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ORIGINAL RESEARCH

Monotonicity and concavity properties of the Gaussian hypergeometric functions, with applications

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Abstract This paper deals with the monotonicity and concavity properties of certain functions involving the Gaussian hypergeometric function. With these results, we not only obtain sharp bounds for the ratio of hypergeometric functions which extend recently discovered inequalities for k-balanced hypergeometric functions, and but also give an affirmative answer to an open problem proposed by Qiu and Vuorinen. In addition, as by-products, some monotonicity theorems for complete p-elliptic integrals and inequalities for generalized Grötzsch ring function are established.

Keywords Gaussian hypergeometric function · Complete p-elliptic integrals · Ratio function · Monotonicity · Concavity

Mathematics Subject Classification 33C05 · 33E05

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1 Introduction

Throughout this paper, let $\mathbb{N}$ ($\mathbb{R}$) be the set of positive numbers (real numbers, resp.), and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, for $x, y > 0$, we denote by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \gamma = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) \approx 0.5772$$

and

$$R(x, y) = -\psi(x) - \psi(y) - 2\gamma$$

the gamma function, the psi function, the beta function, the Euler-Mascheroni constant and the Ramanujan R-function, respectively. For $a \in \mathbb{R}$ with $a \neq 0$ and $n \in \mathbb{N}$, let

$$(a, 0) = 1, \quad (a, n) = a(a+1)(a+2)\cdots(a+n-1)$$

be the shifted factorial.

The classical Gaussian hypergeometric function (series) [1,13,38] is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^\infty \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}$$

with $a, b, c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$. In the particular case $c = a + b + k$ ($k \in \mathbb{N}_0$), the function $F(a, b; c; x)$ is called *k-balanced* [5,14]. It is clear to see that $F(a, b; c; x)$ converges for $x \in (-1, 1)$. When x tends to 1, it satisfies the following properties

(i) $c > a + b$ (cf. [38, p. 49]):

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}; \tag{1.1}$$

(ii) $c = a + b$ (cf. [1, 15.3.10]):

$$B(a, b)F(a, b; c; x) + \log(1-x) = R(a, b) + O[(1-x)\log(1-x)], \quad x \mapsto 1; \tag{1.2}$$

(iii) $c < a + b$ (cf. [28, (1.2)]):

$$F(a, b; c; x) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}(1+o(1)), \quad x \mapsto 1. \tag{1.3}$$

Let $r \in (0, 1)$, then the well-known Legendre's complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ of the first and second kinds are respectively defined by

$$\mathscr{K}(r) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-r^2t^2)}} dt = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2} d\theta,$$

and

$$\mathscr{E}(r) = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}} dt = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{1/2} d\theta.$$

They are the particular cases of the Gaussian hypergeometric functions, that is,

$$\mathscr{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad \mathscr{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right).$$

In the past years, the Gaussian hypergeometric function and the complete elliptic integrals have been research hotspots in special functions, and they have been widely applied in many mathematical branches including differential equation theory, number theory, complex analysis and hyperbolic geometry [1,8,13,21,27,29,40,45, 60].

Recently, the functions $F(a, b; c; x)$, $\mathscr{K}(r)$ and $\mathscr{E}(r)$ have frequently occurred in the theory of conformal and quasiconformal maps and Ramanujan modular equation [7–9,11,34,36,45,52]. For example, using the functions $\mathscr{K}(r)$ and $\mathscr{E}(r)$, the conformal modulus of the classical Grötzsch ring in the plane is $\mu(r) = \pi\mathscr{K}(\sqrt{1-r^2})/[2\mathscr{K}(r)]$, and the exterior modulus of a quadrilateral, which is over a Jordan domain containing the point at infinity and conformally equivalent to a quadrilateral over the outside of a rectangle, can be expressed by $R(r) = 2r[\mathscr{E}(\sqrt{1-r^2}) - r\mathscr{K}(\sqrt{1-r^2})]/\{(1-r^2)[\mathscr{E}(r) - (1-r)\mathscr{K}(r)]\}$ for certain parameter $r(0 < r < 1)$(See [6,19]). By employing $F(a, b; c; x)$, the Ramanujan's generalized modular equation with signature $1/a$ $(a \in (0, 1))$ and degree p $(p > 1)$ can be written as

$$\frac{F(a, 1-a; 1; 1-s^2)}{F(a, 1-a; 1; s^2)} = p\frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)}.$$

Because of their importance, numerous analytic properties and functional inequalities involving $F(a, b; c; x)$, $\mathscr{K}(r)$, $\mathscr{E}(r)$ and other related functions have been proved by mathematicians [2,3,5,10,12,15,26,30,32,35,46–49,51,53,54,56,62]. Furthermore, some generalized forms of $\mathscr{K}(r)$ and $\mathscr{E}(r)$ have been introduced and studied deeply in many fields of mathematics and physics [8,17,18,25,31,41,43,57,59].

In 2017, Alzer and Richards [4] found the best values $\alpha = 1/4$ and $\beta = 0$ such that inequality

$$\frac{1}{1+\alpha r} < \frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})} < \frac{1}{1+\beta r} \tag{1.4}$$

holds for all $r \in (0, 1)$, and then extended (1.4) to the case of Borweins' generalized elliptic integrals [18, p. 178]. Moreover, at the end of the paper [4], the authors pointed out that Ismail [4, p.1669] asked whether inequality (1.4) can be extended to the zero-balanced hypergeometric function.

Motivated by Ismail's suggestion, in 2019, Richards [39] proved the following Theorem 1.1, which is stronger than the corresponding generalization of (1.4).

Theorem 1.1 ([39, Theorem 3.1]) *Let $a, b > 0$. Then inequality*

$$\frac{1}{(1+r)^{\alpha(a,b)}} < \frac{F(a, b; a+b; r^2)}{F(a, b; a+b; r)} < \frac{1}{(1+r)^{\beta(a,b)}} \tag{1.5}$$

holds for all $r \in (0, 1)$ with the optimal exponents $\alpha(a, b) = ab/(a+b)$ and $\beta(a, b) = 0$.

Very recently, Barnard, Richards and Sliheet [14] extended Theorem 1.1 to the case of k-balanced hypergeometric functions further.

Theorem 1.2 ([14, Theorem 1]) *Let $p, q \in (1, +\infty)$, $a, b \in (-1, +\infty)$ and $k \in \mathbb{N}_0$ with $a+b+k > 0$. Then the following statements hold true:*

(1) *If $ab > 0$, then the double inequality*

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{\alpha(a,b,k)} < \frac{F(a, b; a+b+k; r^p)}{F(a, b; a+b+k; r^{p/q})} < \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\beta(a,b,k)} \tag{1.6}$$

holds for all $r \in (0, 1)$ with the optimal exponents $\alpha(a, b, k) = ab/(a+b+k)$ and $\beta(a, b, k) = 0$.

(2) *If $ab < 0$ and $k \geq 1$, then inequality*

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{\alpha(a,b,k)} > \frac{F(a, b; a+b+k; r^p)}{F(a, b; a+b+k; r^{p/q})} > \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\beta(a,b,k)} \tag{1.7}$$

is valid for all $r \in (0, 1)$ with the optimal exponents $\alpha(a, b, k) = ab/(a+b+k)$ and $\beta(a, b, k) = 0$.

It is not difficult to find that the monotonicity of the function $F(x) = (1-x)^d F(a, b; c; x)$ is the key ingredient in the proofs of inequalities (1.5)–(1.7). Therefore, one main purpose of this paper is to present monotonicity theorem of the function $F(x) = (1-x)^d F(a, b; c; x)$ for all $a, b, c \in (0, +\infty)$ and $d \in \mathbb{R}$. As an application, Theorems 1.1 and 1.2 will be extended to the ratio of general hypergeometric function $F(a, b; c; x)$ with $a, b, c > 0$.

It is apparent from (1.2) that the zero-balanced hypergeometric function has logarithmic singular value at $x = 1$, and therefore its asymptotic property is paid more attention to by researchers.

In 1999, Qiu and Vuorinen [36, Theorem 2.1] considered the ratio function $x \mapsto F(a, b; a + b; x)/\log[e^{R(a,b)}/(1-x)]$, and obtained the following

Theorem 1.3 *([36, Theorem 2.1]) Let $a, b \in (0, +\infty)$ with $R(a, b) \geq 0$. Then the function*

$$H(x) = \frac{F(a, b; a+b; x)}{\log[e^{R(a,b)}/(1-x)]} \tag{1.8}$$

is strictly decreasing from $(0, 1)$ onto $(1/B(a, b), 1/R(a, b))$. In particular, $\mathscr{K}(\sqrt{1-r^2})/\log(4/r)$ is strictly decreasing from $(0, 1)$ onto $(1, \pi/\log 16)$. Moreover, for $a, b \in (0, +\infty)$ with $R(a, b) \geq 0$, inequality

$$1 < \frac{B(a, b)F(a, b; a+b; x)}{\log\left[e^{R(a,b)}/(1-x)\right]} < \frac{B(a, b)}{R(a, b)} \tag{1.9}$$

holds for all $x \in (0, 1)$.

Later, Qiu and Vuorinen [35], and Wang, Chu and Song [48] studied the monotonicity property of the function

$$x \to \frac{B(a, b)F(a, b; a+b; x) + \log(1-x) - R(a, b)}{[(1-x)/x]\log[1/(1-x)]}, \quad x \in (0, 1)$$

for arbitrary $(a, b) \in \{(a, b)|a > 0, b > 0\}$ and obtained some refinements of inequality (1.9).

For the special case $F(a, 1-a; 1; x)$ with $a \in (0, 1)$, Yang and Tian [57] and Huang, Qiu and Ma [25] proved the monotonicity of the following function

$$x \to \frac{(p+x)F(a, 1-a; 1; x)}{\log\left[e^{R(a,1-a)}/(1-x)\right]}, \quad x \in (0, 1)$$

with any $p \in \mathbb{R}$. Consequently, a family of sharp inequalities for $F(a, 1-a; 1; x)$ was derived.

Very recently, Wang, Chu and Zhang [50] found the maximal subregions of $\{(a, b)|a > 0, b > 0\}$ such that the function

$$x \to \frac{B(a, b)F(a, b; a+b; x) - \log[e^{R(a,b)}/(1-x)]}{(1-x)\log[e^{R(a,b)}/(1-x)]}, \quad x \in (0, 1)$$

is monotone on the whole interval $(0, 1)$, and got several new estimates for $F(a, b; a + b; x)$, in which the bounding functions are in the form of $[1 + \lambda(1-x)]\log[e^{R(a,b)}/(1-x)]/B(a, b)$, and λ is a constant depending on a and b.

In this paper, we will still focus on the function $H(x)$ defined in (1.8), and continue to study its convexity and concavity properties for each $(a, b) \in \{(a, b)|a > 0, b > 0\}$. Besides, we shall also investigate the monotonicity of the ratio of $F(a, b; c; x)(a, b, c > 0)$ to $\log[d/(1-x)](d > 1)$ on $(0, 1)$. In fact, another purpose of this paper is to answer to the following Question 1.4, which is naturally raised from Theorem 1.3.

Question 1.4 (1) *For any fixed $(a, b) \in \{(a, b)|a, b > 0, R(a, b) \geq 0\}$, is the function $H(x)$ concave or convex on $(0, 1)$?*

(2) *What about the monotonicity of the ratio function $G(x) = F(a, b; c; x)/\log[d/(1-x)]$ with $a, b, c > 0$ and $d \in [1, +\infty)$?*

It is worthy mentioning that there was an open problem concerning the convexity and concavity properties of $H(x)$ on $(0, 1)$, which is proposed by Qiu and Vuorinen in [36].

Problem 1.5 [36, Open problem 4] or, [25, Remark 5.4(2)] *Theorem 1.4 in* [33] *shows that the function* $r \mapsto \mathscr{K}(r)/\log(4/\sqrt{1-r^2})$ *not only decreases, but also has concave property on the whole interval* $(0, 1)$*, and while* $r \mapsto \mathscr{K}(\sqrt{1-r^2})/\log(4/r)$ *is strictly convex on* $(0, 1)$*. It is natural to ask whether there are the analogues of the concavity of* $r \mapsto \mathscr{K}(r)/\log(4/\sqrt{1-r^2})$ *and the convexity of* $r \mapsto \mathscr{K}(\sqrt{1-r^2})/\log(4/r)$ *for* $H(r^2)$ *and* $H(1-r^2)$*, respectively.*

With the complete answer to Question 1.4 (See Theorem 2.4 in Sect. 2), we shall also solve the above Problem 1.5 partly.

This paper is organized as follows. The main results of this paper are listed in Sect. 2, and their complete proofs will be finished in Sect. 3. Applying the monotonicity property of $F(x)$ and its related lemma, several monotonicity theorems for Takeuchi's complete p-elliptic integrals will be obtained in Sect. 4. Additionally, in the final section, we shall derive new sharp upper and lower bounds for generalized Grötzsch ring function with two parameters.

The following two monotonicity criterions will be used in our proofs. For more their applications, we refer to the readers to the literatures [3,34,55,61].

Lemma 1.6 [11, Theorem 1.25] *Let* $-\infty < a < b < \infty$*,* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and be differentiable on* (a, b) *such that* $g'(x) \neq 0$ *on* (a, b)*. If* $f'(x)/g'(x)$ *is strictly increasing (decreasing) on* (a, b)*, then both the functions* $[f(x) - f(a)]/[g(x) - g(a)]$ *and* $[f(x) - f(b)]/[g(x) - g(b)]$ *are strictly increasing (decreasing resp.) on* (a, b)

Lemma 1.7 [55, Theorem 2.1] *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *with* $b_n > 0$ *for all* $n \in \mathbb{N}_0$*. Let* $h(x) = f(x)/g(x)$ *and* $H_{f,g}(x) = [f'(x)/g'(x)]g(x) - f(x)$*, then the following statements are true:*

(1) *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing (decreasing), then* $h(x)$ *is strictly increasing (decreasing resp.) on* $(0, r)$*;*
(2) *If the non-constant sequence* $\{a_n/b_n\}$ *is increasing (decreasing) for* $0 < n \le n_0$ *and decreasing (increasing) for* $n > n_0$*, then the function* h *is strictly increasing (decreasing) on* $(0, r)$ *if and only if* $H_{f,g}(r^-) \ge (\le) 0$*. While if* $H_{f,g}(r^-) < (>) 0$*, then there exists* $\xi \in (0, r)$ *such that* $h(x)$ *is strictly increasing (decreasing resp.) on* $(0, \xi)$ *and strictly decreasing (increasing resp.) on* (ξ, r)*.*

2 Main results

Theorem 2.1 *Let* $a, b, c \in (0, +\infty)$*,* $d \in \mathbb{R}$*, and*

$$F(x) = (1-x)^d F(a, b; c; x), \ \ x \in (0, 1). \tag{2.1}$$

Then F *has the following monotonicity properties:*

(1) *If* $a+b \le c$*, then* $F(x)$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, 1)$ *if and only if* $d \ge (ab)/c$*;* $F(x)$ *is strictly increasing on* $(0, 1)$ *if and only if* $d \le 0$ *(Exactly, if* $d < 0$ *or* $d = 0$ *&* $a + b = c$*, then* $F(x) \in (1, +\infty)$*; if* $d = 0$ *&* $a+b < c$*, then* $F(x) \in (1, \Gamma(c)\Gamma(c-a-b)/[\Gamma(c-a)\Gamma(c-b)])$*); Moreover, if* $0 < d < (ab)/c$*, then there exists a unique* $x_1 \in (0, 1)$*, such that* $F(x)$ *is strictly increasing on* $(0, x_1)$ *and strictly decreasing on* $(x_1, 1)$*;*
(2) *If* $a + b > c$ *and either* $c < \min\{a, b\}$ *or* $c > \max\{a, b\}$*, then* $F(x)$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, 1)$ *if and only if* $d \ge (ab)/c$*;* $F(x)$ *is strictly increasing on* $(0, 1)$ *if and only if* $d \le a + b - c$ *(Exactly, if* $d = a + b - c$*, then* $F(x) \in (1, \Gamma(c)\Gamma(a + b - c)/[\Gamma(a)\Gamma(b)])$*; if* $d < a + b - c$*, then* $F(x) \in (1, +\infty)$*); Moreover, if* $a + b - c < d < (ab)/c$*, then there exists a unique* $x_2 \in (0, 1)$*, such that* $F(x)$ *is strictly increasing on* $(0, x_2)$ *and strictly decreasing on* $(x_2, 1)$*;*
(3) *If* $a + b > c$ *and* $\min\{a, b\} < c < \max\{a, b\}$*, then* $F(x)$ *is strictly decreasing on* $(0, 1)$ *if and only if* $d \ge a+b-c$ *(Exactly, if* $d = a+b-c$*, then* $F(x) \in (\Gamma(c)\Gamma(a+b-c)/[\Gamma(a)\Gamma(b)], 1)$*; if* $d > a+b-c$*, then* $F(x) \in (0, 1)$ *);* $F(x)$ *is strictly increasing from* $(0, 1)$ *onto* $(1, +\infty)$ *if and only if* $d \le (ab)/c$*; Moreover, if* $(ab)/c < d < a + b - c$*, then there exists a unique* $x_3 \in (0, 1)$*, such that* $F(x)$ *is strictly decreasing on* $(0, x_3)$ *and strictly increasing on* $(x_3, 1)$*;*
(4) *In the remaining cases, that is,* $a = c$ *(or* $b = c$*), so that* $F(x) = (1-x)^{d-b}$*(*$F(x) = (1-x)^{d-a}$ *resp.), so that* $F(x)$ *is strictly increasing on* $(0, 1)$ *if and only if* $d < b$ *(*$d < a$ *resp.), and strictly decreasing on* $(0, 1)$ *if and only if* $d > b$ *(*$d > a$ *resp.).*

Corollary 2.2 *Suppose $a, b, c \in (0, +\infty)$, $p, q \in (1, +\infty)$. Then the following statements hold*

(1) *If $a + b \leq c$, then inequality*

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{\alpha(a,b,c)} < \frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})} < \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\beta(a,b,c)} \tag{2.2}$$

holds for all $r \in (0, 1)$ with the optimal exponents $\alpha(a, b, c) = ab/c$ and $\beta(a, b, c) = 0$.

(2) *If $a + b > c$ and either $c < \min\{a, b\}$ or $c > \max\{a, b\}$, then inequality*

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{\alpha(a,b,c)} < \frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})} < \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\beta(a,b,c)} \tag{2.3}$$

holds for all $r \in (0, 1)$ with the optimal exponents $\alpha(a, b, c) = ab/c$ and $\beta(a, b, c) = a + b - c$.

(3) *If $a + b > c$ and $\min\{a, b\} < c < \max\{a, b\}$, then inequality*

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{\alpha(a,b,c)} < \frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})} < \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\beta(a,b,c)} \tag{2.4}$$

holds for all $r \in (0, 1)$ with the optimal exponents $\lambda(a, b, c) = a + b - c$ and $\beta(a, b, c) = ab/c$.

(4) *In the remaining cases, that is, $a = c$ or $b = c$, then*

$$\frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})} = \left(\frac{1-r^{p/q}}{1-r^p}\right)^{a+b-c}. \tag{2.5}$$

Theorem 2.3 *Let $a, b, c \in (0, +\infty)$, $d \in [1, +\infty)$, and*

$$G(x) = \frac{F(a,b;c;x)}{\log[d/(1-x)]}. \tag{2.6}$$

Then G has the following monotonicity properties:

(1) *If $c - a - b = 0$ with $R(a, b) \geq 0$, then $G(x)$ is strictly decreasing on $(0, 1)$ if and only if $1 \leq d \leq e^{R(a,b)}$, strictly increasing on $(0, 1)$ if and only if $d \geq e^{\frac{c}{ab}}$, and there exists a unique $x_0^* \in (0, 1)$ such that $G(x)$ is strictly decreasing on $(0, x_0^*)$, and strictly increasing on $(x_0^*, 1)$ if $e^{R(a,b)} < d < e^{\frac{c}{ab}}$;*

(2) *If $c-a-b < 0$ or if $c-a-b = 0$ with $R(a, b) < 0$, then $G(x)$ is strictly increasing on $(0, 1)$ if and only if $d \geq e^{\frac{c}{ab}}$, and there exists $x_1^* \in (0, 1)$ such that $G(x)$ is strictly decreasing on $(0, x_1^*)$, and strictly increasing on $(x_1^*, 1)$ if $1 \leq d < e^{\frac{c}{ab}}$;*

(3) *If $c - a - b \geq ab$, then $G(x)$ is strictly decreasing on $(0, 1)$ if and only if $1 \leq d \leq e^{\frac{c}{ab}}$, and there exists $x_2^* \in (0, 1)$ such that $G(x)$ is strictly increasing on $(0, x_2^*)$, and strictly decreasing on $(x_2^*, 1)$ if $d > e^{\frac{c}{ab}}$;*

(4) *In the remaining case $0 < c-a-b < ab$, $G(x)$ is strictly decreasing on $(0, 1)$ if and only if $1 \leq d \leq \phi(\zeta)$, and $G(x)$ is not monotone on $(0, 1)$ if $d > \phi(\zeta)$ provided that $\phi(\zeta) > 1$. Here $\phi(\zeta)$ is defined in Lemma 3.5. (Otherwise if $\phi(\zeta) < 1$, then $G(x)$ is not monotone on $(0, 1)$.)*

Furthermore,

$$G(0^+) = \begin{cases} 1/\log d, & d > 1, \\ +\infty, & d = 1, \end{cases} \quad G(1^-) = \begin{cases} +\infty, & c-a-b < 0, \\ 1/B(a,b), & c-a-b = 0, \\ 0, & c-a-b > 0. \end{cases}$$

Theorem 2.4 *For $(a, b) \in D = \{(a, b)|a, b > 0, R(a, b) \geq 0\}$, the function*

$$H(x) = \frac{F(a,b;a+b;x)}{\log[e^{R(a,b)}/(1-x)]}$$

is convex on $(0, 1)$ if and only if $(a, b) = (1, 1)$, and is concave on $(0, 1)$ if and only if $(a, b) \in D_1 \cup D_2$. Here

$$D_1 = \{(a, b)|a, b > 0, R(a + 2, b + 2) + 1/a + 1/b - 2 \geq 0\},$$
$$D_2 = \{(a, b)|a, b > 0, R(a, b) \geq 0, R(a + 2, b + 2) + 1/a + 1/b - 2 < 0, M(a, b) \leq 0\},$$

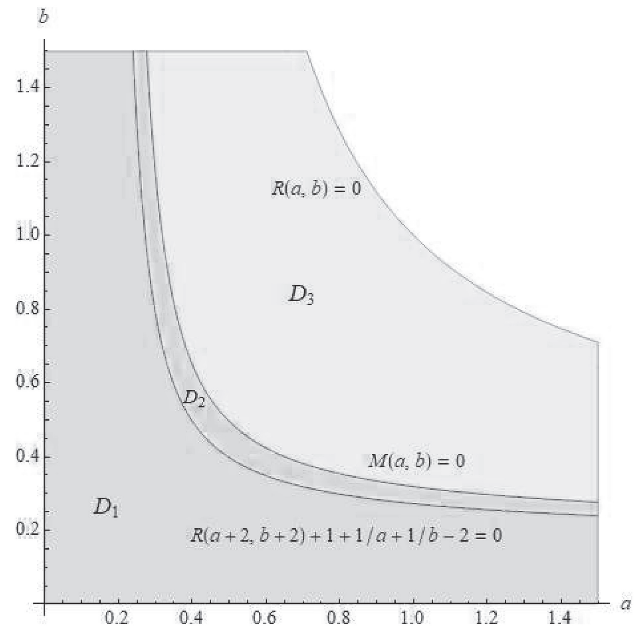

Fig. 1 The regions D_1, D_2, D_3

and

$$M(a,b)=\frac{a+b+1}{a^2b^2}[abR(a,b)-a-b][R(a,b)-2]+R(a,b)^2.$$

Furthermore, if $(a,b)\in D_3=D\setminus\{D_1\cup D_2\cup\{(a,b)|a=b=1\}\}=\{(a,b)|a,b>0,\ R(a,b)\geq 0,\ R(a+2,b+2)+1/a+1/b-2<0,\ M(a,b)>0\}\setminus\{(1,1)\}$*, the function* $H(x)$ *is neither convex nor concave on the whole interval* $(0,1)$*. In particular, for* $(a,b)\in D_1\cup D_2$*, inequality*

$$\frac{1}{R(a,b)}-\left[\frac{1}{R(a,b)}-\frac{1}{B(a,b)}\right]x<\frac{F(a,b;a+b;x)}{\log[e^{R(a,b)}/(1-x)]}<\frac{1}{R(a,b)} \tag{2.7}$$

holds for all $x\in(0,1)$.

Remark 2.5 Since the function $a\mapsto R(a+2,b+2)+1/a+1/b-2$ is strictly decreasing on $(0,+\infty)$ and $R(a+2,b+2)+1/a+1/b-2=R(a,b)-2-(a+b+2)/[(a+1)(b+1)]$, we obtain that $D_1\subset D$. Also it is easy to check that $R(1,1)=0$, $M(1,1)=12$ and $D_1\cup D_2\cup D_3\cup\{(a,b)|a=b=1\}=D$ (cf. Figure 1).

The following corollary is derived immediately from Theorem 2.4 and Lemma 1.30 in [11], which partly solves Problem 1.5.

Corollary 2.6 *If* $(a,b)\in D_1\cup D_2$*, then the function*

$$H(r^2)=\frac{F(a,b;a+b;r^2)}{\log[e^{R(a,b)}/(1-r^2)]}$$

is strictly concave on $(0,1)$.

3 Proofs of main results

In this section, we will prove our main results stated in Sect. 2.

Lemma 3.1 *Let* $x\in(0,1)$*,* $a,b,c\in(0,+\infty)$ *and the function*

$$\varphi(x)=\frac{ab(1-x)F(a+1,b+1;c+1;x)}{cF(a,b;c;x)}. \tag{3.1}$$

Then we have

(1) *If* $a+b\le c$*, then* $\varphi(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,ab/c)$*;*
(2) *If* $a+b>c$ *and either* $c<\min\{a,b\}$ *or* $c>\max\{a,b\}$*, then* $\varphi(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(a+b-c,ab/c)$*;*
(3) *If* $a+b>c$ *and* $\min\{a,b\}<c<\max\{a,b\}$*, then* $\varphi(x)$ *is strictly increasing from* $(0,1)$ *onto* $(ab/c,a+b-c)$*;*
(4) *In the remaining cases, that is,* $a=c$ *(or* $b=c$*), then* $\varphi(x)\equiv b$ *(*$\varphi(x)\equiv a$ *resp.).*

Proof Clearly $\varphi(0^+)=ab/c$. Provided that $a+b\le c-1$, $c-1<a+b<c$, $a+b=c$, and $a+b>c$ respectively, from (1.1)–(1.3) and (3.1) one has

$$\lim_{x\to1^-}\varphi(x)=\lim_{x\to1^-}\frac{\frac{ab}{c}(1-x)F(a+1,b+1;c+1;x)}{F(a,b;c;x)}=\lim_{x\to1^-}\frac{ab}{c}(1-x)\frac{\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}}{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}}$$
$$=\lim_{x\to1^-}\frac{ab}{c}(1-x)\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c)\Gamma(c-a-b)}=0, \tag{3.2}$$
$$\lim_{x\to1^-}\varphi(x)=\lim_{x\to1^-}\frac{\frac{ab}{c}(1-x)^{c-b-a}F(c-a,c-b;c+1;x)}{F(a,b;c;x)}=0, \tag{3.3}$$
$$\lim_{x\to1^-}\varphi(x)=\lim_{x\to1^-}\frac{\frac{ab}{c}(1-x)^{c-b-a}F(c-a,c-b;c+1;x)}{F(a,b;c;x)}$$
$$=\lim_{x\to1^-}\frac{\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(a+b-c+1)}{\Gamma(a+1)\Gamma(b+1)}}{B(a,b)\log[1/(1-x)]}=0 \tag{3.4}$$

and

$$\lim_{x\to1^-}\varphi(x)=\lim_{x\to1^-}\frac{\frac{ab}{c}F(c-a,c-b;c+1;x)}{F(c-a,c-b;c;x)}$$
$$=\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(a+b-c+1)}{\Gamma(a+1)\Gamma(b+1)}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)}=a+b-c. \tag{3.5}$$

Employing the series expansion of hypergeometric function, we get

$$\varphi(x)=\frac{(1-x)\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+1)}\frac{x^n}{n!}}{\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}}$$
$$=\frac{\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+1)}\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+1)}\frac{x^{n+1}}{n!}}{\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}}$$
$$=\frac{\frac{ab}{c}+\sum_{n=1}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+1)}\frac{x^n}{n!}-\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{(n-1)!}}{\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}}$$
$$=\frac{\frac{ab}{c}+\sum_{n=1}^{\infty}\frac{(a,n)(b,n)[(a+b-c)n+ab]}{(c,n+1)}\frac{x^n}{n!}}{\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}}=:\frac{\sum_{n=0}^{\infty}A_n^*x^n}{\sum_{n=0}^{\infty}B_n^*x^n}, \tag{3.6}$$

where

$$A_n^*=\frac{(a,n)(b,n)}{(c,n+1)}\frac{(a+b-c)n+ab}{n!},\quad B_n^*=\frac{(a,n)(b,n)}{(c,n)}\frac{1}{n!}$$

for $n\ge0$.

It is easy to check that $B_n^* \geq 0$ for $n \geq 0$, and

$$\frac{A_n^*}{B_n^*} = a + b - c + \frac{(a-c)(b-c)}{n+c}. \tag{3.7}$$

(1) If $a + b \leq c$, then $a < c$ and $b < c$, and therefore the sequence $\{A_n^*/B_n^*\}_{n\geq 1}$ is strictly decreasing on n by (3.7). Applying Lemma 1.7(1), $\varphi(x)$ is strictly decreasing on $(0, 1)$. According to (3.2)–(3.4), the range of φ follows.

(2) If $a + b > c$ and either $c < \min\{a, b\}$ or $c > \max\{a, b\}$, then the sequence $\{A_n^*/B_n^*\}_{n\geq 1}$ is also strictly decreasing on n by (3.7), and therefore $\varphi(x)$ is strictly decreasing on $(0, 1)$. By (3.5), the range of φ in this case is $(a + b - c, ab/c)$.

(3) If $a + b > c$ and $\min\{a, b\} < c < \max\{a, b\}$, then the sequence $\{A_n^*/B_n^*\}_{n\geq 1}$ is strictly increasing on n by (3.7). This, together with Lemma 1.7(1) and (3.5), leads to the conclusion that $\varphi(x)$ is strictly increasing from $(0, 1)$ onto $(ab/c, a + b - c)$.

(4) If $a = c$ $(b = c)$ then $\{A_n^*/B_n^*\} = b$ ($\{A_n^*/B_n^*\} = a$ resp.). Thus from (3.6) we clearly see that $\varphi(x) \equiv b$ ($\varphi(x) \equiv a$ resp.). □

Remark 3.2 Lemma 3.1(1) has been already proved by Küstner in [22, Theroem 1.5], in which the author showed that the logarithmic derivative of $F(a, b; c; x)$ is a Markov function. The integral representations of some ratios of the Gaussian hypergeometric functions, such as (3.1), have been given in [20].

Proof of Theorem 2.1 If $a = c$ $(b = c)$ then $F(x)$ reduces to $(1-x)^{d-b}$ ($(1-x)^{d-a}$ resp.), which yields that part (4) is valid. Next, we assume that $a \neq c$ and $b \neq c$. It is obvious to see that $F(0^+) = 1$. Differentiating F gives

$$\begin{aligned} F'(x) =&(1-x)^{d-1} F(a, b; c; x) \left[\frac{\frac{ab}{c}(1-x)F(a+1, b+1; c+1; x)}{F(a, b; c; x)} - d \right] \\ =&(1-x)^{d-1} F(a, b; c; x) \left[\varphi(x) - d\right], \end{aligned}$$

where $\varphi(x)$ is defined in (3.1).

Therefore, the monotonicity properties of $F(x)$ directly follow from Lemma 3.1(1)-(3). For the limiting value of $F(x)$ at 1, we investigate the following two cases.

Case 1 $a + b \leq c$. Then

$$F(1^-) = \lim_{x\to 1^-} (1-x)^d F(a, b; c; x) = \begin{cases} 0, & d > 0, \\ +\infty, & d < 0 \ or\ d = 0 \ \&\ a + b = c, \\ \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, & d = 0 \ \&\ a + b < c. \end{cases}$$

Case 2 $a + b > c$. Then

$$F(1^-) = \lim_{x\to 1^-} (1-x)^{d+c-a-b} F(c-a, c-b; c; x) = \begin{cases} \dfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, & d = a + b - c, \\ +\infty, & d < a + b - c, \\ 0, & d > a + b - c. \end{cases}$$

This completes the proof. □

Proof of Corollary 2.2 We only prove the assertion of part (1). The remaining proofs of parts (2)–(4) are similar.

For part (1), if $a+b \leq c$, then Theorem 2.1(1) shows that $F(x) = (1-x)^d F(a, b; c; x)$ is strictly decreasing (increasing) on $(0, 1)$ if and only if $d \geq ab/c$ ($d \leq 0$ resp.). Thus inequalities

$$(1-r^p)^{ab/c} F(a, b; c; r^p) > (1-r^{p/q})^{ab/c} F(a, b; c; r^{p/q})$$

and

$$(1-r^p)^0 F(a, b; c; r^p) < (1-r^{p/q})^0 F(a, b; c; r^{p/q}),$$

that is

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{ab/c} < \frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})}, \quad \frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})} < \left(\frac{1-r^{p/q}}{1-r^p}\right)^0$$

take place for all $r \in (0,1)$ since $r^p < r^{p/q}$ for $p, q \in (1, +\infty)$.

Combining with the fact that the function

$$\lambda \mapsto \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\lambda}, \quad \lambda \in \mathbb{R}$$

is strictly decreasing for fixed $r \in (0,1)$, $p, q \in (1, +\infty)$, we obtain that the double inequality (2.2) holds for all $r \in (0,1)$ if $\alpha(a,b,c) \geq ab/c$ and $\beta(a,b,c) \leq 0$.

In what follows we shall verify the sharpness of the parameters $\alpha(a,b,c) = ab/c$ and $\beta(a,b,c) = 0$. In fact, if $d \in (0, ab/c)$, then it also follows from Theorem 2.1(1) that $F(x) = (1-x)^d F(a,b;c;x)$ is piecewise monotone, and has maximal value at $x = x_1$. Setting $r_1^* = x_1^{1/p}$ and $r_1^{**} = x_1^{q/p}$ for fixed $p, q \in (1, +\infty)$, we have $r_1^*, r_1^{**} \in (0,1)$, and

$$F(x_1) = (1-r_1^{*p})^d F(a,b;c;r_1^{*p}) > (1-r_1^{*p/q})^d F(a,b;c;r_1^{*p/q}),$$
$$F(x_1) = (1-r_1^{**p/q})^d F(a,b;c;r_1^{**p/q}) > (1-r_1^{**p})^d F(a,b;c;r_1^{**p}).$$

Equivalently,

$$\frac{F(a,b;c;r_1^{*p})}{F(a,b;c;r_1^{*p/q})} > \left(\frac{1-r_1^{*p/q}}{1-r_1^{*p}}\right)^d,$$
$$\frac{F(a,b;c;r_1^{**p})}{F(a,b;c;r_1^{**p/q})} < \left(\frac{1-r_1^{**p/q}}{1-r_1^{**p}}\right)^d.$$

The above inequalities imply that both ab/c and 0 are optimal. The proof is complete. □

Remark 3.3 Suppose $a, b \in (-1, 0)$, $c \in (0, +\infty)$ and $p, q \in (1, +\infty)$ in Corollary 2.2. Then it follows from (3.7) together with the proof of Lemma 3.1 that $\{A_n^*/B_n^*\}_{n\geq 1}$ is also strictly decreasing on n, so that $\varphi(x)$, defined by (3.1), is also strictly decreasing from $(0,1)$ onto $(0, ab/c)$ by application of Lemma 1.7(1). Combining with the proof of Theorem 2.1, we obtain that the function $F(x) = (1-x)^d F(a,b;c;x) (d \in \mathbb{R})$ is strictly decreasing on $(0,1)$ if and only if $d \geq ab/c$, is strictly increasing on $(0,1)$ if and only if $d \leq 0$. Therefore, it is easy to show that, for $a, b \in (-1, 0)$, $c \in (0, +\infty)$ and $p, q \in (1, +\infty)$, inequality

$$\left(\frac{1-r^{p/q}}{1-r^p}\right)^{\alpha(a,b,c)} < \frac{F(a,b;c;r^p)}{F(a,b;c;r^{p/q})} < \left(\frac{1-r^{p/q}}{1-r^p}\right)^{\beta(a,b,c)}$$

also holds for all $r \in (0,1)$ with optimal exponents $\alpha(a,b,c) = ab/c$ and $\beta(a,b,c) = 0$. Until now, Theorem 1.2(1) has been extended completely.

Corollary 3.4 *For* $a, b, c \in (0, +\infty)$*, the function*

$$\hat{F}(x) = (1-x)F(a+1, b+1; c+1; x) \tag{3.8}$$

is strictly decreasing on $(0,1)$ *if and only if* $c - a - b \geq ab$*, is strictly increasing on* $(0,1)$ *if and only if* $c - a - b \leq 0$*. Moreover, if* $0 < c - a - b < ab$*, then there exists* $x_0 \in (0,1)$ *such that* $\hat{F}(x)$ *is strictly increasing on* $(0, x_0)$ *and strictly decreasing on* $(x_0, 1)$.

Proof If $a = c$ $(b = c)$, then $\hat{F}(x)$ becomes $(1-x)^{-b}$ ($(1-x)^{-a}$ resp.), $\hat{F}(x)$ is strictly increasing on $(0,1)$ since $a, b > 0$.

Under the assumption of $a \neq c$ and $b \neq c$, we divide the proof into two cases.

Case 1 $a+1+b+1 \leq c+1$, namely $c - a - b \geq 1$. Then from Theorem 2.1(1) one has that $\hat{F}(x)$ is strictly decreasing on $(0,1)$ if and only if $1 \geq (a+1)(b+1)/(c+1)$, or, equivalently, $c - a - b \geq ab$. Moreover, if $c-a-b<ab$, then there exists $x_0^*\in(0,1)$ such that $\hat{F}(x)$ is strictly increasing on $(0,x_0^*)$, and strictly decreasing on $(x_0^*,1)$.

Case 2 $a+1+b+1>c+1$, namely $c-a-b<1$. Then it follows from Theorem 2.1(2)-(3) that $\hat{F}(x)$ is strictly decreasing on $(0,1)$ if and only if $1\geq\max\{(a+1)(b+1)/(c+1),a+b+1-c\}$(or, equivalently, $c-a-b\geq ab$), and strictly increasing if and only if $1\leq\min\{(a+1)(b+1)/(c+1),a+b+1-c\}$(or $c-a-b\leq 0$). Moreover, if $0<c-a-b<ab$ (or $a+b+1-c<1<(a+1)(b+1)/(c+1)$), then there exists $x_0\in(0,1)$ such that $\hat{F}(x)$ is strictly increasing on $(0,x_0)$, and strictly decreasing on $(x_0,1)$.

With the above arguments, it is easy to obtain that $\hat{F}(x)$ is strictly decreasing on $(0,1)$ if and only if $c-a-b\geq ab$, and strictly increasing if and only if $c-a-b\leq 0$. While if $0<c-a-b<ab$, then $\hat{F}(x)$ is piecewise monotone, first increasing then decreasing. □

Lemma 3.5 *Let $a,b,c\in(0,+\infty)$. Define ϕ on $(0,1)$ by*

$$\phi(x)=(1-x)\exp\left\{\frac{cF(a,b;c;x)}{ab(1-x)F(a+1,b+1;c+1;x)}\right\}. \tag{3.9}$$

Then we have

(1) *If $c-a-b\leq 0$, then $\phi(x)$ is strictly decreasing on $(0,1)$. The range of ϕ is $(0,e^{\frac{c}{ab}})$ when $c-a-b<0$, while the range is $(e^{R(a,b)},e^{\frac{c}{ab}})$ when $c-a-b=0$;*
(2) *If $c-a-b\geq ab$, then $\phi(x)$ is strictly increasing from $(0,1)$ onto $(e^{\frac{c}{ab}},+\infty)$;*
(3) *In the remaining case, that is, $0<c-a-b<ab$, then there exists $\zeta\in(0,1)$ such that ϕ is strictly decreasing on $(0,\zeta)$, and strictly increasing on $(\zeta,1)$, in which case the range of ϕ is $(\phi(\zeta),+\infty)$.*

Proof Logarithmic differentiation gives

$$\begin{aligned}\frac{\phi'(x)}{\phi(x)}=&\frac{c}{ab}\frac{(ab/c)F(a+1,b+1;c+1;x)(1-x)F(a+1,b+1;c+1;x)}{(1-x)^2[F(a+1,b+1;c+1;x)]^2}\\&-\frac{c}{ab}\frac{F(a,b;c;x)\frac{d[(1-x)F(a+1,b+1;c+1;x)]}{dx}}{(1-x)^2[F(a+1,b+1;c+1;x)]^2}-\frac{1}{1-x}\\=&-\frac{c}{ab}\frac{F(a,b;c;x)}{(1-x)^2[F(a+1,b+1;c+1;x)]^2}\frac{d[(1-x)F(a+1,b+1;c+1;x)]}{dx}.\end{aligned}$$

Corollary 3.4 implies that $\phi'(x)<0$ for all $x\in(0,1)$ if and only if $c-a-b\leq 0$, and $\phi'(x)>0$ for all $x\in(0,1)$ if and only if $c-a-b\geq ab$. Furthermore, if $0<c-a-b<ab$, then there exists $\zeta\in(0,1)$ such that $\phi'(x)<0$ for $x\in(0,\zeta)$, and $\phi'(x)>0$ for $x\in(\zeta,1)$. Thus the monotonicity properties in parts (1)–(3) follows.

Clearly $\phi(0^+)=e^{\frac{c}{ab}}$, and using (1.1)–(1.3) and (3.2)–(3.5) we have

$$\begin{aligned}&\lim_{x\to1^-}\phi(x)=\lim_{x\to1^-}(1-x)\lim_{x\to1^-}\exp\left[\frac{cF(a,b;c;x)}{ab(1-x)F(a+1,b+1;c+1;x)}\right]=0\cdot e^{1/(a+b-c)}=0,\\&\lim_{x\to1^-}\phi(x)=\lim_{x\to1^-}\exp\left[\frac{cF(a,b;c;x)}{ab(1-x)F(a+1,b+1;c+1;x)}+\log(1-x)\right]\\&\quad=\lim_{x\to1^-}\exp\left[\frac{(a+b)F(a,b;a+b;x)}{abF(a,b;a+b+1;x)}+\log(1-x)\right]\\&\quad=\lim_{x\to1^-}\exp\{B(a,b)F(a,b;a+b;x)+\log(1-x)+o(1)\}=e^{R(a,b)},\\&\lim_{x\to1^-}\phi(x)=+\infty,\end{aligned}$$

provided that $a+b>c$, $a+b=c$, and $a+b<c$, respectively. □

Proof of Theorem 2.3 By (2.6) one has $G(0^+)=1/\log d$ for $d>1$, and $G(0^+)=+\infty$ for $d=1$. Simple computations yield

$$G(1^-)=\begin{cases}+\infty, & c-a-b<0,\\ \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}, & c-a-b=0,\\ 0, & c-a-b>0.\end{cases}$$

By differentiation, one has

$$G'(x) = \frac{\frac{ab}{c}F(a+1,b+1;c+1;x)\log\left(\frac{d}{1-x}\right) - F(a,b;c;x)\frac{1}{1-x}}{\left[\log\left(\frac{d}{1-x}\right)\right]^2}$$
$$= \frac{abF(a+1,b+1;c+1;x)}{c\left[\log\left(\frac{d}{1-x}\right)\right]^2}\left[\log d - \log\phi(x)\right]. \tag{3.10}$$

where $\phi(x)$ is defined in (3.9).

If $c-a-b<0$, or if $c-a-b=0$ with $R(a,b)<0$, then it follows from (3.10) together with Lemma 3.5(1) that $G'(x)$ is positive on $(0,1)$, namely $G(x)$ is strictly increasing if and only if $d \geq \sup\{\phi(x), x \in (0,1)\} = e^{\frac{c}{ab}}$, and and there exists $x_1^* \in (0,1)$ such that $G(x)$ is strictly decreasing on $(0, x_1^*)$, and strictly increasing on $(x_1^*, 1)$ if $1 \leq d < e^{\frac{c}{ab}}$. Thus part (2) holds.

If $0 < c-a-b < ab$, then from (3.10) and Lemma 3.5(3) we conclude that $G(x)$ is strictly decreasing on $(0,1)$ if and only if $1 \leq d \leq \phi(\zeta)$, and $G(x)$ is not monotone on $(0,1)$ if $d > \phi(\zeta)$, provided that $\inf\{\phi(x), x \in (0,1)\} = \phi(\zeta) \geq 1$. Otherwise, if $\phi(\zeta) < 1$, then it follows that the function $G(x)$ is not monotone on $(0,1)$ for any $d \geq 1$. Therefore, part (4) is true.

For parts (1) and (3), it is obvious that $G'(x)$ is negative or positive according to

$$d \leq \inf\{\phi(x), x \in (0,1)\} = \begin{cases} e^{R(a,b)}, & c-a-b=0 \;\&\; R(a,b) \geq 0, \\ e^{\frac{c}{ab}}, & c-a-b \geq ab \end{cases}$$

and

$$d \geq \sup\{\phi(x), x \in (0,1)\} = \begin{cases} e^{\frac{c}{ab}}, & c-a-b=0 \;\&\; R(a,b) \geq 0, \\ +\infty, & c-a-b \geq ab. \end{cases}$$

The proof is complete. □

Lemma 3.6 [50, Lemma 3.5] *If $a, b > 0$ with $1/a + 1/b \leq 2$, then $R(a,b) \leq 0$, with equality if and only if $a = b = 1$.*

Lemma 3.7 *Suppose $a, b > 0$ with $R(a,b) > 0$. Define f and g on $(0,1)$ by*

$$f(x) = R(a,b) - 2 + \log\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} A_n x^n, \tag{3.11}$$

$$g(x) = F(a+1, b+1; a+b+2; x) = \sum_{n=0}^{\infty} B_n x^n. \tag{3.12}$$

Then the following statements hold

(1) *The non-constant sequence $\{A_n/B_n\}$ is strictly decreasing on $n \in \mathbb{N}$, where*

$$A_n = \begin{cases} R(a,b)-2, & n=0, \\ \frac{1}{n}, & n \geq 1, \end{cases} \qquad B_n = \frac{(a+1,n)(b+1,n)}{(a+b+2,n)n!}. \tag{3.13}$$

(2) *The limiting value of the auxiliary function $H_{f,g}(x) = [f'(x)/g'(x)]/g(x) - f(x)$ at $x = 1$ is*

$$H_{f,g}(1^-) = \frac{2ab-a-b}{ab}. \tag{3.14}$$

Proof Due to equations (3.11) and (3.12), we get (3.13) immediately. For $n \geq 1$, if we let

$$C_n = \frac{B_n}{A_n} = \frac{(a+1,n)(b+1,n)}{(a+b+2,n)(n-1)!},$$

then

$$\frac{C_{n+1}}{C_n} = \frac{(n+a+1)(n+b+1)}{n(n+a+b+2)} = 1 + \frac{(a+1)(b+1)}{n(n+a+b+2)} > 1.$$

Therefore, part (1) holds.

For part (2), since the Gaussian hypergeometric function $F(a, b; c; x)$ satisfies the following asymptotic expansion as $x \to 1$(cf. [37, (2.10)])

$$\begin{aligned}F(a, b; a+b+1; x) =& \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \\ &+ \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)}(x-1)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n!)(n+1)!}v_n(1-x)^n,\end{aligned}$$

where

$$v_n = \psi(n+1) + \psi(n+2) - \psi(n+a+1) - \psi(n+b+1) - \log(1-x),$$

then, together with (1.2) and (1.3), we get

$$\begin{aligned}&\lim_{x\to1^-} H_{f,g}(x) = \lim_{x\to1^-}\left[\frac{f'(x)}{g'(x)}g(x) - f(x)\right] \\ &= \lim_{x\to1^-}\left\{\frac{a+b+2}{(a+1)(b+1)F(a+1, b+1, a+b+3; x)}F(a+1, b+1, a+b+2; x)\right. \\ &\left.+ \log(1-x) - R(a, b) + 2\right\} \\ &= \lim_{x\to1^-}\left\{B(a+1, b+1)\frac{[\Gamma(a+b+3)]/[\Gamma(b+2)\Gamma(a+2)] - F(a+1, b+1, a+b+3; x)}{F(a+1, b+1, a+b+3; x)}\right. \\ &\left.\times F(a+1, b+1, a+b+2; x)\right\} \\ &+ \lim_{x\to1^-}\big[B(a+1, b+1)F(a+1, b+1; a+b+2; x) + \log(1-x)\big] - R(a, b) + 2 \\ &= 0 + R(a+1, b+1) - R(a, b) + 2 = \frac{2ab-a-b}{ab}.\end{aligned}$$

The proof is complete ◻

Proof of Theorem 2.4 Differentiating H gives

$$\begin{aligned}H'(x) =& \frac{[ab/(a+b)]F(a+1, b+1; a+b+1; x)\log[e^{R(a,b)}/(1-x)] - F(a, b; a+b; x)/(1-x)}{\{\log[e^{R(a,b)}/(1-x)]\}^2} \\ =& -\frac{-[ab/(a+b)]F(a, b; a+b+1; x)\log[e^{R(a,b)}/(1-x)] + F(a, b; a+b; x)}{(1-x)\{\log[e^{R(a,b)}/(1-x)]\}^2} \\ \equiv& -\frac{1}{h(x)},\end{aligned} \tag{3.15}$$

where

$$h(x) = \frac{(1-x)\{\log[e^{R(a,b)}/(1-x)]\}^2}{-[ab/(a+b)]F(a, b; a+b+1; x)\log[e^{R(a,b)}/(1-x)] + F(a, b; a+b; x)}. \tag{3.16}$$

Let

$$h_1(x) = (1-x)\{\log[e^{R(a,b)}/(1-x)]\}^2$$

and

$$h_2(x) = -\frac{ab}{a+b}F(a,b;a+b+1;x)\log[e^{R(a,b)}/(1-x)] + F(a,b;a+b;x),$$

then $h(x) = h_1(x)/h_2(x)$, $h_1(1^-) = h_2(1^-) = 0$, and by differentiation we have

$$\begin{aligned}
h_1'(x) &= \log[e^{R(a,b)}/(1-x)]\left[2 - \log[e^{R(a,b)}/(1-x)]\right],\\
h_2'(x) &= -\frac{a^2b^2}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)\log[e^{R(a,b)}/(1-x)]\\
&\quad -\frac{ab}{a+b}F(a,b;a+b+1;x)\frac{1}{1-x} + \frac{ab}{a+b}F(a+1,b+1,a+b+1;x)\\
&= -\frac{a^2b^2}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)\log[e^{R(a,b)}/(1-x)].
\end{aligned}$$

So that $h_2(x) > h_2(1^-) = 0$ for $x \in (0,1)$, and

$$\frac{h_1'(x)}{h_2'(x)} = \frac{(a+b)(a+b+1)}{a^2b^2}\frac{R(a,b)-2+\log[1/(1-x)]}{F(a+1,b+1;a+b+2;x)} \equiv \frac{(a+b)(a+b+1)}{a^2b^2}h_3(x),$$

where

$$h_3(x) \equiv \frac{R(a,b)-2+\log[1/(1-x)]}{F(a+1,b+1;a+b+2;x)} = \frac{f(x)}{g(x)} \equiv \frac{\sum_{n=0}^{\infty}A_nx^n}{\sum_{n=0}^{\infty}B_nx^n},$$

and $f(x)$, $g(x)$, A_n, B_n are defined as in Lemma 3.7.

Next we divide the proof into two cases.

Case 1 $A_0/B_0 \geq A_1/B_1$. Then $R(a,b)-2 \geq (a+b+2)/[(a+1)(b+1)]$, namely $R(a+2,b+2)+1/a+1/b-2 \geq 0$. This, in conjunction with Lemma 3.7, shows that the sequence $\{A_n/B_n\}_{n\geq 0}$ is strictly decreasing on $n \in \mathbb{N}_0$, so that $h_3(x)$ is strictly decreasing on $(0,1)$ by application of Lemma 1.7(1). Furthermore, by Lemma 1.6, one has that $h(x) = h_1(x)/h_2(x)$ is also strictly decreasing on $(0,1)$. Consequently, the concavity of $H(x)$ on $(0,1)$ follows from equations (3.15) and (3.16).

Case 2 $A_0/B_0 = R(a,b)-2 < A_1/B_1 = (a+b+2)/[(a+1)(b+1)]$, namely $R(a+2,b+2)+1/a+1/b-2 < 0$. Then the sequence $\{A_n/B_n\}_{n\geq 0}$ is strictly increasing for $0 \leq n \leq 1$, and then strictly decreasing for $n > 1$.

Subcase 2.1 $(a,b) = (1,1)$. Then from Lemma 3.7(2) we obtain that $H_{f,g}(1^-) = 0$. Applying Lemma 1.7(2), $h_3(x)$ is strictly increasing on $(0,1)$, and so is $h(x)$ by application of Lemma 1.6. Therefore, $H(x)$ is convex on $(0,1)$ by (3.15). Indeed, if $(a,b) = (1,1)$, then $H(x) = 1/x$, and the convexity of $H(x)$ on $(0,1)$ is clear.

Subcase 2.2 $(a,b) \neq (1,1)$. Then Lemmas 3.6 and 3.7 show that $H_{f,g}(1^-) = (2ab-a-b)/(ab) < 0$. This, together with Lemma 1.7(2), implies that there exists $\xi \in (0,1)$ such that $h_3(x)$ is strictly increasing on $(0,\xi)$, and strictly decreasing on $(\xi,1)$.

Since

$$\lim_{x\to 1^-}\frac{h_1(x)}{h_2(x)} = \lim_{x\to 1^-}\frac{h_1{}'(x)}{h_2{}'(x)} = \lim_{x\to 1^-}\frac{(a+b)(a+b+1)}{a^2b^2}h_3(x) = \frac{B(a,b)}{ab},$$

then

$$\lim_{x\to 1^-}H_{h_1,h_2}(x) = \lim_{x\to 1^-}\left[\frac{h_1{}'(x)}{h_2'(x)}h_2(x) - h_1(x)\right] = 0. \tag{3.17}$$

Moreover, one has

$$\lim_{x\to 0^+}H_{h_1,h_2}(x) = -\left(\frac{a+b+1}{a^2b^2}\right)[abR(a,b)-a-b][R(a,b)-2] - R(a,b)^2 = -M(a,b). \tag{3.18}$$

Noting that

$$h'(x) = \left[\frac{h_1(x)}{h_2(x)}\right]' = \frac{h_2'(x)}{h_2(x)^2} H_{h_1,h_2}(x), \tag{3.19}$$

$$H'_{h_1,h_2}(x) = \left[\frac{h_1'(x)}{h_2'(x)}\right]' h_2(x) = \frac{(a+b)(a+b+1)}{a^2b^2} h_3'(x) h_2(x). \tag{3.20}$$

It follows from (3.17)-(3.20) together with the piecewise monotonicity of $h_3(x)$ that the quotient $h(x) = h_1(x)/h_2(x)$ is positive and strictly decreasing on (0, 1) if $M(a,b) \le 0$, and is piecewise monotone if $M(a,b) > 0$. Therefore, $H(x)$ is concave on (0, 1) for $(a,b) \in D_1 \cup D_2$, and $H(x)$ is neither convex nor concave on (0, 1) for $(a,b) \in D_3$. The remaining assertions in Theorem 2.4 are clear. The proof is complete. □

4 Monotonicity theorems for complete *p*-elliptic integrals

Since 2016, a new form of the generalized complete elliptic integrals, named complete p-elliptic integrals, have been introduced by Takeuchi in [41,43]. Some interesting topics, such as Legendre-type relations, generalized arithmetic-geometric mean, the computational formulas for π, and monotonicity theorems for complete p-elliptic integrals, have been discussed in [16,42–44,58,61]. Indeed, for $p \in (1,+\infty)$ and $r \in (0,1)$, let $\pi_p = 2\pi/[p\sin(\pi/p)]$. Then the complete p-elliptic integrals $\mathscr{K}_p(r)$ and $\mathscr{E}_p(r)$ of the first and second kinds are, respectively, defined by

$$\mathscr{K}_p(r) = \int_0^1 \frac{dt}{(1-t^p)^{1/p}(1-r^pt^p)^{1-1/p}} = \frac{\pi_p}{2} F\left(1-\frac{1}{p}, \frac{1}{p}; 1; r^p\right) \tag{4.1}$$

and

$$\mathscr{E}_p(r) = \int_0^1 \left(\frac{1-r^pt^p}{1-t^p}\right)^{\frac{1}{p}} dt = \frac{\pi_p}{2} F\left(-\frac{1}{p}, \frac{1}{p}; 1; r^p\right). \tag{4.2}$$

In the particular case $p = 2$, $\mathscr{K}_p(r)$ and $\mathscr{E}_p(r)$ become $\mathscr{K}(r)$ and $\mathscr{E}(r)$, respectively.

In particular, in 2017, Zhang [61] extended several monotonicity properties of complete elliptic integrals of the first and second kinds, occurred in [11, Chapter 3], to the case of complete p-elliptic integrals. Consequently, functional inequalities for the complete p-elliptic integrals and other related special functions have been obtained.

In this section, utilizing Lemma 3.1 and Theorem 2.1, we shall also derive some new monotonicity theorems involving $\mathscr{K}_p(r)$ and $\mathscr{E}_p(r)$. In our opinion, these theorems will play an important role in investigating analytic properties of certain combinations of complete p-elliptic integrals and elementary functions. Throughout this section, let $p \in (1,+\infty)$ and denote by $r' = \sqrt[p]{1-r^p}$ for $r \in (0,1)$.

Theorem 4.1 *Let $p \in (1,+\infty)$, $r' = \sqrt[p]{1-r^p}$ for $r \in (0,1)$. Then we have*

(1) *The function*

$$g_1(r) = \frac{\mathscr{E}_p(r) - r'^p\mathscr{K}_p(r) - (p-1)r'^p[\mathscr{K}_p(r) - \mathscr{E}_p(r)]}{r^p[\mathscr{K}_p(r) - \mathscr{E}_p(r)]}$$

is strictly decreasing from (0, 1) *onto* $(0, (p^2-1)/(2p))$*;*

(2) *The function*

$$g_2(r) = \frac{\mathscr{E}_p(r) - r'^p\mathscr{K}_p(r) - (p-1)r'^p[\mathscr{K}_p(r) - \mathscr{E}_p(r)]}{r^p[\mathscr{E}_p(r) - r'^p\mathscr{K}_p(r)]}$$

is strictly increasing from (0, 1) *onto* $((p+1)/(2p), 1)$*;*

(3) *The function*

$$g_3(r)=\frac{2p\left\{\mathscr{E}_p(r)-r'^p\mathscr{K}_p(r)-(p-1)r'^p\left[\mathscr{K}_p(r)-\mathscr{E}_p(r)\right]\right\}-(p+1)r^p\left[\mathscr{E}_p(r)-r'^p\mathscr{K}_p(r)\right]}{r^p\left\{(p-1)\left[\mathscr{K}_p(r)-\mathscr{E}_p(r)\right]-\left[\mathscr{E}_p(r)-r'^p\mathscr{K}_p(r)\right]\right\}}$$

is strictly decreasing from $(0,1)$ *onto* $(0,(2p^2+p-1)/(3p))$*;*

(4) *The function*

$$g_4(r)=\frac{(p^2-1)r'^pr^p\left[\mathscr{K}_p(r)-\mathscr{E}_p(r)\right]-2pr'^p\left\{\mathscr{E}_p(r)-r'^p\mathscr{K}_p(r)-(p-1)r'^p\left[\mathscr{K}_p(r)-\mathscr{E}_p(r)\right]\right\}}{r^p\left\{\mathscr{E}_p(r)-r'^p\mathscr{K}_p(r)-(p-1)r'^p\left[\mathscr{K}_p(r)-\mathscr{E}_p(r)\right]\right\}}$$

is strictly decreasing from $(0,1)$ *onto* $(0,(p^2-1)/(3p))$.

Proof Using (4.1) and (4.2) to simplify g_1, we obtain

$$\begin{aligned} g_1(r)=&\frac{\frac{\pi_p}{2}\sum_{n=0}^{\infty}\frac{(1-1/p,n)(1/p,n+1)}{(n+2)!n!}\left(p-\frac{1}{p}\right)r^{(n+2)p}}{\frac{\pi_p}{2}\sum_{n=0}^{\infty}\frac{(1-1/p,n)(1/p,n+1)}{(2,n)n!}r^{(n+2)p}}\\ =&\frac{p(1-1/p^2)F(1-1/p,1+1/p;3;r^p)}{2F(1-1/p,1+1/p;2;r^p)}. \end{aligned}\tag{4.3}$$

According to (4.3) and (1.3), it is not difficult to check that $g_1(r)=p\varphi(x)$ with $a=1-1/p$, $b=1+1/p$, $c=2$ and $x=r^p$, where $\varphi(x)$ is defined in Lemma 3.1. Therefore, by Lemma 3.1, part (1) takes place.

Similarly, by tedious computations, one has $1-g_2(r)=p\varphi(x)$ with $a=1-1/p$, $b=1/p$, $c=2$ and $x=r^p$; $g_3(r)=p\varphi(x)$ with $a=2-1/p$, $b=1+1/p$, $c=3$ and $x=r^p$; $g_4(r)=p\varphi(x)$ with $a=1-1/p$, $b=1+1/p$, $c=3$ and $x=r^p$. Applying Lemma 3.1, we also obtain the assertions in parts (2)-(4). □

The following theorem reduces to Theorem 15 in [3] for $p=2$.

Theorem 4.2 *Suppose* $p\in(1,+\infty)$, $c\in\mathbb{R}$. *Let*

$$G_c^{\{1\}}(r)=\frac{r'^c[\mathscr{K}_p(r)-\mathscr{E}_p(r)]}{r^p},\quad r\in(0,1).$$

Then the following statements hold

(1) $G_c^{\{1\}}(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\pi_p/(2p))$ *if and only if* $c\ge(p^2-1)/(2p)$*;*
(2) $G_c^{\{1\}}(r)$ *is strictly increasing from* $(0,1)$ *onto* $(\pi_p/(2p),+\infty)$ *if and only if* $c\le 0$*;*
(3) *If* $0<c<(p^2-1)/(2p)$*, then there exists* $\lambda^{\{1\}}\in(0,1)$ *such that* $G_c^{\{1\}}(r)$ *is strictly increasing on* $(0,\lambda^{\{1\}})$ *and strictly decreasing on* $(\lambda^{\{1\}},1)$.

Proof Theorem 4.2 directly follows from Theorem 2.1 and the fact that

$$\begin{aligned} \frac{\mathscr{K}_p(r)-\mathscr{E}_p(r)}{r^p}&=\frac{\pi_p}{2r^p}\left(\sum_{n=0}^{\infty}\frac{(1/p,n)(1-1/p,n)}{n!}\frac{r^{np}}{n!}-\sum_{n=0}^{\infty}\frac{(1/p,n)(-1/p,n)}{n!}\frac{r^{np}}{n!}\right)\\ &=\frac{\pi_p}{2r^p}\sum_{n=0}^{\infty}\frac{(1-1/p,n)(1/p,n+1)}{n!}\frac{r^{(n+1)p}}{(n+1)!}\\ &=\frac{\pi_p}{2p}\sum_{n=0}^{\infty}\frac{(1-1/p,n)(1+1/p,n)}{(2,n)}\frac{r^{np}}{n!}\\ &=\frac{\pi_p}{2p}F\left(1-\frac{1}{p},1+\frac{1}{p};2;r^p\right). \end{aligned}$$

This completes the proof. □

Analogously, employing the identities

$$\frac{\mathscr{E}_p(r) - r'^p \mathscr{K}_p(r)}{r^p} = \frac{\pi_p}{2}\left(1 - \frac{1}{p}\right) F\left(1 - \frac{1}{p}, \frac{1}{p}; 2; r^p\right),$$

$$\frac{(p-1)\left[\mathscr{K}_p(r) - \mathscr{E}_p(r)\right] - \left[\mathscr{E}_p(r) - r'^p \mathscr{K}_p(r)\right]}{r^{2p}} = \frac{\pi_p}{4p^2}(p-1)^2 F\left(1 + \frac{1}{p}, 2 - \frac{1}{p}; 3; r^p\right)$$

and

$$\frac{\mathscr{E}_p(r) - r'^p \mathscr{K}_p(r) - (p-1) r'^p \left[\mathscr{K}_p(r) - \mathscr{E}_p(r)\right]}{r^{2p}} = \frac{\pi_p}{4}\left(1 - \frac{1}{p^2}\right) F\left(1 - \frac{1}{p}, 1 + \frac{1}{p}; 3; r^p\right),$$

and applying Theorem 2.1, we also obtain the following Theorems 4.3–4.5 immediately.

Theorem 4.3 *Suppose $p \in (1, +\infty)$, $c \in \mathbb{R}$. Let*

$$G_c^{\{2\}}(r) = \frac{r'^c[\mathscr{E}_p(r) - r'^p \mathscr{K}_p(r)]}{r^p}, \quad r \in (0, 1).$$

Then the following statements hold

(1) $G_c^{\{2\}}(r)$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, \pi_p(p-1)/(2p))$ *if and only if* $c \geq (p-1)/(2p)$*;*
(2) $G_c^{\{2\}}(r)$ *is strictly increasing on* $(0, 1)$ *if and only if* $c \leq 0$*, in which case the range of* $G_c^{\{2\}}(r)$ *is* $(\pi_p(p-1)/(2p), +\infty)$ *if* $c < 0$*, and while the range is* $(\pi_p(p-1)/(2p), 1)$ *if* $c = 0$*;*
(3) *If* $0 < c < (p-1)/(2p)$*, then there exists* $\lambda^{\{2\}} \in (0, 1)$ *such that* $G_c^{\{2\}}(r)$ *is strictly increasing on* $(0, \lambda^{\{2\}})$ *and strictly decreasing on* $(\lambda^{\{2\}}, 1)$.

Theorem 4.4 *Suppose $p \in (1, +\infty)$, $c \in \mathbb{R}$. Let*

$$G_c^{\{3\}}(r) = \frac{r'^c\left\{(p-1)\left[\mathscr{K}_p(r) - \mathscr{E}_p(r)\right] - \left[\mathscr{E}_p(r) - r'^p \mathscr{K}_p(r)\right]\right\}}{r^{2p}}, \quad r \in (0, 1).$$

Then the following statements hold

(1) $G_c^{\{3\}}(r)$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, \pi_p(p-1)^2/(4p^2))$ *if and only if* $c \geq (2p^2 + p - 1)/(3p)$*;*
(2) $G_c^{\{3\}}(r)$ *is strictly increasing from* $(0, 1)$ *onto* $(\pi_p(p-1)^2/(4p^2), +\infty)$ *if and only if* $c \leq 0$*;*
(3) *If* $0 < c < (2p^2 + p - 1)/(3p)$*, then there exists* $\lambda^{\{3\}} \in (0, 1)$ *such that* $G_c^{\{3\}}(r)$ *is strictly increasing on* $(0, \lambda^{\{3\}})$ *and strictly decreasing on* $(\lambda^{\{3\}}, 1)$.

Theorem 4.5 *Suppose $p \in (1, +\infty)$, $c \in \mathbb{R}$. Let*

$$G_c^{\{4\}}(r) = \frac{r'^c\left\{\mathscr{E}_p(r) - r'^p \mathscr{K}_p(r) - (p-1) r'^p \left[\mathscr{K}_p(r) - \mathscr{E}_p(r)\right]\right\}}{r^{2p}}, \quad r \in (0, 1).$$

Then the following statements hold

(1) $G_c^{\{4\}}(r)$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, \pi_p(p^2-1)/(4p^2))$ *if and only if* $c \geq (p^2-1)/(3p)$*;*
(2) $G_c^{\{4\}}(r)$ *is strictly increasing on* $(0, 1)$ *if and only if* $c \leq 0$*; in which case the range of* $G_c^{\{4\}}(r)$ *is* $(\pi_p(p^2-1)/(4p^2), +\infty)$ *if* $c < 0$*, and while the range is* $(\pi_p(p^2-1)/(4p^2), 1)$ *if* $c = 0$*;*
(3) *If* $0 < c < (p^2-1)/(3p)$*, then there exists* $\lambda^{\{4\}} \in (0, 1)$ *such that* $G_c^{\{4\}}(r)$ *is strictly increasing on* $(0, \lambda^{\{4\}})$ *and strictly decreasing on* $(\lambda^{\{4\}}, 1)$.

5 Sharp bounds for generalized Grötzsch ring function

Given $a, b \in (0, +\infty)$, the generalized Grötzsch ring function with two parameters is defined by (cf. [17,23,24])

$$\mu_{a,b}(r) = \frac{B(a,b)}{2}\frac{F(a,b;a+b;1-r^2)}{F(a,b;a+b;r^2)} \tag{5.1}$$

for $r \in (0,1)$. In particular, when $a = b = 1/2$, $\mu_{a,b}(r)$ becomes the classical Grötzsch ring function (capacity) $\mu(r)$.

In this section, we present some new inequalities for $\mu_{a,b}(r)$, which extend Exercise 5.68(36) in [11].

Theorem 5.1 *Suppose $a, b \in (0, +\infty)$. Then we have*

(1) *If $R(a,b) \geq 0$, then inequality*

$$\frac{B(a,b)}{2}\frac{\log[e^{R(a,b)}/r^2]}{\log\left[e^{(a+b)/(ab)}/(1-r^2)\right]} \leq \mu_{a,b}(r) \leq \frac{B(a,b)}{2}\frac{\log[e^{(a+b)/(ab)}/r^2]}{\log\left[e^{R(a,b)}/(1-r^2)\right]} \tag{5.2}$$

holds for all $r \in (0,1)$, and

$$\frac{R(a,b)}{2}\frac{\log[e^{R(a,b)}/r^2]}{\log\left[e^{R(a,b)}/(1-r^2)\right]} \leq \mu_{a,b}(r) \leq \frac{B(a,b)^2}{2R(a,b)}\frac{\log[e^{R(a,b)}/r^2]}{\log\left[e^{R(a,b)}/(1-r^2)\right]} \tag{5.3}$$

holds for all $r \in (0,1)$;

(2) *Inequality*

$$\frac{abB(a,b)^2}{2(a+b)}\frac{\log[e^{(a+b)/(ab)}/r^2]}{\log\left[e^{(a+b)/(ab)}/(1-r^2)\right]} \leq \mu_{a,b}(r) \leq \frac{a+b}{2ab}\frac{\log[e^{(a+b)/(ab)}/r^2]}{\log\left[e^{(a+b)/(ab)}/(1-r^2)\right]} \tag{5.4}$$

holds for all $r \in (0,1)$.

Proof Let

$$f_1(r) = \frac{F(a,b;a+b;r^2)}{\log\left[e^{(a+b)/(ab)}/(1-r^2)\right]}, \quad a, b \in (0,+\infty),$$

$$f_2(r) = \frac{F(a,b;a+b;r^2)}{\log\left[e^{R(a,b)}/(1-r^2)\right]}, \quad a, b \in (0,+\infty) \ \&\ R(a,b) \geq 0.$$

Then it follows from Theorem 2.3(1) that the function $f_1(r)$ is strictly increasing from $(0,1)$ onto $(ab/(a+b), 1/B(a,b))$, while $f_2(r)$ is strictly decreasing from $(0,1)$ onto $(1/B(a,b), 1/R(a,b))$.

For part (1), in view of inequalities $f_1(r) < 1/B(a,b) < f_2(\sqrt{1-r^2})$ and $f_1(\sqrt{1-r^2}) < 1/B(a,b) < f_2(r)$ for all $r \in (0,1)$, we obtain

$$\frac{F(a,b;a+b;1-r^2)}{F(a,b;a+b;r^2)} \geq \frac{\log\left[e^{R(a,b)}/r^2\right]}{\log\left[e^{(a+b)/(ab)}/(1-r^2)\right]}, \quad \frac{F(a,b;a+b;1-r^2)}{F(a,b;a+b;r^2)} \leq \frac{\log\left[e^{(a+b)/(ab)}/r^2\right]}{\log\left[e^{R(a,b)}/(1-r^2)\right]},$$

which gives inequality (5.2) by (5.1).

To obtain (5.3), we consider the ratio function $f_2(\sqrt{1-r^2})/f_2(r)$. Notice that this function $r \mapsto f_2(\sqrt{1-r^2})/f_2(r)$ is strictly increasing from $(0,1)$ onto $(R(a,b)/B(a,b), B(a,b)/R(a,b))$. Hence, we conclude that, for all $r \in (0,1)$,

$$\frac{R(a,b)}{B(a,b)}\frac{\log\left[e^{R(a,b)}/r^2\right]}{\log\left[e^{R(a,b)}/(1-r^2)\right]} \leq \frac{F(a,b;a+b;1-r^2)}{F(a,b;a+b;r^2)} \leq \frac{B(a,b)}{R(a,b)}\frac{\log\left[e^{R(a,b)}/r^2\right]}{\log\left[e^{R(a,b)}/(1-r^2)\right]}.$$

By (5.1), inequality (5.3) follows, thus part (1) is true.

For part (2), since the function $f_1(\sqrt{1-r^2})/f_1(r)$ is strictly decreasing from $(0,1)$ onto $([abB(a,b)]/(a+b), (a+b)/[abB(a,b)])$, then we derive the inequality (5.4) using a similar argument as in the proof of inequality (5.3). □

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