(q^3)}{\psi(q)}+q^{\frac{1}{3}}\frac{f^3(-q^3)}{f(-q)}-q^{\frac{2}{3}}\frac{f^3(-q^6)}{f(-q^2)}. \tag{1.4}$$

The identity (1.3) is equivalent to [6, eq. (viii)] and (1.4) is equivalent to (2.9) of [4]. It is interesting to note that (1.1)-(1.4) offer 3-dissection of $\varphi^3(q)$, $\psi^3(q)$, $1/\varphi(q)$ and $1/\psi(q)$ respectively. For details on dissection see [8, p. 14]. In our proof, we require the following identities:

$$\frac{\varphi^3(q)}{\varphi(q^3)}=1+6\left\{\frac{q}{1-q}+\frac{q^2}{1+q^2}-\frac{q^4}{1+q^4}-\frac{q^5}{1-q^5}+\cdots\right\}, \tag{1.5}$$

$$\frac{\psi^3(q)}{\psi(q^3)}=1+3\left\{\frac{q}{1-q}-\frac{q^5}{1-q^5}+\frac{q^7}{1-q^7}-\frac{q^{11}}{1-q^{11}}+\cdots\right\}, \tag{1.6}$$

$$\frac{\varphi^3(q^3)}{\varphi(q)}=1-2\left\{\frac{q}{1+q}-\frac{q^2}{1-q^2}+\frac{q^4}{1-q^4}-\frac{q^5}{1+q^5}+\cdots\right\} \tag{1.7}$$

and

$$q\frac{\psi^3(q^3)}{\psi(q)}=\frac{q}{1-q^2}-\frac{q^2}{1-q^4}+\frac{q^4}{1-q^8}-\frac{q^5}{1-q^{10}}+\cdots. \tag{1.8}$$

The identities (1.5) and (1.6) are recorded by Ramanujan in Chapter 19 of his second notebook [10, p. 229]. Berndt [2, pp. 228-229] proved (1.5) and (1.6) by employing Jacobi's triple product identity and logarithmic derivatives of theta functions. S. Cooper has given a proof of (1.5)-(1.8) in [5, Theorem 3.21]. For completeness, in this article, we recall the same proof given by S. Cooper for (1.5) and (1.6). For (1.7) and (1.8), we offer a slightly different proof. In Section 2, we prove (1.1)-(1.8).

Let

$$\sum_{n=0}^{\infty}a_3(n)q^n=\frac{f^3(-q^3)}{f(-q)}.$$

Then $a_3(n)$ denotes the number of 3-core partitions of n (For more details see [7,11]). In [7], A. Granville and K. Ono have proved that

$$a_3(n)=d_{1,3}(3n+1)-d_{2,3}(3n+1),$$

where $d_{r,3}$ denotes the number of divisors of n which are congruent to r modulo 3, by employing the theory of modular forms. M. D. Hirschhorn, J. A. Sellers [9], give an elementary proof for $a_3(n)$. Further they obtained the following explicit formula for $a_3(n)$:

$$a_3(n)=\begin{cases}\prod\limits_{\alpha_i}(\alpha_i+1), & \text{if } \beta_j\text{'s are even}\\ 0, & \text{otherwise,}\end{cases} \tag{1.9}$$

where $3n+1=\prod\limits_{p_i\equiv 1 \pmod 3}p_i^{\alpha_i}\cdot\prod\limits_{q_j\equiv 2 \pmod 3}q_j^{\beta_j}$ is the prime factorization of $3n+1$.

Also, L. Wang [11] proved the above by using Ramanujan's ${}_1\psi_1$ summation formula. At the end of this article, we offer two more distinct formulas for $a_3(n)$ which are straight forward from our intermediate identities.

Now close this section by recalling definitions and certain identities. As usual for any complex numbers a and q with $|q|<1$,

$$(a;q)_\infty=\prod_{n=0}^{\infty}\left(1-aq^n\right),$$

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

and

$$[a;q]_\infty = (a;q)_\infty \left(\frac{q}{a};q\right)_\infty.$$

The following is the famous Ramanujan's ${}_1\psi_1$ summation formula [2, p. 34]:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{[az;q]_\infty \left(\frac{b}{a};q\right)_\infty (q;q)_\infty}{(z;q)_\infty \left(\frac{b}{az};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty},$$

where $a,\ b,\ z$ and q are complex numbers with $|\frac{b}{a}| < |z| < 1$ and $|q| < 1$. Setting $b = aq$ in the above, we find that

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{[az;q]_\infty (q;q)_\infty^2}{[z;q]_\infty [a;q]_\infty}. \tag{1.10}$$

In [1], G. E. Andrews, R. Lewis and Z-G Liu have deduced the following identity:

If $a,\ b,\ c \neq q^n$ (for any $n \in \mathbb{Z}$) are non-zero complex numbers with $abc \neq q^n$. Then

$$\begin{aligned}\frac{[ab;q]_\infty [bc;q]_\infty [ca;q]_\infty (q;q)_\infty^2}{[a;q]_\infty [b;q]_\infty [c;q]_\infty [abc;q]_\infty} &= 1 + \sum_{n=0}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{a^{-1}q^n}{1-a^{-1}q^n} + \sum_{n=0}^{\infty} \frac{bq^n}{1-bq^n} \\ &\quad - \sum_{n=1}^{\infty} \frac{b^{-1}q^n}{1-b^{-1}q^n} + \sum_{n=0}^{\infty} \frac{cq^n}{1-cq^n} - \sum_{n=1}^{\infty} \frac{c^{-1}q^n}{1-c^{-1}q^n} \\ &\quad - \sum_{n=0}^{\infty} \frac{abcq^n}{1-abcq^n} + \sum_{n=1}^{\infty} \frac{(abc)^{-1}q^n}{1-(abc)^{-1}q^n}. \end{aligned} \tag{1.11}$$

Later, it came to known that the above identity was existing much before [1] and was known to Halphen [5, p. 71]. Ramanujan's theta functions [2, p. 36] are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{1.12}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.13}$$

and

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty. \tag{1.14}$$

2 Proofs of (1.1)-(1.8)

Proof of (1.5) [5, pp. 189-190] Replacing q by q^3 in (1.11) and then setting $a = b = c = -q$, we find that

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = 1 - 6\sum_{n=0}^{\infty} \frac{q^{3n+1}}{1+q^{3n+1}} + 6\sum_{n=0}^{\infty} \frac{q^{3n+2}}{1+q^{3n+2}}. \tag{2.1}$$

Changing q to $-q$ in the above, we obtain (1.5). □

Proof of (1.6) [5, pp. 189-190] Replacing q by q^6 in (1.11) and then setting $a = b = c = q$, we arrive at (1.6). □

Proof of (1.7) Replacing q by q^3 in (1.10) and then setting $a=-1$ and $z=q$, we find that

$$\frac{\varphi^3(-q^3)}{\varphi(-q)}=2\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{3n}}. \tag{2.2}$$

Expanding each of the summands of right hand side of (2.2) into geometric series, interchange the order of summands, summing the inner geometric series and then changing q to $-q$, we obtain the proof of (1.7). □

Proof of (1.8) Replacing q by q^6 in (1.10) and then setting $a=q^2$ and $z=q^3$, we obtain (1.8). □

Proof of (1.1) Expanding each of the summands of right hand side of (2.1) into geometric series, interchange the order of summands and then summing the inner geometric series, we obtain

$$\frac{\varphi^3(-q)}{\varphi(-q^3)}=1+6\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1-q^{3n}}-6\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{1-q^{3n}}.$$

From the above, we have

$$\frac{\varphi^3(-q)}{\varphi(-q^3)}-\frac{\varphi^3(-q^3)}{\varphi(-q^9)}=6\left[-\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n+1}}{1-q^{9n+3}}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n+2}}{1-q^{9n+6}}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{6n-4}}{1-q^{9n-6}}\right.$$
$$\left.-\sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{6n-2}}{1-q^{9n-3}}\right].$$

This implies

$$\frac{\varphi^3(-q)}{\varphi(-q^3)}-\frac{\varphi^3(-q^3)}{\varphi(-q^9)}=6\left[\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n+2}}{1-q^{9n+6}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n+1}}{1-q^{9n+3}}\right]. \tag{2.3}$$

Changing q to q^9 in (1.10) and then setting $a=q^6,\ z=-q^3$, we deduce that

$$2\frac{f^3(-q^{18})}{f(-q^6)}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n}}{1-q^{9n+6}}. \tag{2.4}$$

Changing q to q^9 in (1.10) and then setting $a=q^3,\ z=-q^3$, we find that

$$\frac{f^3(-q^9)}{f(-q^3)}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n}}{1-q^{9n+3}}. \tag{2.5}$$

Employing (2.4) and (2.5) in (2.3), we obtain

$$\frac{\varphi^3(-q)}{\varphi(-q^3)}=\frac{\varphi^3(-q^3)}{\varphi(-q^9)}-6q\frac{f^3(-q^9)}{f(-q^3)}+12q^2\frac{f^3(-q^{18})}{f(-q^6)}.$$

Changing q to $-q^{\frac{1}{3}}$ in the above, we obtain (1.1). □

Proof of (1.2) Expanding each of the summands of right hand side of (1.6) into geometric series, interchange the order of summands and then summing the inner geometric series, we obtain

$$\frac{\psi^3(q)}{\psi(q^3)}=1+3\left[\sum_{n=1}^{\infty}\frac{q^n}{1-q^{6n}}-\sum_{n=1}^{\infty}\frac{q^{5n}}{1-q^{6n}}\right].$$

From the above, we deduce that

$$\frac{\psi^3(q)}{\psi(q^3)}-\frac{\psi^3(q^3)}{\psi(q^9)}=3\left[\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}}{1-q^{18n+6}}+\sum_{n=-\infty}^{\infty}\frac{q^{3n+2}}{1-q^{18n+12}}\right].$$

Replace q by $q^{\frac{1}{3}}$ in the above, we find that

$$\frac{\psi^3(q^{\frac{1}{3}})}{\psi(q)} - \frac{\psi^3(q)}{\psi(q^3)} = 3q^{\frac{1}{3}} \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{6n+2}} + 3q^{\frac{2}{3}} \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{6n+4}}. \tag{2.6}$$

Changing q to q^6 in (1.10) and then setting $a = q^2,\ z = q$, we obtain that

$$\frac{f^3(-q^3)}{f(-q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{6n+2}}. \tag{2.7}$$

Changing q to q^6 in (1.10) and then setting $a = q^4,\ z = q$, we find that

$$\frac{f^3(-q^6)}{f(-q^2)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{6n+4}}. \tag{2.8}$$

By substituting (2.7) and (2.8) into (2.6), we obtain (1.2). □

Proof of (1.3) From (2.2), we deduce that

$$\frac{\varphi^3(-q^3)}{\varphi(-q)} - \frac{\varphi^3(-q^9)}{\varphi(-q^3)} = 2q \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1+q^{9n+3}} + 2q^2 \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1+q^{9n+3}}. \tag{2.9}$$

Replace q by q^9 in (1.10) and then setting $a = -q^3,\ z = q^3$ we find that

$$\frac{f^3(-q^9)}{f(-q^3)} = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1+q^{9n+3}}. \tag{2.10}$$

Replace q to q^9 in (1.10) and then setting $a = -q^3,\ z = q^6$, we obtain that

$$2\frac{f^3(-q^{18})}{f(-q^6)} = \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1+q^{9n+3}}. \tag{2.11}$$

Employing (2.10) and (2.11) in (2.9) and then changing q to $-q^{\frac{1}{3}}$, we arrive at (1.3). □

Proof of (1.4) Expanding each of the summands of right hand side of (1.8) into geometric series, interchange the order of summands and then summing the inner geometric series, we obtain

$$q\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{6n+3}} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1-q^{6n+3}}.$$

From the above, we deduce that

$$q\frac{\psi^3(q^3)}{\psi(q)} - q^3\frac{\psi^3(q^9)}{\psi(q^3)} = q \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1-q^{18n+3}} - q^2 \sum_{n=-\infty}^{\infty} \frac{q^{12n}}{1-q^{18n+3}}. \tag{2.12}$$

Changing q to q^{18} in (1.10) and then setting $a = q^3,\ z = q^6$, we obtain that

$$\frac{f^3(-q^9)}{f(-q^3)} = \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1-q^{18n+3}}. \tag{2.13}$$

Changing q to q^{18} in (1.10) and then setting $a = q^3,\ z = q^{12}$, we find that

$$\frac{f^3(-q^{18})}{f(-q^6)} = \sum_{n=-\infty}^{\infty} \frac{q^{12n}}{1-q^{18n+3}}. \tag{2.14}$$

By substituting (2.13) and (2.14) into (2.12) and then changing q to $q^{\frac{1}{3}}$, we obtain (1.4). □

Remark If we expanding each of the summands of right hand side of (2.5) into geometric series, interchange the order of summands and then summing the inner geometric series, we obtain (2.10). Similarly by changing q to q^3 in (2.7) and then expanding each of the summands of right hand side into geometric series, interchange the order of summands and then summing the inner geometric series, we obtain (2.13). From these two different pattern of series, we obtain the following two distinct formulas for 3-core partitions.

$$a_3(n) = d_{1,6}(3n+1) - d_{5,6}(3n+1)$$

and

$$a_3(n) = d_{1,6}(3n+1) + d_{2,6}(3n+1) - d_{4,6}(3n+1) - d_{5,6}(3n+1),$$

where $d_{r,6}(n)$ denotes the number of divisors of n which are congruent to r modulo 6.

From the above two formulas, we conclude that

$$d_{2,6}(3n+1) = d_{4,6}(3n+1).$$

Further, we obtained the following explicit formula for $a_3(n)$:

$$a_3(n) = \begin{cases} \prod\limits_{\alpha_i}(\alpha_i + 1), & \text{if } \beta_j\text{'s are even} \\ 0, & \text{otherwise,} \end{cases}$$

where $3n + 1 = \prod\limits_{p_i \equiv 1 \pmod 6} p_i^{\alpha_i} \cdot \prod\limits_{q_j \equiv 2 \pmod 6} q_j^{\beta_j}$ is the prime factorization of $3n + 1$.

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