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ORIGINAL RESEARCH

Cayley graphs of groupoids and generalized fat-trees

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Abstract Recall that for a graph which is a Cayley graph of some group, using the group theoretical structure of the graph we can use algebraic methods for studying the network and its properties. As the main result of this note, we investigate a similar result for asymmetric multigraphs and graphs. Specially, for a fat-tree (tree) $\mathcal{F}$, we present an algebraic structure on $\mathcal{F}$ induced by a Cayley multigraph of a power-associative groupoid $\mathcal{S}_{\mathcal{F}}$.

Keywords Generalized fat-trees · Cayley graphs of groupoids

Mathematics Subject Classification Primary: 05C25; Secondary: 94C15, 05C38

1 introduction and main results

Akers and Krishnamurthy in [1], suggested Cayley graphs of groups as possible interconnection networks, because these graphs are always vertex-transitive (vertex symmetric) and so they have high fault-tolerance. On the other hand, there are many graphs which are not vertex-transitive but they are widely used as interconnection networks (e.g. fat-trees and generalized fat-trees). Recall that Cayley graphs of semigroups and groupoids are very well-studied structures and as it is remarked in [16], although these graphs are not necessarily vertex-transitive, but they have the benefit of being asymmetric. These graphs are closely related to finite state automata and have several valuable applications (see [10], [15], [16] or [38]) and they have their own symmetric properties (see for example, [5], [8]- [27], [30], [31], [36], [37]).

Now it is natural to ask “Is it possible to use Cayley graphs of semigroups or groupoids in designing asymmetric graphs (multigraphs) with high fault-tolerance which enable us to use algebraic methods in studying asymmetric networks?". Answering this question enables us to use algebraic structures and operations between semigroups or groupoids for constructing new algebraic and more complicated networks. In this paper, we prove that for every fat-tree $\mathcal{F}$, there exists an algebraic structure on $\mathcal{F}$ induced by a Cayley multigraph of a power-associative groupoid $\mathcal{S}_{\mathcal{F}}$ (see Theorem 1.2).

To state our main results, we recall some necessary facts and notions. Throughout this paper, all sets are assumed to be finite. Recall that a *multigraph* $\Gamma = (V, E)$, is a set $V = V(\Gamma)$ of vertices, together with a multiset $E = E(\Gamma)$ of ordered pairs of vertices. The elements of E are called the *edges* or *arcs* of Γ. Also note that by a graph we mean a directed graph without multiple edges but so that loops are allowed.

Recall that every fat-tree $\mathcal{F}$ with root r is an undirected multigraph such that the underlying graph of $\mathcal{F}$ is a tree with root r. Furthermore, note that in a fat-tree, we suppose that the number of links going down to the

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children of a node in the bottom level (layer) is equal to the number of links going up to its parent in the upper level. Therefore the links toward the top of a fat-tree get fatter and specially the number of links connected to the root of a fat-tree is equal to the number of leaves of the fat-tree. Furthermore, it is supposed that each pair of vertices of a fat-tree which are not leaves, have the same number of children. Finally, note that in a fat-tree as a topology for interconnection networks (graphs which are used for connecting processors in a system of parallel computing), it is assumed that the processors are as the leaves of the network and the other vertices are corresponded to the switches in the network which are used for connecting the processors (see [33], [35], [40] or [41]).

Recall that a set $\mathcal{S}$ with a binary composition $\cdot_{\mathcal{S}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is called a *groupoid* and it is denoted by $(\mathcal{S}, \cdot_{\mathcal{S}})$. Note that if there is no ambiguity about $\mathcal{S}$, we write $(\mathcal{S}, \cdot)$. Furthermore, if there is no ambiguity about the composition, for simplicity we denote a groupoid by $\mathcal{S}$, and we write $s_1 s_2$ instead of $s_1 \cdot s_2$. An element z in a groupoid $\mathcal{S}$ is called a *zero element*, if for every $g \in \mathcal{S}$ we have $z = gz = zg$. From now on, we denote the zero element of a groupoid by 0. If the binary composition is associative (i.e. for every $s_1, s_2, s_3 \in \mathcal{S}$ we have $(s_1 s_2)s_3 = s_1(s_2 s_3)$), then $\mathcal{S}$ is called a *semigroup*. Recall that a groupoid $\mathcal{S}$ with a power-associative composition, is called a *power-associative groupoid*. For a groupoid $\mathcal{S}$, by $\mathcal{S}^1$ we mean the groupoid defined on the set $\mathcal{S}\dot{\cup}\{1\}$ with the composition $x_1 \cdot_{\mathcal{S}^1} x_2 = x_1 \cdot_{\mathcal{S}} x_2$, if $x_1, x_2 \in \mathcal{S}$ and $1 \cdot_{\mathcal{S}^1} x_1 = x_1 \cdot_{\mathcal{S}^1} 1 = x_1$.

Let $\mathcal{S}$ be a groupoid and C be a subset of $\mathcal{S}$. The graph $\mathrm{Cay}(\mathcal{S}, C) = (\mathcal{S}, E(C))$ such that, for $x, y \in \mathcal{S}$ we have $(x, y) \in E(C)$ if and only if $cx = y$ for some $c \in C$, is called the Cayley (color) graph of $\mathcal{S}$ with connection set C. We say that the arc $e = (x, y)$ has *color* c, if $cx = y$ (see [31, Definition7.3.1]). Note that for $s, s', s'' \in \mathcal{S}$, if $s's = s''s$, then we can not necessarily conclude that $s' = s''$. Since we usually want $\mathrm{Cay}(\mathcal{S}, C)$ to be a graph, it is usually supposed that $\mathrm{Cay}(\mathcal{S}, C)$ has no multiple edges but they can have possible loops. However if we suppose that $E(\mathrm{Cay}(\mathcal{S}, C))$ is a multiset, then we can use Cayley multigraphs of groupoids (semigroups), to give algebraic structures to multigraphs. From now on, in order to prevent confusions, we denote the corresponding multigraph of $\mathrm{Cay}(\mathcal{S}, C)$ by $\mathrm{MCay}(\mathcal{S}, C)$.

Recall that for a directed graph Γ, if the ordered pair $(u, v) \in E(\Gamma)$, then u is called an *in-neighbor of* v. Also note that for every undirected graph Γ and a vertex v of Γ, by *the degree of* v, we mean the number of edges which are incident to v, and we will denote it by $\deg_\Gamma(v)$. Note that for every natural number m, the path of length m is denoted by P_m. Also note that the star graph with m vertices is denoted by S_m. For an undirected graph Γ and for every two vertices $u, v \in V$, by $d_\Gamma(u, v)$ we mean the distance between u and v which is equal to the length of the shortest path from u to v in Γ.

From now on, by *the vertices at level i of a tree $\mathcal{F}$*, we mean the elements of the set $\{x \in V(\mathcal{F}) \mid d_{\mathcal{F}}(r, x) = i - 1\}$. If all leaves of $\mathcal{F}$ are at level m, then $\mathcal{F}$ is called a *tree with m levels and root r*. For every two adjacent vertices $v, v' \in V(\mathcal{F})$ such that v is at some level i and v' is at level $i + 1$, the vertex v is called *the parent* of v and v' is called a *child (descendant)* of v. Recall that for a pair of distinct vertices v and v', the farthest ancestor in common of v and v' from r, is called *the lowest ancestor in common of v and v'*.

1.1 Main results

In this paper, first we introduce the following family of multigraphs which clearly contains the family of fat-trees. Then we show that all these graphs can be considered as Cayley graphs of some power associative groupoids.

Definition 1.1 By a *g-fat-tree* $\mathcal{F}$ with m levels and root r, we mean a multigraph $\mathcal{F}$ with the following properties.

1. By replacing multiple edges between every two adjacent vertices of $\mathcal{F}$, the new simple graph is a tree with m levels and root r (equivalently, the underlying undirected graph of $\mathcal{F}$ is a tree with m levels and root r);
2. For every vertex x at a level $2 \leq i \leq m - 1$, the number of edges from x to its parent, is equal to the number of edges from x to all its children (descendants).

Also note that for a vertex x of $\mathcal{F}$ which is at level 2, by *the branch of $\mathcal{F}$ which contains* x, we mean the connected component of $\mathcal{F} \setminus \{r\}$ which contains x.

For a g-fat-tree $\mathcal{F}$, note that we do not assume that a parent must have more than one child. Also note that two vertices which are not the leaves of $\mathcal{F}$, may have different numbers of children.

Now for every g-fat-tree $\mathcal{F}$, we use the Cayley graphs of groupoids, and amalgamation of semigroups and groupoids (see Definition 1.3), to present an algebraic structure on $\mathcal{F}$ by proving the following result. Our main idea for proving Theorem 1.2 is to use monogenic semigroups and natural amalgamated groupoids (see Definition 1.3) in a recursive process to construct a groupoid $\mathcal{S}_{\mathcal{F}}$ which satisfy in conditions of Theorem 1.2.

Theorem 1.2 *For every g-fat-tree $\mathcal{F}$ with m levels and root r, there exist a power-associative groupoid $\mathcal{S}_{\mathcal{F}}$ with zero and a subset $L_{\mathcal{F}} \subseteq \mathcal{S}_{\mathcal{F}}$ which satisfy the following conditions.*

(i) *The underlying undirected multigraph of $MCay(\mathcal{S}_{\mathcal{F}}, L_{\mathcal{F}}) \setminus \{0\}$ is isomorphic to $\mathcal{F}$.*
(ii) *The elements of $L_{\mathcal{F}}$ are corresponded to the leaves of $MCay(\mathcal{S}_{\mathcal{F}}, L_{\mathcal{F}})$.*
(iii) *For every $b \in L_{\mathcal{F}}$, the path $b - b^2 - \cdots - b^m = r$ is the unique path from b to r.*
(iv) *For every $v \in V(\mathcal{F})$, there exists at least one $b \in L_{\mathcal{F}}$ such that the unique path from b to r contains v and specially $L_{\mathcal{F}}$ is a generating set of $\mathcal{S}_{\mathcal{F}}$.*
(v) *For every $b, b' \in L_{\mathcal{F}}$, let c be the lowest ancestor in common of b and b'. Then $c = b_1^{j_1} = b_2^{j_2}$, where*

$$j_1 = \min\{i \in \mathbb{N} \mid b_2 \in N_p(b_1^i)\}, \text{ and } j_2 = \min\{i \in \mathbb{N} \mid b_1 \in N_p(b_2^i)\}$$

in which $N_p(g) = \{u \in \mathcal{S}_{\mathcal{F}} \mid g \text{ is a power of } u \text{ and } u \neq g\}$ for every $g \in \mathcal{S}_{\mathcal{F}}$.

1.2 Natural Amalgamated Groupoid

In this subsection, after recalling some notions and facts needed in the sequel, we present the algebraic structure which is the foundation of our study and will be used for proving Theorem 1.2. Recall that a semigroup S is called *monogenic* if S is generated by a single element $a \in S$. Let m be the smallest natural number such that $a^m = a^{m+q}$ for some $q \in \mathbb{N}$. Let p be the smallest natural number such that $a^m = a^{m+p}$. The number m is called *the index* or *height* of a and p the *period* of a (see [29]).

Definition 1.3 (i) Let $\{S_i \mid i \in I\}$ be a family of semigroups and mappings $h_i : U \to S_i$ that are injective homomorphisms for all $i \in I$, where U is a semigroup. The set $G = \cup_{i \in I} \overline{S_i}$, where $\overline{S_i} = [S_i \setminus h_i(U)] \cup U$ with a partial binary operation defined in such a way that the natural mappings $\phi_i : \overline{S_i} \to S_i$ are isomorphisms, is called *the groupoid of the amalgam* $[S_i; U; h_i]_{i \in I}$. Exactly similar to the above, the groupoid of amalgam $[\mathcal{S}_i; U; h_i]_{i \in I}$ can be defined where U and each $\mathcal{S}_i$ are groupoids with zero and each $h_i : U \to \mathcal{S}_i$ is an injective homomorphism (see [2], [6] or [32]).

Now suppose that $\{\mathcal{S}_i \mid i \in I\}$ is a family of groupoids with zero. Also suppose that U is a groupoid with zero and $h_i : U \to \mathcal{S}_i$ are injective homomorphisms such that for every i, $h_i(U)$ contains the zero element of $\mathcal{S}_i$ and let $\phi_i : \overline{\mathcal{S}_i} \to \mathcal{S}_i$ be natural mappings, where $\overline{\mathcal{S}_i} = [\mathcal{S}_i \setminus h_i(U)] \cup U$ for every i. Let $\mathcal{S}$ be the groupoid on the set $G = \cup_{i \in I} \overline{\mathcal{S}_i}$ with the composition function $\cdot : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ defined by

$$g \cdot_{\mathcal{S}} g' = \begin{cases} \phi_i^{-1}(g \cdot_{\mathcal{S}_i} g'), & g, g' \in \overline{\mathcal{S}_i} \setminus U \\ \phi_i^{-1}(g \cdot_{\mathcal{S}_i} h_i(g')), & g \in \overline{\mathcal{S}_i} \setminus U \text{ and } g' \in U \\ \phi_i^{-1}(h_i(g) \cdot_{\mathcal{S}_i} g'), & g \in U \text{ and } g' \in \overline{\mathcal{S}_i} \setminus U \\ g \cdot_U g', & g, g' \in U \\ 0, & \text{otherwise} \end{cases}$$

It is straightforward to see that the natural mappings ϕ_i are isomorphisms and so $\mathcal{S}$ is the groupoid of the amalgam $[\mathcal{S}_i; U; h_i]_{i \in I}$. Furthermore, it is clear that for every $g \in \mathcal{S}$, we have $0g = g0 = 0$. From now on, for every family of groupoid with zero $\{\mathcal{S}_i \mid i \in I\}$ and injective homomorphisms $\{h_i : U \to \mathcal{S}_i\}_{i \in I}$ with the above properties, by their *natural amalgamated groupoid*, we mean the above groupoid (Fig. 1).

2 g-Fat-trees and Cayley graphs of groupoids

In this section, we present our recursive process for constructing the algebraic structure over a g-fat-tree using amalgamated of groupoids.

Definition 2.1 Let $m, n, m' \in \mathbb{N}$.

(i) Consider P_m and replace every edge of P_m by n parallel edges. From now on, we denote this multigraph by P_m^n (note that P_m^1 is P_m and for every $n > 1$, P_m^n is an undirected multigraph).

(ii) By $PS(m, k)$, we mean the multigraph obtained by identifying one of the endpoints of P_m^k by the center of S_{k+1} (see Fig 2).

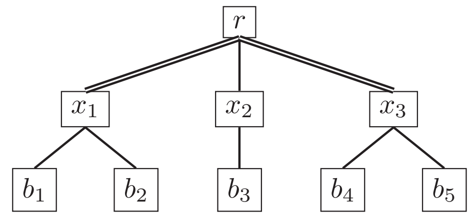

Fig. 1 An example of a g-fat-tree with 2 levels and root r

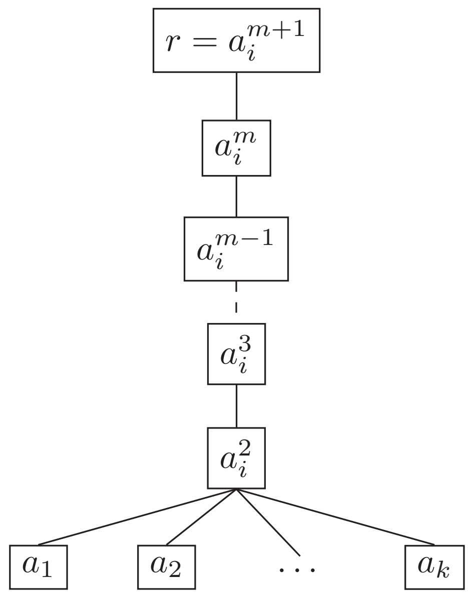

Fig. 2 The underlying tree of $PS(m, k)$

Lemma 2.2 *Consider $PS(m, k)$, where $m, k \in \mathbb{N}$. Then there exist a family $\{S_i = \langle a_i \rangle \mid 1 \leq i \leq k\}$ of monogenic semigroups with height $m + 2$ and period 1, a semigroup U, and injective homomorphisms $h_i : U \to S_i \setminus \{a_i\}$ such that the underlying undirected graph of $MCay(\mathcal{S}, L) \setminus \{0\}$ is isomorphic to $PS(m, k)$, where $\mathcal{S}$ is the natural amalgamated groupoid of $[S_i; U; h_i]_{i \in I}$ and $L = \{a_1, \cdots, a_k\}$. Furthermore, $\mathcal{S}$ is power-associative.*

Proof Let T be a monogenic semigroup with height $m + 2$ and period 1. If T is generated by t, then $U = \{t^2, \dots, t^{m+2}\}$ is a subsemigroup of T. For $1 \leq i \leq k$, let $S_i = \langle a_i \rangle$ be a monogenic semigroup with height $m + 2$ and period 1. Also for every i, let $h_i : U \to S_i$ be defined by $h_i(t^j) = a_i^j$, for every $2 \leq j \leq m + 2$ (see Fig. 2). It is clear that for every $1 \leq i \leq k$, a_i^{m+2} is the zero element of $\mathcal{S}_i$ and $h_i(t^{m+2}) = a_i^{m+2}$. Furthermore, the natural mappings from $\overline{S_i}$ to S_i are isomorphisms for every i. By the assumption, $\mathcal{S}$ is the natural amalgamated groupoid of $[S_i; U; h_i]_{i \in I}$ and $L = \{a_1, \dots, a_k\}$.

Note that $\mathcal{S} = U \cup L$, t^{m+2} is the zero element of $\mathcal{S}$ and for every $j \leq k$, $a_j \cdot_{\mathcal{S}} a_j = \phi^{-1}(a_j^2) = t^2$. Also by induction we can show that for each $i \geq 2$, we have

$$a_j \cdot_{\mathcal{S}} (\underbrace{a_j \cdot_{\mathcal{S}} \cdots \cdot_{\mathcal{S}} a_j}_{i}) = \phi_j^{-1}(a_j \cdot_{\mathcal{S}_j} a_j^i) = \phi_j^{-1}(a_j^{i+1}) = t^{i+1}.$$

Since $\overline{S_i}$ is power-associative for every i, the groupoid $\mathcal{S}$ is power-associative and furthermore, in $\Gamma = \mathrm{MCay}(\mathcal{S}, L) \setminus \{0\}$, there exists $|L| = k$ edges from t^i to t^{i+1}, where $2 \leq i \leq m$. Also for each a_j, where $1 \leq j \leq k$, there exists exactly one edge from a_j to t^2 in Γ. Finally, t^{m+1} is not the tail of any edge in Γ. We note that there exists no other edge in Γ. Now we can easily see that Γ is isomorphic to $PS(m, k)$. □

Notation 2.3 *For every $m, k \in \mathbb{N}$, by $\mathcal{S}_{PS(m,k)}$ we mean the groupoid with zero $\mathcal{S}$ which was defined in the proof of the above lemma such that $PS(m, k)$ is isomorphic to the underlying undirected multigraph of $MCay(\mathcal{S}_{PS(m,k)}, L) \setminus \{0\}$. Also we denote the set L defined in that proof by $L_{PS(m,k)}$.*

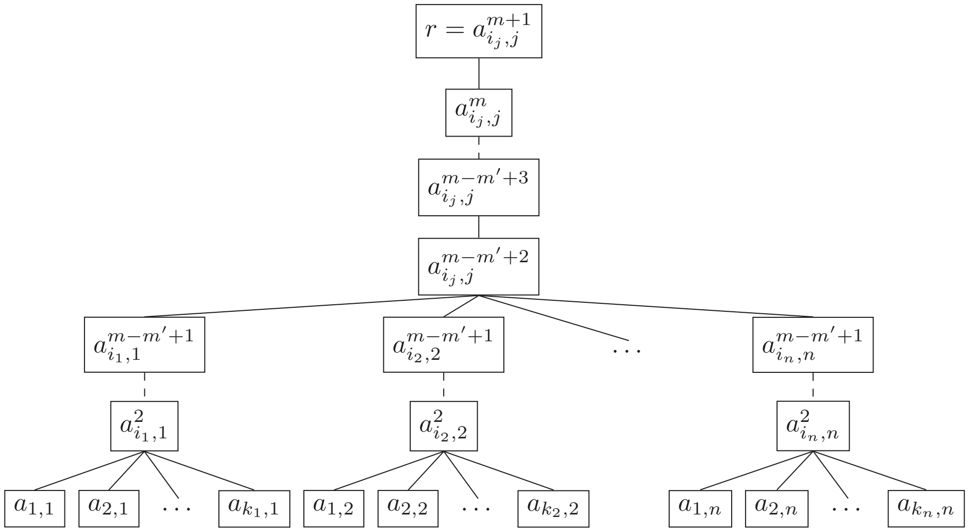

Fig. 3 The underlying tree of $(PS(m, k_1) \wedge PS(m, k_2) \wedge \cdots \wedge PS(m, k_n); m')$

Definition 2.4 Let $m, m', k_1, \dots k_n \in \mathbb{N}$ and $2 \leq m' < m$. Also for $k = \sum_{i=1}^{n} k_i$ consider P_m^k and let v be one of its leaves. For every $1 \leq j \leq n$, consider $PS(m - m', k_j)$ and identify all roots of $PS(m - m', k_i)$ by v. From now on, by $(PS(m, k_1) \wedge \cdots \wedge PS(m, k_n); m')$ we denote this multigraph (see Fig. 3).

Lemma 2.5 *Let $\Lambda = (PS(m, k_1) \wedge \cdots \wedge PS(m, k_n); m')$. Then there exist a semigroup U and injective homomorphisms $h_i : U \to \mathcal{S}_{PS(m,k_i)}$ for every i, such that the underlying undirected multigraph of $MCay(\mathcal{S}, L) \setminus \{0\}$ is isomorphic to Λ, where $\mathcal{S}$ is the natural amalgamated groupoid of $[\mathcal{S}_{PS(m,k_i)}; U; h_i]_{i \in I}$ and $L = \cup_{i=1}^{n} L_{PS(m,k_i)}$. Furthermore, $\mathcal{S}$ is power-associative.*

Proof The proof is similar to the proof of Lemma 2.2. Suppose that for every $1 \leq i \leq n$, $\mathcal{S}_{PS(m,k_i)} = U_i \cup L_{PS(m,k_i)}$, where $U_i = \{t_i^2, \dots, t_i^{m+2}\}$. Let S be a monogenic semigroup with height $m + 2$ and period 1 which is generated by s. Suppose that $U' = \{s^{m'}, \dots, s^{m+2}\}$. Also for every i, let $h_i : U' \to \mathcal{S}_{PS(m,k_i)}$ be defined by $h_i(s^j) = t_i^j$ for every $m' \leq j \leq m + 2$. For every $1 \leq i \leq n$, let $\overline{\mathcal{S}_{PS(m,k_i)}} = (\mathcal{S}_{PS(m,k_i)} \setminus h_i(U')) \cup U'$ and for every $1 \leq i \leq k$, let $\phi_i : \overline{\mathcal{S}_{PS(m,k_i)}} \to \mathcal{S}_{PS(m,k_i)}$ be the natural mapping. For every $g, g' \in \mathcal{S} = \cup_{i=1}^{n} \overline{\mathcal{S}_{PS(m,k_i)}}$ let

$$g \cdot_{\mathcal{S}} g' = \begin{cases} \phi_i^{-1}(g \cdot_{\mathcal{S}_{PS(m,k_i)}} g'), & g, g' \in \overline{\mathcal{S}_{PS(m,k_i)}} \setminus U \\ \phi_i^{-1}(g \cdot_{\mathcal{S}_{PS(m,k_i)}} h_i(g')), & g \in \overline{\mathcal{S}_{PS(m,k_i)}} \setminus U \text{ and } g' \in U \\ \phi_i^{-1}(h_i(g) \cdot_{\mathcal{S}_{PS(m,k_i)}} g'), & g \in U \text{ and } g' \in \overline{\mathcal{S}_{PS(m,k_i)}} \setminus U \\ g \cdot_U g', & g, g' \in U \\ 0, & \text{otherwise} \end{cases}$$

Now it is straightforward to see that $\mathcal{S}$ with the above composition is the natural amalgamated groupoid $\{\mathcal{S}_{PS(m,k_i)} \mid 1 \leq i \leq n\}$ and injective homomorphisms $h_i : U \to \mathcal{S}_{PS(m,k_i)}$. Since for every $1 \leq i \leq n$, $\mathcal{S}_{PS(m,k_i)}$ is power-associative, the groupoid $\mathcal{S}$ is power-associative. Similarly to the proof of Lemma 2.2, we can show that the underlying undirected multigraph of $\mathrm{MCay}(\mathcal{S}, L) \setminus \{0\}$ is isomorphic to Λ, where $L = \cup_{i=1}^{n} L_{PS(m,k_i)}$. □

Note that for every set A, by $|A|$ we mean the cardinality of A. Now by the above discussion, we can conclude our main theorem.

Proof of Theorem 1.2 Let $\mathcal{F}$ be a g-fat-tree with m levels and root r. Let S be a monogenic semigroup with height $m + 2$ and period 1, which is generated by an element s. First, note that in Lemma 2.5, $k_1, \dots, k_n$ are arbitrary natural numbers and specially, the case $k_1 = \cdots = k_n = 1$ is permitted.

Choose an arbitrary branch B_1 of $\mathcal{F}$. For every $1 \leq i \leq m - 1$, let A_i be the set of vertices v at level i of $\mathcal{F}$ such that $\deg(v) > 2$ (equivalently, let A_i consists of all vertices at level i which are ancestor in common of some elements in lower levels). Let $A_{m-1} = \{x_1, \dots, x_{t_1}\}$. For every $1 \leq j \leq t_1$, let x_j be the ancestor in common of k_j leaves of $\mathcal{F}$ and suppose that $S_j = \mathcal{S}_{PS(m,k_j)}$. Let $C = \{a_1, \ldots, a_{s_1}\}$ be the set of vertices at level $m-1$ which are not in A_{m-1}. For every $1 \leq j \leq s_1$, let S_{j+t_1} be a monogenic semigroup with height $m+1$ and period 1, which is not generated by S_{j+t_1}. Note that $\mathrm{MCay}(S_{j+t_1}, \{s_{j+t_1}) \setminus \{0\}$ is isomorphic to $PS(m, 1)$. Now if $A_{m-2} \neq \emptyset$, then for each $x \in A_{m-2}$, we consider the corresponding groupoids of its children $\mathcal{S}_{PS(m,k_1)}, \ldots, \mathcal{S}_{PS(m,k_d)}$, where $d = \deg(x) - 1$ and using Lemma 2.5 we can construct a groupoid $\mathcal{S}_x$ and $L_x \subset \mathcal{S}_x$ such that $\mathrm{MCay}(\mathcal{S}_x, L_x) \setminus \{0\}$ is isomorphic to $(PS(m, k_1) \wedge \cdots \wedge PS(m, k_d); m-2)$. Now by repeating this process and using amalgamated groupoids, we can construct a groupoid $\mathcal{S}_{\mathcal{F}}$ and $L_{\mathcal{F}} \subset \mathcal{S}_{\mathcal{F}}$ which satisfies the conditions (i)-(v). □

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