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ORIGINAL RESEARCH

Applied neat reducts on some classes of algebras

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Abstract A significant focal idea presented in [2,3], is that of neat reducts. In this paper, we apply neat reducts on some classes of algebras like cylindric algebras, quasi-polyadic algebras and quasi-polyadic equality algebras.

Keywords Neat reducts · Substitution algebras · Quasi-polyadic algebras · Quasi-polyadic equality algebras · Cylindric algebras

Mathematics Subject Classification 03C05 · 03G27 · 03C48 · 03G15

1 Notations and definitions

The notion of neat reducts is an old venerable notion in algebraic logic which is introduced by Leon Henkin back in the fifties. Also, one can find that the discussion of this idea is exhaustive and itemized in [2,3]. This idea demonstrated valuable in at least two regards. Analyzing the number of variables showing up in proofs of first order formulas [5], and characterizing the class of representable algebras; those algebras that are isomorphic to veritable algebras of relations.

Our notation is in conformity with the three monographs on the subject [2–4]. A cylindric set algebra may be regarded as a kind of multi-dimensional Boolean set algebra. Indeed, associated with each cylindric set algebra is an ordinal number α which indicates the dimensionality of the algebra. Whereas the elements of a Boolean set algebra may be subsets of a quite arbitrary set V, the elements of a cylindric set algebra of dimension α are subsets of the cartesian power ${}^{\alpha}U$ of some set U. In a Boolean set algebra the empty set 0 and the unit set V are treated as distinguished elements. In a cylindric set algebra we treat as distinguished elements, not only the empty set 0 and the whole space ${}^{\alpha}U$, but also certain additional point sets D_{ij} (for any $i, j < \alpha$), called the *diagonal sets*. The set D_{ij} consists of all those points x of ${}^{\alpha}U$ whose i-th coordinate x_i equals the j-th coordinate x_j. Thus, in case $i \neq j$, the diagonal sets D_{ij} are seen to be the hyperplanes defined by the equations $x_i = x_j$.

In a cylindric set algebra, just as in a Boolean set algebra,we are concerned with the set-theoretical operations of forming unions $X \cup Y$, intersections $X \cap Y$, and complements $\sim X$ of members of the algebra. In the case of a Boolean set algebra these are the only fundamental operations. In a cylindric set algebra, however, we single out certain further fundamental operations called *cylindrifications*. In fact, with each ordinal $i < \alpha$ we correlate the unary operations C_i, the i-th cylindrification, which, when applied to a subset X of the space ${}^{\alpha}U$, produces the cylinder $\mathsf{C}_i X$ swept out by all translations of X parallel to the i-th coordinate axis; in other words, a point y

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of ${}^{\alpha}U$ belongs to the cylinder $\mathsf{C}_i X$ if and only if it can be obtained from some point x of X by changing at most the i-th coordinate. Thus, e.g., we have for any $i, j, \mu < \alpha$:

$$\begin{aligned}
&\mathsf{C}_i 0 = 0,\\
&\mathsf{C}_i {}^{\alpha}U = {}^{\alpha}U,\\
&\mathsf{C}_i \mathsf{D}_{ij} = \mathsf{C}_j \mathsf{D}_{ij} = {}^{\alpha}U,\\
&\mathsf{C}_\mu \mathsf{D}_{ij} = \mathsf{D}_{ij}, \quad \text{in case} \quad \mu \neq i, j.
\end{aligned}$$

Now let A be any family of subsets of ${}^{\alpha}U$ which contains as elements all the distinguished sets - $0, {}^{\alpha}U$, and D_{ij} for $i, j < \alpha$ - and which is closed under all the fundamental operations $\cup, \cap, \setminus$ and C_i, for $i < j$.

Definition 1.1 [2] The structure $\mathfrak{A} = (A, \cup, \cap, \setminus, 0, {}^{\alpha}U, \mathsf{C}_i, \mathsf{D}_{ij})_{i,j<\alpha}$ is then called a *cylindric set algebra of dimension* α; A is, of course, the universe (the set of all elements) of this algebra.

We use the notation $\mathbf{Cs}_\alpha$ to refer to the *class of all cylindric set algebra of dimension* α. There are many equations that hold identically in every cylindric set algebra of a given dimension; in addition to all of the Boolean identities, we may mention:

$$\begin{aligned}
&\mathsf{C}_i 0 = 0,\\
&X \cap \mathsf{C}_i X = X,\\
&\mathsf{C}_i (X \cap \mathsf{C}_i Y) = \mathsf{C}_i X \cap \mathsf{C}_i Y
\end{aligned}$$

as simple examples. Certain of these identities have been selected to serve as axioms for the abstractly defined notion of cylindric algebras. Thus, a cylindric algebra of dimension α ($\mathbf{CA}_\alpha$, for short) can be described as any system $(B, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha}$ similar to $\mathbf{Cs}_\alpha$'s which satisfies a certain set of prescribed equational identities. The identities which serve as axioms for defining $\mathbf{CA}_\alpha$'s have been selected from all identities holding in every $\mathbf{Cs}_\alpha$ because they are simple. More precisely:

Definition 1.2 [2] By *a cylindric algebra of dimension* α, where α is any ordinal number, we mean an algebraic structure

$$\mathfrak{A} = (A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha}$$

where $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra such that 0, 1, and d_{ij} are distinguished elements of A (for all $i, j < \alpha$), $-$ and c_i are unary operations on A (for all $i < \alpha$), $+$ and $\cdot$ are binary operations on A, and such that the following postulates are satisfies for any $x, y \in A$ and any $i, j, \mu < \alpha$:

(C_1) $\mathsf{c}_i 0 = 0$;
(C_2) $x \leq \mathsf{c}_i x$ $(i.e., x + \mathsf{c}_i x = \mathsf{c}_i x)$;
(C_3) $\mathsf{c}_i (x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$;
(C_4) $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$;
(C_5) $\mathsf{d}_{ii} = 1$;
(C_6) if $i \neq j, \mu$, then $\mathsf{d}_{j\mu} = \mathsf{c}_i (\mathsf{d}_{ji} \cdot \mathsf{d}_{i\mu})$;
(C_7) if $i \neq j$, then $c_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = 0$.

A fundamental notion in the representation theory of cylindric algebras is that of neat reducts. Since it will also play a role in connection with some of the amalgamation results addressed herein, we recall this notion. A neat reduct of a cylindric algebra A, is an algebra of lesser dimension obtained from A by overlooking some of its operations and discarding some of its elements. Neat reducts in cylindric algebra have received a lot of attention recently (see [7–11]). More precisely:

Definition 1.3 [2]

(i) Let $\alpha < \beta$ be ordinals and let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{i,j<\beta}$ be a $\mathbf{CA}_\beta$. For $x \in A$, Δx, referred to as the *dimension set of* x (cf. [2] 1.6.1) is defined to be $\{i \in \beta : c_i x \neq x\}$. We now recall from [2] definition 2.6.28 that the *neat* α*-reduct* of $\mathfrak{A}$, or $\mathcal{N}r_\alpha \mathfrak{A}$ for short, is the $\mathbf{CA}_\alpha$ whose domain is the set of all α*-dimensional* elements of $\mathfrak{A}$ defined by

$$Nr_\alpha A = \{x \in A : \Delta x \subseteq \alpha\}.$$

The operations of $\mathcal{N}r_\alpha\mathfrak{A}$ are those of the α-dimensional reduct $\mathfrak{Rd}_\alpha\mathfrak{A} = \langle A, +^{\mathfrak{A}}, -^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, \mathsf{c}_i^{\mathfrak{A}}, \mathsf{d}_{ij}^{\mathfrak{A}}\rangle_{i,j<\alpha}$ of $\mathfrak{A}$, restricted to $\mathcal{N}r_\alpha\mathfrak{A}$. When no confusion is likely to ensue, we omit the superscript $\mathfrak{A}$. $Nr_\alpha A$, as easily checked, is closed under the indicated operations and thus is a $\mathbf{CA}_\alpha$. $\mathcal{N}r_\alpha\mathfrak{A}$ is thus a special subreduct of $\mathfrak{A}$, i.e., a special subalgebra of a reduct in the universal algebraic sense.

(ii) When $\mathfrak{A} \in \mathbf{CA}_\alpha$ is a subalgebra of $\mathcal{N}r_\alpha\mathfrak{B}$ for some $\mathfrak{B} \in \mathbf{CA}_\beta$, then we say that $\mathfrak{A}$ neatly embeds in $\mathfrak{B}$. If furthermore A generates $\mathfrak{B}$, then we say that $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$.

(iii) For a class $L \subseteq \mathbf{CA}_\beta$, we let $Nr_\alpha L$ be the following subclass of $\mathbf{CA}_\alpha$:

$$Nr_\alpha L = \{\mathcal{N}r_\alpha\mathfrak{A} : \mathfrak{A} \in L\}.$$

We start by defining the concrete versions of Pinter's algebras [6]. Let α be an ordinal. Let U be a set. Then we define for $i, j < \alpha$ and $X \subseteq {}^\alpha U$:

$$\begin{aligned}
\mathsf{c}_i X &= \{s \in {}^\alpha U : \exists t \in X, t(j) = t(i) \text{ for all } i \neq j\},\\
\mathsf{s}_i^j X &= \{s \in {}^\alpha U : s \circ [i|j] \in X\},\\
\mathsf{s}_{[i,j]} X &= \{s \in {}^\alpha U : s \circ [i,j] \in X\}\\
\mathsf{d}_{ij} &= \{s \in {}^\alpha U : s_i = s_j\}
\end{aligned}$$

$[i|j]$ is the replacement on α that takes i to j and leaves every other thing fixed, while $[i, j]$ is the transposition interchanging i and j. The extra non -Boolean operations we deal with are as specified above.

Definition 1.4 [2] By *a substitution algebras of dimension* α, briefly a $\mathbf{SC}_\alpha$ such that α is any ordinal number, we mean an algebraic structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{s}_i^j\rangle_{i,j<\alpha}$$

where $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra and $\mathsf{c}_i, \mathsf{s}_i^j$ are unary operations on A $(i, j < \alpha)$ satisfying the following equations for all $i, j, \mu, l < \alpha$:

(s_1) $\mathsf{c}_i 0 = 0$, $x \leq \mathsf{c}_i x$, $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$, and $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$,
(s_2) $\mathsf{s}_i^i x = x$,
(s_3) s_j^i are boolean endomorphisms,
(s_4) $\mathsf{s}_j^i\mathsf{c}_i x = \mathsf{c}_i x$,
(s_5) $\mathsf{c}_i\mathsf{s}_j^i x = \mathsf{s}_j^i x$ whenever $i \neq j$,
(s_6) $\mathsf{s}_j^i\mathsf{c}_\mu x = \mathsf{c}_\mu\mathsf{s}_j^i x$, whenever $\mu \notin \{i, j\}$,
(s_7) $\mathsf{c}_i\mathsf{s}_i^j x = \mathsf{c}_j\mathsf{s}_j^i x$,
(s_8) $\mathsf{s}_i^j\mathsf{s}_\mu^l x = \mathsf{s}_\mu^l\mathsf{s}_i^j x$, whenever $|\{i, j, \mu, l\}| = 4$,
(s_9) $\mathsf{s}_i^l\mathsf{s}_l^j x = \mathsf{s}_i^l\mathsf{s}_i^j x$.

Now, let **QA** denotes quasi-polyadic algebras and **QEA** denotes quasi-polyadic algebras with equality. For U a set and $X \subseteq {}^\alpha U$ and $i, j < \alpha$,

$$\mathsf{p}_{ij} X = \{s \in {}^\alpha U : s \circ [i, j] \in X\}.$$

Here $[i, j]$ is the transposition interchanging i and j. On the other hand, the diagonal element d_{ij} is the set $\{s \in {}^\alpha U : s_i = s_j\}$. More precisely:

Definition 1.5 [1] By a quasi polyadic algebra of dimension α, briefly a $\mathbf{QA}_\alpha$, we mean an algebra

$$\mathfrak{A} = (A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{s}_j^i, \mathsf{p}_{ij})_{i,j<\alpha}$$

where $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra, c_i, s_j^i and p_{ij} are unary operations on A, (for $i, j < \alpha$) and the following postulates are satisfied for all $i, j, \mu < \alpha$:

(Q_1) $\mathsf{s}_i^i = \mathsf{p}_{ii} = Id$, and $\mathsf{p}_{ij} = \mathsf{p}_{ji}$,
(Q_2) $x \leq \mathsf{c}_i x$,
(Q_3) $\mathsf{c}_i(x + y) = \mathsf{c}_i x + \mathsf{c}_i y$,

(Q_4) $\mathsf{s}^i_j\mathsf{c}_i x = \mathsf{c}_i x$,
(Q_5) $\mathsf{c}_i\mathsf{s}^i_j x = \mathsf{s}^i_j x$ if $i \neq j$,
(Q_6) $\mathsf{s}^i_j\mathsf{c}_\mu x = \mathsf{c}_\mu\mathsf{s}^i_j x$ if $\mu \notin \{i, j\}$,
(Q_7) s^i_j and p_{ij} are Boolean endomorphisms,
(Q_8) $\mathsf{p}_{ij}\mathsf{p}_{ij} x = x$,
(Q_9) $\mathsf{p}_{ij}\mathsf{p}_{i\mu} = \mathsf{p}_{j\mu}\mathsf{p}_{ij} x$ if $|\{i, j, \mu\}| = 3$,
(Q_{10}) $\mathsf{p}_{ij}\mathsf{s}^j_i x = \mathsf{s}^i_j x$.

Definition 1.6 [1] By a quasi polyadic equality algebra of dimension α, briefly a $\mathbf{QEA}_\alpha$, we mean an algebra

$$\mathfrak{B} = (\mathfrak{A}, \mathsf{d}_{ij})_{i,j<\alpha}$$

where $\mathfrak{A}$ is a $\mathbf{QA}_\alpha$, d_{ij} is a constant (for $i, j < \alpha$) and the following postulates are satisfied for all $i, j, k < \alpha$:

(Q_{11}) $\mathsf{s}^i_j\mathsf{d}_{ij} = 1$,
(Q_{12}) $x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j x$.

2 Basic results

In this section, we give some examples of applying the neat reduct on classes of algebras like cylindric algebras, quasi-polyadic algebras and quasi-polyadic equality algebras and get another algebras.

Proposition 2.1 *In every cylindric algebra* $\mathfrak{A} = (A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha}$, *the following properties holds for any* $x, y \in A$ *and any* $i, j, \mu < \alpha$:

(a) $\mathsf{d}_{ij} = \mathsf{d}_{ji}$
(b) $\mathsf{c}_i\mathsf{c}_i y = \mathsf{c}_i y$.
(c) $\mathsf{c}_\mu\mathsf{d}_{ij} = \mathsf{d}_{ij}$.
(d) $\mathsf{c}_i\mathsf{d}_{ij} = 1$.
(e) $x \cdot \mathsf{c}_i y = o$ *iff* $y \cdot \mathsf{c}_i x = 0$.
(f) $\mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu} = \mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu}$, *whenever* $j \notin \{i, \mu\}$

Proof (a) $\mathsf{d}_{ij} = \mathsf{d}_{ji}$

Replacing x with d_{ji} in (C_7) we get $c_i(\mathsf{d}_{ij} \cdot \mathsf{d}_{ji}) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -\mathsf{d}_{ji}) = 0$. But by (C_6), (C_5) we have

$$\mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{d}_{ji}) = \mathsf{d}_{ii} = 1 \tag{1}$$

Also, since we have from (C_1), (C_2)

$$\mathsf{c}_i x = 0 \quad \text{iff} \quad x = 0. \tag{2}$$

Hence, from (2), (1) we obtain $\mathsf{d}_{ij} \cdot -\mathsf{d}_{ji} = 0$ i.e., $\mathsf{d}_{ij} \leq \mathsf{d}_{ji}$.

Similarly, we can get the opposite inclusion.

(b) $\mathsf{c}_i\mathsf{c}_i y = \mathsf{c}_i y$.

Taking $x = 1$ in (C_3) we get (b), since $\mathsf{c}_i 1 = 1$ from (C_2).

(c) $\mathsf{c}_\mu\mathsf{d}_{ij} = \mathsf{d}_{ij}$.

We get (c) by using (C_6) and (b) as follows

$$\mathsf{c}_\mu\mathsf{d}_{ij} = \mathsf{c}_\mu\mathsf{c}_\mu(\mathsf{d}_{i\mu} \cdot \mathsf{d}_{\mu j}) = \mathsf{c}_\mu(\mathsf{d}_{i\mu} \cdot \mathsf{d}_{\mu j}) = \mathsf{d}_{ij}$$

(d) $\mathsf{c}_i\mathsf{d}_{ij} = 1$.

We get (d) by using (a), (C_6) and (C_5) as follows

$$\mathsf{c}_i\mathsf{d}_{ij} = \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{d}_{ij}) = \mathsf{c}_i(\mathsf{d}_{ji} \cdot \mathsf{d}_{ij}) = \mathsf{d}_{jj} = 1$$

(e) $x.\mathsf{c}_i y = o$ iff $y.\mathsf{c}_i x = 0$.

By (2) and (C_3) we obtain $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y = \mathsf{c}_i(y \cdot \mathsf{c}_i x)$. Then, we get (e) by using (2).

(f) $\mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu} = \mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu}$, whenever $j \notin \{i, \mu\}$
From C_2 and C_6 we get $\mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu} \leq \mathsf{c}_j(\mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu}) = \mathsf{d}_{i\mu}$, so we obtain

$$\mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu} \leq \mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu}. \tag{3}$$

Similarly, again from C_2, (a) and C_6 we get $\mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu} \leq \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu}) = \mathsf{d}_{j\mu}$, so we get

$$\mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu} \leq \mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu}. \tag{4}$$

Thus, from (3) and (4) we obtain $\mathsf{d}_{ij} \cdot \mathsf{d}_{j\mu} = \mathsf{d}_{ij} \cdot \mathsf{d}_{i\mu}$ □

Theorem 2.2 *If* $\mathfrak{A} \in \mathbf{CA}_\alpha$ *, then* $\mathfrak{Rd}_{\mathbf{SC}}\mathfrak{A} \models (s_1) - (s_9)$

Proof We define

$$\mathsf{s}^i_j x = \begin{cases} \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) & i \neq j \\ x & i = j \end{cases}$$

We note that C_1, C_2, C_3 and C_4 are in s_1. So, the rest axioms as follow: (s_4)

$$\begin{aligned} \mathsf{s}^i_j \mathsf{c}_i x &= \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{c}_i x) \\ &= \mathsf{c}_i \mathsf{d}_{ij} \cdot \mathsf{c}_i x \quad \text{by} \quad (C_3) \\ &= 1 \cdot \mathsf{c}_i x \quad \text{by} \quad (d) \\ &= \mathsf{c}_i x. \end{aligned}$$

(s_5)

$$\begin{aligned} \mathsf{c}_i \mathsf{s}^i_j x &= \mathsf{c}_i \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \\ &= \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \quad \text{by} \quad (b) \\ &= \mathsf{s}^i_j x. \end{aligned}$$

(s_6)

$$\begin{aligned} \mathsf{s}^i_j \mathsf{c}_\mu x &= \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{c}_\mu x) \\ &= \mathsf{c}_i(\mathsf{c}_\mu \mathsf{d}_{ij} \cdot \mathsf{c}_\mu x) \quad \text{by} \quad (c) \\ &= \mathsf{c}_i \mathsf{c}_\mu(\mathsf{c}_\mu \mathsf{d}_{ij} \cdot x) \quad \text{by} \quad (C_3) \\ &= \mathsf{c}_i \mathsf{c}_\mu(\mathsf{d}_{ij} \cdot x) \quad \text{by} \quad (c) \\ &= \mathsf{c}_\mu \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \quad \text{by} \quad (C_4) \\ &= \mathsf{c}_\mu \mathsf{s}^i_j x \end{aligned}$$

(s_7)

$$\begin{aligned} \mathsf{c}_i \mathsf{s}^j_i x &= \mathsf{c}_i \mathsf{c}_j(\mathsf{d}_{ji} \cdot x) \\ &= \mathsf{c}_j \mathsf{c}_i(\mathsf{d}_{ji} \cdot x) \quad \text{by} \quad (C_4) \\ &= \mathsf{c}_j \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \quad \text{by} \quad (a) \\ &= \mathsf{c}_j \mathsf{s}^i_j x \end{aligned}$$

(s_8)

$$\begin{aligned} \mathsf{s}^j_i \mathsf{s}^l_\mu x &= \mathsf{c}_j(\mathsf{s}^l_\mu x \cdot \mathsf{d}_{ji}) \\ &= \mathsf{c}_j(\mathsf{c}_l(x \cdot \mathsf{d}_{l\mu}) \cdot \mathsf{d}_{ji}) \\ &= \mathsf{c}_j(\mathsf{c}_l(x \cdot \mathsf{d}_{l\mu}) \cdot \mathsf{c}_l \mathsf{d}_{ji}) \quad \text{by} \quad (c) \\ &= \mathsf{c}_j \mathsf{c}_l((x \cdot \mathsf{d}_{l\mu}) \cdot \mathsf{c}_l \mathsf{d}_{ji}) \quad \text{by} \quad (C_3) \\ &= \mathsf{c}_l \mathsf{c}_j(x \cdot \mathsf{d}_{l\mu} \cdot \mathsf{d}_{ji}) \quad \text{by} \quad (C_4), (c) \\ &= \mathsf{c}_l \mathsf{c}_j(x \cdot \mathsf{d}_{ij} \cdot \mathsf{d}_{l\mu}) \quad \text{by} \quad (a) \\ &= \mathsf{c}_l \mathsf{c}_j((x \cdot \mathsf{d}_{ij}) \cdot \mathsf{c}_j \mathsf{d}_{l\mu}) \quad \text{by} \quad (c) \\ &= \mathsf{c}_l(\mathsf{c}_j(x \cdot \mathsf{d}_{ij}) \cdot \mathsf{c}_j \mathsf{d}_{l\mu}) \quad \text{by} \quad (C_3) \\ &= \mathsf{c}_l(\mathsf{s}^j_i x \cdot \mathsf{d}_{l\mu}) \quad \text{by} \quad (c) \\ &= \mathsf{s}^l_\mu \mathsf{s}^j_i x \end{aligned}$$

(s_9)

$$\begin{aligned}
\mathsf{s}^l_i\mathsf{s}^j_l x &= \mathsf{c}_l(\mathsf{s}^j_l x\cdot \mathsf{d}_{li})\\
&= \mathsf{c}_l(\mathsf{c}_j(x\cdot \mathsf{d}_{jl})\cdot \mathsf{d}_{li})\\
&= \mathsf{c}_l(\mathsf{c}_j(x\cdot \mathsf{d}_{jl})\cdot \mathsf{c}_j\mathsf{d}_{li}) \quad \text{by} \quad (c)\\
&= \mathsf{c}_l\mathsf{c}_j((x\cdot \mathsf{d}_{jl})\cdot \mathsf{c}_j\mathsf{d}_{li}) \quad \text{by} \quad (C_3)\\
&= \mathsf{c}_l\mathsf{c}_j(x\cdot \mathsf{d}_{jl}\cdot \mathsf{d}_{li}) \quad \text{by} \quad (c)\\
&= \mathsf{c}_l\mathsf{c}_j(x\cdot \mathsf{d}_{ji}\cdot \mathsf{d}_{li}) \quad \text{by} \quad (f)\\
&= \mathsf{c}_l\mathsf{c}_j((x\cdot \mathsf{d}_{ji})\cdot \mathsf{c}_j\mathsf{d}_{li}) \quad \text{by} \quad (c)\\
&= \mathsf{c}_l(\mathsf{c}_j(x\cdot \mathsf{d}_{ji})\cdot \mathsf{c}_j\mathsf{d}_{li}) \quad \text{by} \quad (C_3)\\
&= \mathsf{c}_l(\mathsf{c}_j(x\cdot \mathsf{d}_{ji})\cdot \mathsf{d}_{li}) \quad \text{by} \quad (c)\\
&= \mathsf{c}_l(\mathsf{s}^j_i x\cdot \mathsf{d}_{li})\\
&= \mathsf{s}^l_i\mathsf{s}^j_i x
\end{aligned}$$

Finally, we want to prove that s^i_j is Boolean endomorphisms. To do this we first need to show that

$$\mathsf{c}_i(x+y) = \mathsf{c}_i x + \mathsf{c}_i y \tag{5}$$

as follows: Since $\mathsf{c}_i(x+y)\cdot -\mathsf{c}_i(x+y) = 0$ we see that $(x+y)\cdot \mathsf{c}_i - \mathsf{c}_i(x+y) = 0$ by (e). Hence, $x\cdot \mathsf{c}_i - \mathsf{c}_i(x+y) = 0$ and $y\cdot \mathsf{c}_i - \mathsf{c}_i(x+y) = 0$. Also, by (e) we have $\mathsf{c}_i x\cdot -\mathsf{c}_i(x+y) = 0$, $\mathsf{c}_i y\cdot -\mathsf{c}_i(x+y) = 0$ which implies to $\mathsf{c}_i x \le \mathsf{c}_i(x+y)$, $\mathsf{c}_i y \le \mathsf{c}_i(x+y)$. So, $\mathsf{c}_i(x+y)$ is an upper bound of $\mathsf{c}_i x$, $\mathsf{c}_i y$. Now, let z be any other upper bound of these elements, i.e., $\mathsf{c}_i x \le z$, $\mathsf{c}_i y \le z$. Then, $\mathsf{c}_i x\cdot -z = 0$, $\mathsf{c}_i y\cdot -z = 0$, so $x\cdot \mathsf{c}_i - z = 0$, $y\cdot \mathsf{c}_i - z = 0$ by (e). It follows that $(x+y)\cdot \mathsf{c}_i - z = 0$ and apply (e) again we find $\mathsf{c}_i(x+y)\cdot -z = 0$ which means $\mathsf{c}_i(x+y)\le z$. Thus we obtain (5).

Also, we show

$$\mathsf{c}_i(\mathsf{d}_{ij}\cdot -x) = -\mathsf{c}_i(\mathsf{d}_{ij}\cdot x) \tag{6}$$

From (C_7) we have $\mathsf{c}_i(\mathsf{d}_{ij}\cdot -x)\cdot \mathsf{c}_i(\mathsf{d}_{ij}\cdot x) = 0$ and from (5), (d) we have $\mathsf{c}_i(\mathsf{d}_{ij}\cdot -x) + \mathsf{c}_i(\mathsf{d}_{ij}\cdot x) = \mathsf{c}_i((\mathsf{d}_{ij}\cdot -x) + (\mathsf{d}_{ij}\cdot x)) = \mathsf{c}_i\mathsf{d}_{ij} = 1$. Hence, we get (6) since $\mathfrak{A}$ is Boolean algebra.

Thus, we have

- $\mathsf{s}^i_j - x = \mathsf{c}_i(\mathsf{d}_{ij}\cdot -x) = -\mathsf{c}_i(\mathsf{d}_{ij}\cdot x) = -\mathsf{s}^i_j x$ by (6).
- $\mathsf{s}^i_j(x+y) = \mathsf{c}_i(\mathsf{d}_{ij}\cdot(x+y)) = \mathsf{c}_i((\mathsf{d}_{ij}\cdot x) + (\mathsf{d}_{ij}\cdot y)) = \mathsf{c}_i(\mathsf{d}_{ij}\cdot x) + \mathsf{c}_i(\mathsf{d}_{ij}\cdot y) = \mathsf{s}^i_j x + \mathsf{s}^i_j y$ by (5).
- $\mathsf{s}^i_j 0 = \mathsf{c}_i(\mathsf{d}_{ij}\cdot 0) = \mathsf{c}_i 0 = 0$ by (C_1).
- $\mathsf{s}^i_j 1 = \mathsf{c}_i(\mathsf{d}_{ij}\cdot 1) = \mathsf{c}_i\mathsf{d}_{ij} = 1$ by (d).

So, we get (s_3). Thus, the proof is completed. □

Proposition 2.3 *In every quasi-polyadic algebra $\mathfrak{A} = (A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{s}^i_j, \mathsf{p}_{ij})_{i,j<\alpha}$, the following properties holds for any $x \in A$ and any $i, j, \mu < \alpha$:*

(i) $\mathsf{s}^i_j\mathsf{s}^i_j x = \mathsf{s}^i_j x$.
(ii) $\mathsf{s}^i_j\mathsf{s}^j_i x = \mathsf{s}^i_j x$.
(iii) $\mathsf{s}^j_i\mathsf{s}^j_\mu x = \mathsf{s}^j_\mu x$.
(iv) $\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{j\mu} x = \mathsf{s}^i_j\mathsf{s}^\mu_j x$.
(v) $\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{ji} x = \mathsf{s}^i_j\mathsf{s}^\mu_j x$.
(vi) $\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{\mu i} x = \mathsf{s}^i_j\mathsf{s}^\mu_j x$.

Proof (i) $\mathsf{s}^i_j\mathsf{s}^i_j x = \mathsf{s}^i_j x$.

$$\begin{aligned}
\mathsf{s}^i_j\mathsf{s}^i_j x &= \mathsf{s}^i_j\mathsf{c}_i\mathsf{s}^i_j x \quad \text{by} \quad (Q_5)\\
&= \mathsf{c}_i\mathsf{s}^i_j x \quad \text{by} \quad (Q_4)\\
&= \mathsf{s}^i_j x \quad \text{by} \quad (Q_5).
\end{aligned}$$

(ii) $\mathsf{s}^i_j \mathsf{s}^j_i x = \mathsf{s}^i_j x$.

$$\begin{aligned}\mathsf{s}^i_j \mathsf{s}^j_i x &= \mathsf{p}_{ij} \mathsf{s}^j_i \mathsf{s}^j_i x \quad \text{by} \quad (Q_{10})\\ &= \mathsf{p}_{ij} \mathsf{s}^j_i x \quad \text{by} \quad \text{(i)}\\ &= \mathsf{s}^i_j x \quad \text{by} \quad (Q_{10}).\end{aligned}$$

(iii) $\mathsf{s}^j_i \mathsf{s}^j_\mu x = \mathsf{s}^j_\mu x$.

$$\begin{aligned}\mathsf{s}^j_i \mathsf{s}^j_\mu x &= \mathsf{s}^j_i \mathsf{c}_j \mathsf{s}^j_\mu x \quad \text{by} \quad (Q_5)\\ &= \mathsf{c}_j \mathsf{s}^j_\mu x \quad \text{by} \quad (Q_4)\\ &= \mathsf{s}^j_\mu x \quad \text{by} \quad (Q_5).\end{aligned}$$

(iv) $\mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} x = \mathsf{s}^i_j \mathsf{s}^\mu_j x$. Since,

$$\begin{aligned}\mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu}(x \cdot -\mathsf{s}^j_i x) &\leq \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \mathsf{c}_\mu (x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad (Q_2)\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \mathsf{s}^\mu_j \mathsf{c}_\mu (x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad (Q_4)\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{s}^j_\mu \mathsf{c}_\mu (x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad (Q_{10})\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{c}_\mu (x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad \text{(ii)}\\ &= \mathsf{s}^i_j \mathsf{c}_\mu (x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad (Q_4)\\ &= \mathsf{c}_\mu \mathsf{s}^i_j (x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad (Q_6)\\ &= \mathsf{c}_\mu (\mathsf{s}^i_j x \cdot \mathsf{s}^i_j(-\mathsf{s}^j_i x)) \quad \text{by} \quad (Q_7)\\ &= \mathsf{c}_\mu (\mathsf{s}^i_j x \cdot -\mathsf{s}^i_j(\mathsf{s}^j_i x)) \quad \text{by} \quad (Q_7)\\ &= \mathsf{c}_\mu (\mathsf{s}^i_j x \cdot -\mathsf{s}^i_j x) \quad \text{by} \quad \text{(ii)}\\ &= \mathsf{c}_\mu 0\\ &= 0 \quad \text{by} \quad (s_1).\end{aligned}$$

Hence, we show that

$$\mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \left(x \cdot -\mathsf{s}^j_i x\right) = 0 \quad \text{by} \quad (Q_7). \tag{7}$$

Thus,

$$\begin{aligned}\mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} x &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \left(\left(x \cdot \mathsf{s}^j_i x\right) + \left(x \cdot -\mathsf{s}^j_i x\right)\right)\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \left(x \cdot \mathsf{s}^j_i x\right) + \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu}(x \cdot -\mathsf{s}^j_i x) \quad \text{by} \quad (Q_7)\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu}(x \cdot \mathsf{s}^j_i x) + 0 \quad \text{by} \quad (7)\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} x \cdot \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \mathsf{s}^j_i x \quad \text{by} \quad (Q_7).\end{aligned}$$

Then

$$\begin{aligned}\mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} x &\leq \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \mathsf{s}^j_i x\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} \mathsf{s}^j_\mu \mathsf{s}^j_i x \quad \text{by} \quad \text{(iii)}\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{s}^\mu_j \mathsf{s}^j_i x \quad \text{by} \quad (Q_{10})\\ &= \mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{s}^j_i x \quad \text{by} \quad \text{(i)}\\ &= \mathsf{s}^\mu_j \mathsf{s}^i_j \mathsf{s}^j_i x \quad \text{by} \quad (s_8)\\ &= \mathsf{s}^\mu_j \mathsf{s}^i_j x \quad \text{by} \quad \text{(ii)}.\end{aligned}$$

Thus,

$$\mathsf{s}^i_j \mathsf{s}^\mu_j \mathsf{p}_{j\mu} x \leq \mathsf{s}^i_j \mathsf{s}^\mu_j x \quad \text{by} \quad (s_8). \tag{8}$$

In (8) replacing x with $\mathsf{p}_{j\mu}x$ we get

$$\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{j\mu}\mathsf{p}_{j\mu}x \leq \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{j\mu}x.$$

So, by (Q_8) we obtain

$$\mathsf{s}^i_j\mathsf{s}^\mu_j x \leq \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{j\mu}x. \tag{9}$$

From (8), (9) we get (iv).

(v) $\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{ji}x = \mathsf{s}^i_j\mathsf{s}^\mu_j x$.

$$\begin{aligned}\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{ji}x &= \mathsf{s}^\mu_j\mathsf{s}^i_j\mathsf{p}_{ji}x \quad \text{by} \quad (s_8)\\ &= \mathsf{s}^\mu_j\mathsf{s}^i_j x \quad \text{by} \quad \text{(iv)}\\ &= \mathsf{s}^i_j\mathsf{s}^\mu_j x \quad \text{by} \quad (s_8).\end{aligned}$$

(vi) $\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{\mu i}x = \mathsf{s}^i_j\mathsf{s}^\mu_j x$.

$$\begin{aligned}\mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{\mu i}x &= \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{\mu i}\mathsf{p}_{j\mu}\mathsf{p}_{j\mu}x \quad \text{by} \quad (Q_8)\\ &= \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{j\mu}\mathsf{p}_{ji}\mathsf{p}_{j\mu}x \quad \text{by} \quad (Q_9)\\ &= \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{ji}\mathsf{p}_{j\mu}x \quad \text{by} \quad \text{(iv)}\\ &= \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{p}_{j\mu}x \quad \text{by} \quad \text{(v)}\\ &= \mathsf{s}^i_j\mathsf{s}^\mu_j x \quad \text{by} \quad \text{(iv)}.\end{aligned}$$

□

Theorem 2.4 *If* $\mathfrak{A} \in \mathbf{QA}_\alpha$ *, then* $\mathfrak{Rd}_{\mathbf{SC}}\mathfrak{A} \models (s_1) - (s_9)$

Proof We observe that $x \leq \mathsf{c}_i x$ in (s_1), (s_2), (s_3), (s_4), (s_5) and (s_6) are (Q_2), (Q_1), (Q_7), (Q_4), (Q_5) and (Q_6) respectively. So, we need to show the rest of (s_1), $(s_7) - (s_9)$.

- $\mathsf{c}_i 0 = 0$.
  Since s^i_j is Boolean endomorphism by (Q_7), so $\mathsf{s}^i_j 0 = 0$ and substitute in (Q_5) we get $\mathsf{c}_i 0 = 0$.
- $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$.
  By (Q_3) we see that

$$\text{if} \quad x \leq y \quad \text{then} \quad \mathsf{c}_i x \leq \mathsf{c}_i y \tag{10}$$

Choose $k \notin \{i, j\}$. Then

$$\begin{aligned}\mathsf{c}_i\mathsf{c}_j\mathsf{c}_i x &= \mathsf{c}_i\mathsf{c}_j\mathsf{s}^i_k\mathsf{c}_i x \quad \text{by} \quad (Q_4)\\ &= \mathsf{c}_i\mathsf{s}^i_k\mathsf{c}_j\mathsf{c}_i x \quad \text{by} \quad (Q_6)\\ &= \mathsf{s}^i_k\mathsf{c}_j\mathsf{c}_i x \quad \text{by} \quad (Q_5)\\ &= \mathsf{c}_j\mathsf{s}^i_k\mathsf{c}_i x \quad \text{by} \quad (Q_6)\\ &= \mathsf{c}_j\mathsf{c}_i x \quad \text{by} \quad (Q_4)\end{aligned} \tag{11}$$

And from (Q_2) we have

$$\mathsf{c}_j x \leq \mathsf{c}_j\mathsf{c}_i x, \qquad \mathsf{c}_i\mathsf{c}_j x \leq \mathsf{c}_i\mathsf{c}_j\mathsf{c}_i x \tag{12}$$

So, from (11), (12) we get

$$\mathsf{c}_i\mathsf{c}_j x \leq \mathsf{c}_j\mathsf{c}_i x \tag{13}$$

Since (13) satisfies for all choices of i, j, so

$$\mathsf{c}_j\mathsf{c}_i x \leq \mathsf{c}_i\mathsf{c}_j x \tag{14}$$

Hence, from (13), (14) we get

$$\mathsf{c}_j\mathsf{c}_i x = \mathsf{c}_i\mathsf{c}_j x. \tag{15}$$

- $c_i(x \cdot c_i y) = c_i x \cdot c_i y$.
  From (Q_4) and (Q_2) we observe that $c_i x \in Rg(s^i_j)$. If $y \in Rg(s^i_j)$, then for any $z \in A$ and by (Q_5) we have

$$y = s^i_j z = c_i s^i_j z = s_i y.$$

So, $x \leq y$ implies to $c_i x \leq c_i y = y$ by (10). Hence, $c_i x = min\{y \in Rg(s^i_j) : y \geq x\}$ whenever $i \neq j$. Thus

$$c_i c_i x \leq c_i x \tag{16}$$

And by (Q_2), (10) we have

$$c_i x \leq c_i c_i x \tag{17}$$

Then we get form (16), (17)

$$c_i x = c_i c_i x \tag{18}$$

Also, we have

$$\begin{aligned} s^i_j(-c_i x) &= -s^i_j c_i x \quad \text{by} \quad (Q_7) \\ &= -c_i x \quad \text{by} \quad (Q_4) \end{aligned}$$

Thus $-c_i x \in Rg(s^i_j)$. Therefore

$$c_i(-c_i x) = min\{y \in Rg(s^i_j) : y \geq -c_i x\} = -c_i x \tag{19}$$

Now since $x \cdot c_i y \leq c_i y$, so $c_i(x \cdot c_i y) \leq c_i c_i y$. Hence, by (18) we get

$$c_i(x \cdot c_i y) \leq c_i y. \tag{20}$$

Similarly, from (19) we obtain

$$c_i(x \cdot -c_i y) \leq c_i y(-c_i y) = -c_i y. \tag{21}$$

By (Q_3) we see that

$$c_i x = c_i(x \cdot c_i y + x \cdot -c_i y) = c_i(x \cdot c_i y) + c_i(x \cdot -c_i y).$$

Multiplying both sides by $c_i y$ and using Boolean distributively gives us

$$c_i x \cdot c_i y = c_i(x \cdot c_i y) \cdot c_i y + c_i(x \cdot -c_i y) \cdot c_i y \tag{22}$$

which implies to $c_i x \cdot c_i y = c_i(x \cdot c_i y)$ because the first part of (22) is equal $c_i(x \cdot c_i y)$ by using (20) and the second part is equal 0 by (21).

(s_8) $s^j_i s^l_\mu = s^l_\mu s^j_i$. We recall from [6] the following lemma:

Lemma 2 *Let f and g are Boolean homomorphisms such that $ff = f$, $gg = g$. If $Rg(f)$ and $ker(f)$ closed under g, then $fg = gf$.*

We will apply Lemma 2 to s^l_μ and s^j_i.
Firstly, if $l \notin \{i, j, \mu\}$, then

$$\begin{aligned} s^j_i s^l_\mu x &= s^j_i c_l s^l_\mu x \quad \text{by} \quad (Q_5) \\ &= c_l s^j_i s^l_\mu x \quad \text{by} \quad (Q_6) \\ &= s^l_\mu c_l s^j_i s^l_\mu x \quad \text{by} \quad (Q_4) \\ &\in Rg(s^l_\mu). \end{aligned}$$

Thus $Rg(s^l_\mu)$ closed under s^j_i.

Secondly, if $j \notin \{\mu, l\}$ and $s^l_\mu x = 0$. Then $c_j s^l_\mu x = c_j 0 = 0$ by (s_1). So, by (Q_6) we get

$$s^l_\mu c_j x = 0. \tag{23}$$

Now from (Q_2), (Q_7) and (Q_4) we obtain $\mathsf{s}_i^j x \leq \mathsf{s}_i^j \mathsf{c}_j x = \mathsf{c}_j x$. So, by (Q_7), (23) we get $\mathsf{s}_\mu^l \mathsf{s}_i^j x \leq \mathsf{s}_\mu^l \mathsf{c}_j x = 0$. Thus, $\mathsf{s}_i^j \in ker(\mathsf{s}_\mu^l)$. Hence, by Lemma (2) we obtain (s_8).

Now to show (s_7) we first show that

$$\mathsf{c}_j \mathsf{p}_{ij} x = \mathsf{p}_{ij} \mathsf{c}_i x. \tag{24}$$

Since

$$\begin{aligned}
\mathsf{c}_j \mathsf{p}_{ij} x &\leq \mathsf{c}_j \mathsf{p}_{ij} \mathsf{c}_i x \quad \text{by} \quad (Q_2, Q_7)\\
&= \mathsf{c}_j \mathsf{p}_{ij} \mathsf{s}_j^i \mathsf{c}_i x \quad \text{by} \quad (Q_4)\\
&= \mathsf{c}_j \mathsf{p}_{ji} \mathsf{s}_j^i \mathsf{c}_i x \quad \text{by} \quad (Q_1)\\
&= \mathsf{c}_j \mathsf{s}_i^j \mathsf{c}_i x \quad \text{by} \quad (Q_{10})\\
&= \mathsf{s}_i^j \mathsf{c}_i x \quad \text{by} \quad (Q_5)\\
&= \mathsf{p}_{ji} \mathsf{s}_j^i \mathsf{c}_i x \quad \text{by} \quad (Q_{10})\\
&= \mathsf{p}_{ij} \mathsf{s}_j^i \mathsf{c}_i x \quad \text{by} \quad (Q_1).
\end{aligned}$$

Hence, by (Q_4) we get

$$\mathsf{c}_j \mathsf{p}_{ij} x \leq \mathsf{p}_{ij} \mathsf{c}_i x. \tag{25}$$

Since (25) holds for arbitrary $i, j < \alpha$, so we have

$$\mathsf{c}_i \mathsf{p}_{ij} x \leq \mathsf{p}_{ij} \mathsf{c}_j x. \tag{26}$$

Hence, by (26) and (Q_8) we see that

$$\mathsf{p}_{ij} \mathsf{c}_i \mathsf{p}_{ij} x \leq \mathsf{p}_{ij} \mathsf{p}_{ij} \mathsf{c}_j x = \mathsf{c}_j x. \tag{27}$$

Thus, by (Q_8) and (27) we obtain that

$$\mathsf{p}_{ij} \mathsf{c}_i x = \mathsf{p}_{ij} \mathsf{c}_i \mathsf{p}_{ij} \mathsf{p}_{ij} x \leq \mathsf{c}_j \mathsf{p}_{ij} x. \tag{28}$$

So, (24) follows from (25) and (28).

Hence, we get (s_7) as follows

$$\begin{aligned}
\mathsf{c}_i \mathsf{s}_i^j x &= \mathsf{c}_i \mathsf{c}_j \mathsf{s}_i^j x \quad \text{by} \quad (Q_5)\\
&= \mathsf{c}_j \mathsf{c}_i \mathsf{s}_i^j x \quad \text{by} \quad (15)\\
&= \mathsf{s}_i^j \mathsf{c}_j \mathsf{c}_i \mathsf{s}_i^j x \quad \text{by} \quad (Q_4)\\
&= \mathsf{p}_{ji} \mathsf{s}_j^i \mathsf{c}_j \mathsf{c}_i \mathsf{s}_i^j x \quad \text{by} \quad (Q_{10})\\
&= \mathsf{p}_{ij} \mathsf{s}_j^i \mathsf{c}_i \mathsf{c}_j \mathsf{s}_i^j x \quad \text{by} \quad (15), (Q_1)\\
&= \mathsf{p}_{ij} \mathsf{c}_i \mathsf{c}_j \mathsf{s}_i^j x \quad \text{by} \quad (Q_4)\\
&= \mathsf{p}_{ij} \mathsf{c}_i \mathsf{s}_i^j x \quad \text{by} \quad (Q_5)\\
&= \mathsf{c}_j \mathsf{p}_{ij} \mathsf{s}_i^j x \quad \text{by} \quad (24)\\
&= \mathsf{c}_j \mathsf{s}_j^i x \quad \text{by} \quad (Q_{10}).
\end{aligned}$$

Finally, we show (s_9).

$$\begin{aligned}
\mathsf{s}^i_j\mathsf{s}^\mu_j x &= \mathsf{s}^i_j\mathsf{s}^j_i\mathsf{s}^\mu_j x \quad \text{by} \quad (\text{iv})\\
&= \mathsf{s}^i_j\mathsf{s}^\mu_i\mathsf{s}^j_i x \quad \text{by} \quad (s_8)\\
&= \mathsf{s}^i_j\mathsf{s}^\mu_i\mathsf{s}^j_i\mathsf{s}^i_j x \quad \text{by} \quad (\text{iv})\\
&= \mathsf{s}^i_j\mathsf{s}^j_i\mathsf{s}^\mu_i\mathsf{s}^i_j x \quad \text{by} \quad (s_8)\\
&= \mathsf{s}^i_j\mathsf{s}^\mu_i\mathsf{s}^i_j x \quad \text{by} \quad (\text{iv})\\
&= \mathsf{s}^i_j\mathsf{s}^\mu_j\mathsf{s}^\mu_i\mathsf{s}^i_j x \quad \text{by} \quad (\text{iii})\\
&= \mathsf{s}^\mu_j\mathsf{s}^i_j\mathsf{s}^\mu_i\mathsf{s}^i_j x \quad \text{by} \quad (s_8)\\
&= \mathsf{s}^\mu_j\mathsf{s}^i_j\mathsf{p}_{\mu i}\mathsf{s}^\mu_i\mathsf{s}^i_j x \quad \text{by} \quad (\text{iv})\\
&= \mathsf{s}^\mu_j\mathsf{s}^i_j\mathsf{s}^i_\mu\mathsf{s}^i_j x \quad \text{by} \quad (Q_{10})\\
&= \mathsf{s}^\mu_j\mathsf{s}^i_\mu\mathsf{s}^i_j x \quad \text{by} \quad (\text{iii})\\
&= \mathsf{s}^\mu_j\mathsf{s}^i_j x \quad \text{by} \quad (\text{iii})\\
&= \mathsf{s}^i_j\mathsf{s}^\mu_j x \quad \text{by} \quad (s_8).
\end{aligned}$$

□

Proposition 2.5 *In every quasi-polyadic equality algebra* $\mathfrak{B} = (B, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{s}^i_j, \mathsf{p}_{ij}, \mathsf{d}_{ij})_{i,j<\alpha}$, *the following properties holds for any* $x \in B$ *and any* $i, j, \mu, \nu < \alpha$:

(a') $x \cdot \mathsf{d}_{ij} = \mathsf{s}^i_j x \cdot \mathsf{d}_{ij}$.
(b') $\mathsf{c}_i\mathsf{d}_{ij} = 1$.
(c') $\mathsf{c}_i(x \cdot \mathsf{d}_{ij}) = \mathsf{s}^i_j x$.
(d') $\mathsf{s}^i_\nu\mathsf{d}_{j\mu} = \mathsf{d}_{j\mu}$ *for* $i \notin \{j, \mu\}$.
(e') $\mathsf{s}^i_\mu\mathsf{d}_{ij} = \mathsf{d}_{j\mu}$.

Proof(a') $x \cdot \mathsf{d}_{ij} = \mathsf{s}^i_j x \cdot \mathsf{d}_{ij}$. From (Q_{12}) we see that $x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j x$ and since $x \cdot \mathsf{d}_{ij} \leq \mathsf{d}_{ij}$. Then

$$x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j x \cdot \mathsf{d}_{ij}. \tag{29}$$

On the other hand, from (Q_{12}) we have $-x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j - x$. So,

$$\mathsf{s}^i_j x \cdot -x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j x \cdot \mathsf{s}^i_j - x = 0.$$

Hence $x \cdot \mathsf{d}_{ij} \geq \mathsf{s}^i_j x$ which implies to

$$x \cdot \mathsf{d}_{ij} \geq \mathsf{s}^i_j x \cdot \mathsf{d}_{ij}. \tag{30}$$

Thus, we get (a') from (29) and (30).

(b') $\mathsf{c}_i\mathsf{d}_{ij} = 1$. By (Q_2) we have $\mathsf{d}_{ij} \leq \mathsf{c}_i\mathsf{d}_{ij}$ and by (Q_7) we get $\mathsf{s}^i_j\mathsf{d}_{ij} \leq \mathsf{s}^i_j\mathsf{c}_i\mathsf{d}_{ij}$. Hence, by (Q_{11}) and (Q_4) we obtain $1 = \mathsf{c}_i\mathsf{d}_{ij}$ which gives (b').

(c') $\mathsf{c}_i(x \cdot \mathsf{d}_{ij}) = \mathsf{s}^i_j x$. We note that (c') be as follows

$$\begin{aligned}
\mathsf{c}_i(x \cdot \mathsf{d}_{ij}) &= \mathsf{c}_i(\mathsf{s}^i_j x \cdot \mathsf{d}_{ij}) \qquad \text{by}(a')\\
&= \mathsf{c}_i(\mathsf{c}_i\mathsf{s}^i_j x \cdot \mathsf{d}_{ij}) \qquad \text{by} \quad (Q_5)\\
&= \mathsf{c}_i\mathsf{s}^i_j x \cdot \mathsf{c}_i\mathsf{d}_{ij} \qquad \text{by} \quad (C_3)\\
&= \mathsf{s}^i_j x \cdot 1 \qquad \text{by} \quad (Q_5), (b')\\
&= \mathsf{s}^i_j x.
\end{aligned}$$

(d') $\mathsf{s}^i_\nu \mathsf{d}_{j\mu} = \mathsf{d}_{j\mu}$ for $i \notin \{j, \mu\}$.

We note that

$$\begin{aligned} \mathsf{s}^j_\mu \mathsf{c}_i(-\mathsf{d}_{j\mu}) &= \mathsf{c}_i \mathsf{s}^j_\mu(-\mathsf{d}_{j\mu}) \quad \text{by} \quad (Q_6)\\ &= \mathsf{c}_i(-\mathsf{s}^j_\mu \mathsf{d}_{j\mu}) \quad \text{by} \quad (Q_7)\\ &= \mathsf{c}_i(-1) \quad \text{by} \quad (Q_{11})\\ &= 0. \end{aligned}$$

So,

$$\mathsf{s}^j_\mu(-\mathsf{c}_i(-\mathsf{d}_{j\mu})) = 1. \tag{31}$$

By (Q_{11}) and (a') we see that

$$\mathsf{d}_{ij} = min\{x : \mathsf{s}^i_j x = 1\}. \tag{32}$$

Hence, by (31) and (32) we have $\mathsf{d}_{j\mu} \leq -\mathsf{c}_i(-\mathsf{d}_{j\mu})$ which implies to $\mathsf{c}_i(-\mathsf{d}_{j\mu}) \leq -\mathsf{d}_{j\mu}$. But by (Q_2) we have $-\mathsf{d}_{j\mu} \leq \mathsf{c}_i(-\mathsf{d}_{j\mu})$. So,

$$\mathsf{c}_i(-\mathsf{d}_{j\mu}) = -\mathsf{d}_{j\mu}. \tag{33}$$

Thus

$$\begin{aligned} -\mathsf{s}^i_\nu \mathsf{d}_{j\mu} &= \mathsf{s}^i_\nu \mathsf{c}_i(-\mathsf{d}_{j\mu}) \quad \text{by} \quad (33)\\ &= \mathsf{c}_i(-\mathsf{d}_{j\mu}) \quad \text{by} \quad (Q_4)\\ &= -\mathsf{d}_{j\mu} \quad \text{by} \quad (33). \end{aligned}$$

It follows that (d') holds.

(e') $\mathsf{s}^i_\mu \mathsf{d}_{ij} = \mathsf{d}_{j\mu}$.

We note that from Theorem (2.4) we prove that (s_9) which is $\mathsf{s}^l_i \mathsf{s}^j_l = \mathsf{s}^l_i \mathsf{s}^j_i$ from $(Q_1 - Q_{10})$. So, by (s_9) and (Q_{11}) we get

$$\begin{aligned} \mathsf{s}^j_\mu(\mathsf{s}^i_\mu \mathsf{d}_{ij}) &= \mathsf{s}^j_\mu(\mathsf{s}^i_j \mathsf{d}_{ij}) \quad \text{by} \quad (s_9)\\ &= \mathsf{s}^j_\mu 1 \quad \text{by} \quad (Q_{11})\\ &= 1 \quad \text{by} \quad (Q_7). \end{aligned} \tag{34}$$

Hence, by (34) and (32) we have

$$\mathsf{d}_{j\mu} \leq \mathsf{s}^i_\mu \mathsf{d}_{ij}. \tag{35}$$

Symmetrically,

$$\mathsf{d}_{ij} \leq \mathsf{s}^\mu_i \mathsf{d}_{j\mu}. \tag{36}$$

Thus

$$\begin{aligned} \mathsf{s}^i_\mu \mathsf{d}_{ij} &\leq \mathsf{s}^i_\mu \mathsf{s}^\mu_i \mathsf{d}_{j\mu} \quad \text{by} \quad (36)\\ &= \mathsf{s}^i_\mu \mathsf{d}_{j\mu} \quad \text{by (ii) in the proof of Theorem(22.4)}\\ &= \mathsf{d}_{j\mu} \quad \text{by} \quad (d'). \end{aligned} \tag{37}$$

So, from (35), (37) we obtain (e'). □

Theorem 2.6 *If* $\mathfrak{A} \in \mathbf{QEA}_\alpha$ *, then* $\mathfrak{Rd}_{\mathbf{CA}}\mathfrak{A} \models (C_1) - (C_7)$.

Proof We note that (C_2) is (Q_2) and the proof of (C_1), (C_3), (C_4) as in Theorem (2.4).

For (C_5), taking $i = j$ in (Q_{11}) we get $\mathsf{s}^i_i \mathsf{d}_{ii} = 1$ and replacing x with d_{ii} in (Q_1) we have $\mathsf{s}^i_i \mathsf{d}_{ii} = \mathsf{d}_{ii}$. Thus $\mathsf{d}_{ii} = 1$.

Also, we get (C_6) as follows

$$\begin{aligned} \mathsf{c}_k(\mathsf{d}_{ik}.\mathsf{d}_{kj}) &= \mathsf{s}^k_i \mathsf{d}_{kj} \quad \text{by} \quad (c')\\ &= \mathsf{d}_{ij} \quad \text{by} \quad (e'). \end{aligned}$$

Finally, we get (C_7) from (c') as follows

$$\mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = \mathsf{s}^i_j x \cdot \mathsf{s}^i_j - x = 0.$$

□

3 Conclusion and future work

The aim of this paper was to show how the concept of neat reduct is applicable on classes of cylindric algebras (CA), quasi-polyadic algebras (QA) and quasi-polyadic equality algebras (QEA) to get another algebras such as substitution algebra and cylindric algebra (in this case). It is to be noted that the concept of neat reducts is the bridge connecting Tarski's relation algebra and Tarski's cylindric algebra. Also this concept helps to analyse the number of variables showing up in the proofs of first order formulas and characterize the class of representable algebras.

The future direction is trying to apply the concept of neat reduct on another classes.

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Declarations

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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