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ORIGINAL RESEARCH

The class of meromorphic functions sharing values with their difference polynomials

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Abstract The field of c-periodic meromorphic functions in $\mathbb{C}$ is defined by $\mathcal{M}_c := \{f : f$ is meromorphic in $\mathbb{C}$ and $f(z+c) = f(z)\}$ and the c-shift linear difference polynomial of a meromorphic function f is defined by

$$L_c^n(f) = a_n f(z+nc) + \cdots + a_1 f(z+c) + a_0 f(z),$$

where $a_n(\neq 0), \ldots, a_1, a_0 \in \mathbb{C}$. It is easy to see that if $a_j = \binom{n}{j}(-1)^{n-j}$, then $L_c^n(f) = \Delta_c^n f$, where $\Delta_c^n f$ is a higher difference operator of f. Let

$$\mathcal{S}_c = \{f : f \text{ is meromorphic in } \mathbb{C} \text{ and } L_c^n(f) \equiv f\}.$$

In this paper, we study the value sharing problem between a meromorphic functions f and their linear difference polynomials $L_c^n(f)$ and prove a result generalizing several existing results. In addition, we find the class $\mathcal{S}_c$ completely which gives the positive answers to a conjecture and an open problem in this direction.

Keywords Uniqueness problem · Linear difference polynomials · Meromorphic functions · General solutions · Shared values

Mathematics Subject Classification 30D35

1 Introduction

Throughout the paper, we assume that the reader is familiar with the standard notations and fundamental results of Nevanlinna theory of meromorphic functions (see [15,22,32]). In what follows, a meromorphic function always means meromorphic in the whole complex plane $\mathbb{C}$, and c always means a nonzero complex constant. For a meromorphic function f, we define its shift by $f(z+c)$ and its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z),\ \ \Delta_c^n f(z) = \Delta_c^{n-1}\left(\Delta_c f(z)\right),\ \ n \in \mathbb{N},\ \ n \geq 2.$$

Two non-constant meromorphic functions f and g share the value $a \in \mathbb{C} \cup \{\infty\}$, if $f^{-1}(a) = g^{-1}(a)$. We say that f and g share the value a CM (Counting Multiplicities) if in addition to the sharing of values if $f(z_0) = a$ with multiplicity p implies $g(z_0) = a$ with multiplicity p. If we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). When $a = \infty$, the zeros of $f - a$ means the poles of f.

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The uniqueness theory of entire and meromorphic functions in the purview of sharing values (using value distribution theory of Nevanlinna [29]) has grown up to an extensive subfield of the value distribution theory. Interested readers are referred to the articles [9–11,13,21,23,28,30,31] and references therein. After the development of the difference analogue lemma of logarithmic derivatives, by Halburd and Korhonen [14] in 2006, and Chiang and Feng [6], in 2008, independently, the research findings dealing with the sharing value problems between the shifts $f(z+c)$ or with the difference operators $\Delta_c f$ of meromorphic functions f, gets a new dimension in the literature of meromorphic functions. The reader is referred the articles [16,17] in this regard. In 2013, Jiang and Chen [20] studied two shared value problems for meromorphic functions and proved that if $\Delta_c f$ and f share a, b CM, then $f(z+c) = 2f(z)$.

Recently, Barki *et al.* [5] have established result of for higher order difference operator $\Delta_c^n f$ considering two shared values.

Theorem 1.1 *[5] Let f be a non-constant meromorphic of finite order such that $N(r, f) = S(r, f)$, let c be a constant such that $\Delta_c^n f \not\equiv 0$ and let a, b be two non-zero distinct finite complex constants. If $\Delta_c^n f$ and f share a, b CM, then $\Delta_c^n f = f(z)$.*

However, in [5], Barki *et al.* have obtained the following corollary generalizing the result of Jiang and Chen [20]..

Corollary 1.2 *[5] Let f be a non-constant meromorphic of finite order such that $N(r, f) = S(r, f)$, let c be a constant such that $\Delta_c^n f \not\equiv 0$ and let a, b be two non-zero distinct finite complex constants. If $\Delta_c^n f$ and f share a, b CM, then $f(z+nc) = 2^n f(z)$.*

We define the linear difference polynomial of a meromorphic function f as follows (see [3]):

$$L_c^n(f) = a_n f(z+nc) + \cdots + a_1 f(z+c) + a_0 f(z),$$

where $a_n(\neq 0), \ldots, a_1, a_0 \in \mathbb{C}$. It is easy to see that if $a_j = \binom{n}{j}(-1)^{n-j}$, then $L_c^n(f) = \Delta_c^n f$. Clearly, $L_c^n(f)$ is a general setting of the difference operator $\Delta_c^n f$.

In a number of articles, the higher order difference operators $\Delta_c^n f$ have been studied extensively (for example, see [4,7,8,18,19,24–27]) in view of sharing values with entire or meromorphic functions f but less investigations done till date with the linear difference polynomials $L_c^n(f)$. In fact, what could be the precise form of a meromorphic function f, when sharing values with its linear difference polynomials $L_c^n(f)$, is completely unknown. In 2019, succeeding in particular with $n = 1$ in order to find the specific form of the function, Ahamed [1] investigated value sharing problem (CM sharing) between $L_c^1(f)$ and f and obtained $L_c^1(f) \equiv f$, and showed that f assumes the following form

$$f(z) = \begin{cases} \left(\dfrac{1-a_0}{a_1}\right)^{z/c} \pi_1(z), & \text{when} \quad a_0 + a_1 \neq 1 \\ \pi_2(z), & \text{when} \ a_0 + a_1 = 1, \end{cases}$$

for $\pi_1,\ \pi_2 \in \mathcal{M}_c$.

Remark 1.3 In order to obtain the relation $L_c^1(f) \equiv f$ with the specific form of the function, by the following two examples, it was shown in [1] that the nature of 2 CM sharing of values cannot be replaced by 1 CM+1 IM or 2 IM.

Example 1.4 [1] Let $f(z) = \left(e^z + e^{-z}\right)/2$, and for $c \neq k\pi i$, $k \in \mathbb{Z}$, let

$$L_c^1(f) = \frac{2e^c}{e^{2c}-1} f(z+c) - \frac{2}{e^{2c}-1} f(z).$$

A simple computation shows that $L_c^1(f) = e^z$. It is easy to see that f and $L_c^1(f)$ share the values 1 and -1 IM but f neither in the specific form nor satisfies $L_c^1(f) \equiv f$.

Example 1.5 [1] Let $f(z) = 1 + (e^z - 1)^2$ and $c \in \mathbb{C}$ be such that $e^c = -1$, let

$$L_c^1(f) = \frac{1}{4} f(z+c) - \frac{1}{4} f(z).$$

We see that $L_c^1(f) = e^z$. We verify that f and $L_c^1(f)$ share the values 2 CM and 1 IM, but note that f has neither the specific form nor satisfies $L_c^1(f) \equiv f$.

It was shown in [1] that similar results can be obtained for meromorphic functions also.

Theorem 1.6 *[1] Let f be transcendental meromorphic function of finite order, and a, b be two distinct constants. If $L_c^1(f)(\not\equiv (d_1e^{\alpha}+d_2)/(d_3e^{\beta}+d_4))$, where $d_j \in \mathbb{C}$ and α, β are polynomials in z, and f share a, b and ∞ CM, then $L_c^1(f) \equiv f$. Furthermore, f must be of the following form*

$$f(z) = \begin{cases} \left(\dfrac{1-a_0}{a_1}\right)^{z/c} \pi_1(z), & \text{when } a_1 + a_0 \neq 1 \\ \pi_2(z), & \text{when } a_1 + a_0 = 1, \end{cases}$$

where $\pi_1, \pi_2 \in \mathcal{M}_c$.

To establish an improved version of Theorem 1.6, it is natural to raise the following question.

Question 1.7 Can we prove a result replacing "$L_c^1(f)(\not\equiv (d_1e^{\alpha}+d_2)/(d_3e^{\beta}+d_4))$ and f share a, b and ∞ CM" in Theorem 1.6 by "$L_c^1(f)(\not\equiv 0)$ and f share a and b CM"?

In case of investigating the general meromorphic solutions to the difference equation $L_c^n(f) \equiv f$, recently, Ahamed [2] posed the following conjecture.

Conjecture 1 [2] Let f be a non-constant meromorphic function such that $L_c^n(f) \equiv f$, then f assumes the form

$$f(z) = \alpha_n^{z/c} g_n(z) + \cdots + \alpha_1^{z/c} g_1(z),$$

where g_j $(j = 1, 2, \ldots, n)$ are periodic meromorphic functions of period c.

In [3], Banerjee and Ahamed, also posed the following question with the similar query.

Question 1.8 Let $L_n(f, \Delta) = L_c^n(f) - \left(\sum_{j=0}^{n} a_j\right) f(z)$ and t be a non-zero constant. What would be the general meromorphic solution of the difference equation $L_n(f, \Delta) \equiv tf$?

In this paper, our aim is two fold. In view of Remark 1.3, in one hand, instead of considering 1 CM + 1 IM or 2 IM sharing of values, however, we shall be interested to deal with the 2 CM sharing problems between meromorphic functions f and their linear difference polynomials $L_c^n(f)$. On the other hand, in order to give the positive answers of Conjecture 1 and Question 1.8, our aim is to find the general solutions to the difference equation which is obtained as a conclusion of our main result. In fact, we obtain a corollary of the main result to give a complete answer of Question 1.7.

2 Main results

We define the class $\mathcal{M}_c$ by

$$\mathcal{M}_c := \{f : f \text{ is meromorphic in } \mathbb{C} \text{ and } f(z+c) = f(z)\}.$$

It can be shown that $\mathcal{M}_c$ is a field of meromorphic functions of period c. We state the main result of this paper.

Theorem 2.1 *Let f be a non-constant meromorphic of finite order such that $N(r, f) = S(r, f)$, let c be a constant such that $L_c^n(f) \not\equiv 0$ with $\sum_{j=0}^{n} a_j = 0$ and let a, b be two non-zero distinct finite complex constants. If $L_c^n(f)$ and f share a, b CM, then $L_c^n(f) = f$. Furthermore,*

(i) $f(z) = \rho_1^{z/c}\pi_1(z) + \rho_2^{z/c}\pi_2(z) + \cdots + \rho_n^{z/c}\pi_n(z)$, where $\pi_j(z) \in \mathcal{M}_c$ $(j = 1, 2, \ldots, n)$ and ρ_j $(j = 1, 2, \ldots, n)$ are distinct roots of the equation

$$a_n w^n + \cdots + a_1 w + a_0 - 1 = 0.$$

In particular, if $\rho_j \in \{0, 1\}$ be such that atleast one of ρ_j's are non-zero, then $f \in \mathcal{M}_c$.

(ii) $$f(z)=\left(\sum_{m_1=1}^{N_1-1} z^{m_1}\pi_{m_1}(z)\right)\sigma_1^{z/c}+\cdots+\left(\sum_{m_q=1}^{N_q-1} z^{m_q}\pi_{m_q}(z)\right)\sigma_q^{z/c},$$

where $\pi_j\in\mathcal{M}_c$ *and* $\sigma_1,\sigma_2,\ldots,\sigma_q$ *are multiple roots, of respective multiplicities* $N_1,N_2,\ldots,N_q$, *of the equation*

$$a_nw^n+\cdots+a_1w+a_0-1=0.$$

Remark 2.2 The following observations are clear.

(i) Clearly, part-(i) of Theorem 2.1 answered the conjecture of Ahamed [2] completely.
(ii) In particular, if $\sum_{j=0}^{n} a_j=0$, then it is easy to see that part-(i) of Theorem 2.1 is the complete answer of Question 1.8.
(iii) In view of Example 1.5 and the following example, in Theorem 2.1, we see that 2 CM sharing cannot be replaced by 1 $CM+1$ IM.

Example 2.3 Let $f(z)=1+(e^z-1)^2$ and c be so chosen that $e^c\neq\pm1$ and

$$\begin{cases} a_0=\dfrac{e^{2c}}{\left(e^{2c}-1\right)-\left(e^{2c}+1\right)\left(e^{c}-1\right)}\\ a_1=\dfrac{1-e^{4c}}{\left(e^{2c}-1\right)\left(\left(e^{2c}-1\right)-\left(e^{2c}+1\right)\left(e^{c}-1\right)\right)}\\ a_2=\dfrac{1}{\left(e^{2c}-1\right)\left(\left(e^{2c}-1\right)-\left(e^{2c}+1\right)\left(e^{c}-1\right)\right)}. \end{cases}$$

A simple computation shows that $L_c^2(f)=e^z$ with $\sum_{j=0}^{2} a_j=0$ and $N(r,f)=S(r,f)$. It is easy to verify that $L_c^2(f)$ and f share the values 1 IM and 2 CM but conclusion of Theorem 2.1 fails to hold.

We obtain the following corollary of Theorem 2.1 which gives a more compact form of Theorem 1.1.

Corollary 2.4 *Let f be a non-constant meromorphic of finite order such that $N(r,f)=S(r,f)$, let c be a constant such that $\Delta_c^n f\not\equiv 0$ and let a,b be two non-zero distinct finite complex constants. If $\Delta_c^n f$ and f share a,b CM, then $\Delta_c^n f\equiv f$. Furthermore,*

$$f(z)=\sum_{k=0}^{n}\lambda_k^{z/c}\pi_k(z),\ \textit{where}\ \pi_k\in\mathcal{M}_c \tag{2.1}$$

and $\lambda_k=1+e^{2ki\pi/n}$, $k=0,1,\ldots,n-1$.

Remark 2.5 It is easy to verify that function in (2.1) satisfies $f(z+nc)=2^nf(z)$ of Corollary 1.2.

We obtain the following corollary of Theorem 2.1 in case of when $n=1$.

Corollary 2.6 *Let f be transcendental meromorphic function of finite order, and a,b be two distinct constants. If $L_c^1(f)(\not\equiv 0)$ and f share a and b CM, then $L_c^1(f)\equiv f$. Furthermore,*

$$f(z)=\begin{cases}\left(\dfrac{1-a_0}{a_1}\right)^{z/c}\pi_1(z), & \textit{when}\ a_1+a_0\neq1\\ \pi_2(z), & \textit{when}\ a_1+a_0=1,\end{cases}$$

where $\pi_1,\pi_2\in\mathcal{M}_c$.

Remark 2.7 Clearly, Corollary 2.6 is the complete answer of Question 1.7.

3 Key lemmas

In this section, we present some lemmas which will play key roles to prove the main results of this paper.

Lemma 3.1 *[32] Let f be a non-constant meromorphic function, a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers. Then*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{n} m\left(r, \frac{1}{f - a_j}\right) + O(1). \tag{3.1}$$

Lemma 3.2 [6,14] *Let f be a meromorphic function of finite order and c be a non-zero constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f). \tag{3.2}$$

Lemma 3.3 [5] *Let f be a non-constant meromorphic function in $\mathbb{C}$. Let a_j $(j = 1, 2, \ldots, q)$ be q-distinct complex numbers. Then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \le m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \tag{3.3}$$

To prove our main result, we prove the following lemma for linear difference polynomial $L_c^n(f)$.

Lemma 3.4 *Let f be a meromorphic function and $L_c^n(f)$ be its linear difference polynomial such that $\sum_{j=0}^{n} a_j = 0$. Then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \le m\left(r, \frac{1}{L_c^n(f)}\right) + S(r, f). \tag{3.4}$$

Proof In view of Lemmas 3.1 and 3.2, a simple computation shows that

$$\begin{aligned}
\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) &= m\left(r, \frac{L_c^n(f)}{L_c^n(f)(f - a_j)}\right) + O(1)\\
&\le m\left(r, \frac{1}{Ł_c^n(f)}\right) + m\left(r, \sum_{j=1}^{q} \frac{Ł_c^n(f)}{f - a_j}\right) + O(1)\\
&\le m\left(r, \frac{1}{Ł_c^n(f)}\right) + \sum_{j=1}^{q} m\left(r, \frac{Ł_c^n(f)}{f - a_j}\right) + O(1)\\
&= m\left(r, \sum_{k=0}^{n} a_k \left(\frac{f(z+kc) - a_j}{f(z) - a_j}\right) + \frac{a_j}{f - a_j} \sum_{k=0}^{n} a_k\right)\\
&\quad + m\left(r, \frac{1}{Ł_c^n(f)}\right)\\
&\le m\left(r, \frac{1}{Ł_c^n(f)}\right) + \sum_{k=0}^{n} m\left(r, \frac{f(z+kc) - a_j}{f(z) - a_j}\right) + S(r, f)\\
&\le m\left(r, \frac{1}{Ł_c^n(f)}\right) + S(r, f).
\end{aligned}$$

This completes the proof. □

Lemma 3.5 [16] *Let f be a meromorphic function with order $\sigma(f) = \sigma < \infty$, and c be a fixed non-zero complex number, then for each $\epsilon > 0$,*

$$T(r, f(z+c)) = T(r, f) + O\left(r^{\sigma - 1 + \epsilon}\right) + S(r.f).$$

4 Proof of the main results

This section is devoted to the detailed discussion on our proof of the main results.

Proof of Theorem 2.1 Since $L_c^n(f)$ and f share the values a, b CM and $N(r, f) = S(r, f)$, then by the *Second Fundamental Theorem of Nevanlinna*, we obtain

$$\begin{aligned} T(r, f) &\leq N(r, f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &= N\left(r, \frac{1}{L_c^n(f)-a}\right) + N\left(r, \frac{1}{L_c^n(f)-b}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{L_c^n(f)-f}\right) + S(r, f) \end{aligned} \tag{4.1}$$

In view of the *First Fundamental Theorem of Nevanlinna*, a simple computation shows that

$$\begin{aligned} N\left(r, \frac{1}{L_c^n(f)-f}\right) + S(r, f) &= N\left(r, \frac{f}{L_c^n(f)-f}\right) + S(r, f) \\ &= N\left(r, \frac{1}{\frac{L_c^n(f)}{f}-1}\right) + S(r, f) \leq T\left(r, \frac{1}{\frac{L_c^n(f)}{f}-1}\right) + S(r, f) \\ &\leq T\left(r, \frac{L_c^n(f)}{f}\right) + S(r, f) \\ &\leq m\left(r, \frac{L_c^n(f)}{f}\right) + N\left(r, \frac{L_c^n(f)}{f}\right) + S(r, f) \\ &\leq N\left(r, \frac{L_c^n(f)}{f}\right) + S(r, f) \leq N\left(r, \frac{1}{f}\right) + N\left(r, L_c^n(f)\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f}\right) + 2^n N(r, f) + S(r, f) = N\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq T\left(r, \frac{1}{f}\right) + S(r, f) = T(r, f) + S(r, f). \end{aligned} \tag{4.2}$$

Therefore, in view of (4.1) and (4.2), we obtain

$$\begin{aligned} T(r, f) &= N\left(r, \frac{1}{f}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f). \end{aligned} \tag{4.3}$$

We define

$$\Psi(f) := \frac{\left(L_c^n(f)\right)'}{L_c^n(f)-a} - \frac{f'}{f-a}. \tag{4.4}$$

By the logarithmic derivative lemma, it is easy to see that

$$\begin{aligned} m(r, \Psi(f)) &\leq m\left(r, \frac{\left(L_c^n(f)\right)'}{L_c^n(f)-a}\right) + m\left(r, \frac{f'}{f-a}\right) + O(1) \\ &= S(r, L_c^n(f)) + S(r, f). \end{aligned}$$

On the other hand, a simple computation shows that

$$\begin{aligned} T\left(r, L_c^n(f)\right) &= m\left(r, L_c^n(f)\right) + N\left(r, L_c^n(f)\right) + S(r, f) \\ &\leq m\left(r, \frac{L_c^n(f)}{f}\right) + m(r, f) + \sum_{j=0}^{n} N(r, f) + S(r, f) \\ &\leq m(r, f) + 2^n N(r, f) + S(r, f) \\ &= m(r, f) + S(r, f) \leq T(r, f) + S(r, f). \end{aligned} \tag{4.5}$$

In view of Lemma 3.5, it can be shown that $S\left(r, L_c^n(f)\right) = S(r, f)$, thus it follows that

$$m(r, \Psi) = S(r, f). \tag{4.6}$$

Since Ψ is the logarithmic derivative of $(L_c^n(f) - a)/(f - a)$, the poles of Ψ derive from the poles and zeros of $(L_c^n(f) - a)/(f - a)$. Again, $L_c^n(f)$ and f share a CM, then $(L_c^n(f) - a)/(f - a)$ has no zeros and has atmost $N(r, f)$ poles. Therefore, it follows that

$$N(r, \Psi) = N(r, f) = S(r, f). \tag{4.7}$$

Hence, in view of (4.6) and (4.7), it is easy to see that

$$T(r, \Psi) = m(r, \Psi) + N(r, \Psi) + O(1) = S(r, f).$$

On the other hand, we see that

$$\frac{\Psi}{f-a} = \frac{\left(L_c^n(f)\right)'}{L_c^n(f) - a} \frac{L_c^n(f)}{f-b} - \frac{f'}{(f-a)(f-b)}. \tag{4.8}$$

We suppose that $\Psi \not\equiv 0$. Then, from (4.8), we obtain

$$\begin{aligned} m\left(r, \frac{1}{f-b}\right) &\leq m\left(r, \frac{1}{\Psi}\right) + m\left(r, \frac{\Psi}{f-b}\right) \\ &\leq m\left(r, \frac{1}{\Psi}\right) + m\left(r, \frac{\left(L_c^n(f)\right)'}{L_c^n(f) - a}\right) + m\left(r, \frac{L_c^n(f)}{f-b}\right) \\ &\quad + m\left(r, \frac{f'}{(f-a)(f-b)}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{\Psi}\right) + m\left(r, \frac{\left(L_c^n(f)\right)'}{a}\left(\frac{1}{L_c^n(f) - a} - \frac{1}{L_c^n(f)}\right)\right) \\ &\quad + m\left(r, \frac{L_c^n(f)}{f-b}\right) + m\left(r, \frac{f'}{(a-b)}\left(\frac{1}{f-a} - \frac{1}{f-b}\right)\right) + S(r, f) \\ &\leq T(r, \Psi) + S(r, L_c^n(f)) + S(r, f) = S(r, f) \end{aligned}$$

which shows that

$$m\left(r, \frac{1}{f-b}\right) = S(r, f).$$

By the *First Fundamental Theorem of Nevanlinna*, we obtain

$$T(r, f) = N\left(r, \frac{1}{f-b}\right) + S(r, f). \tag{4.9}$$

Therefore, it follows from (4.3) and (4.9) that

$$N\left(r, \frac{1}{f-a}\right) = S(r, f).$$

By the assumption, since $L_c^n(f)$ and f share the value a CM, then it is easy to see that

$$N\left(r, \frac{1}{L_c^n(f)-a}\right) = N\left(r, \frac{1}{f-a}\right) = S(r, f).$$

Again, since $L_c^n(f)$ and f share the value b CM, then by using (4.5) and (4.9), by *First Fundamental Theorem of Nevanlinna*, we obtain

$$\begin{aligned} m\left(r, \frac{1}{L_c^n(f)-b}\right) + N\left(r, \frac{1}{L_c^n(f)-b}\right) &= T(r, L_c^n(f)) + S(r, f) \\ &\leq T(r, f) + S(r, f) \leq N\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &= N\left(r, \frac{1}{L_c^n(f)-b}\right) + S(r, f) \end{aligned}$$

which in turn shows that

$$m\left(r, \frac{1}{L_c^n(f)-b}\right) = S(r, f).$$

In view of Lemma 3.3, we obtain

$$\begin{aligned} m\left(r, \frac{1}{L_c^n(f)}\right) &+ m\left(r, \frac{1}{L_c^n(f)-a}\right) + m\left(rm \frac{1}{L_c^n(f)-b}\right) \\ &\leq m\left(r, \frac{1}{\left(L_c^n(f)\right)'}\right) + S(r, f). \end{aligned} \tag{4.10}$$

$$m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) \leq m\left(r, \frac{1}{L_c^n(f)}\right) + S(r, f). \tag{4.11}$$

By using (4.3) and the previous inequalities for counting functions, it is easy to obtain

$$N\left(r, \frac{1}{L_c^n(f)-a}\right) + N\left(r, \frac{1}{L_c^n(f)-b}\right) \leq N\left(r, \frac{1}{L_c^n(f)-b}\right) + S(r, f). \tag{4.12}$$

$$N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) \leq T(r, f) + S(r, f). \tag{4.13}$$

Therefore, in view of (4.10) to (4.13), a simple computation shows that

$$\begin{aligned} &T\left(r, \frac{1}{L_c^n(f)-a}\right) + T\left(r, \frac{1}{L_c^n(f)-b}\right) + T\left(r, \frac{1}{f-a}\right) + T\left(r, \frac{1}{f-b}\right) \\ &= m\left(r, \frac{1}{L_c^n(f)-a}\right) + m\left(r, \frac{1}{L_c^n(f)-b}\right) + N\left(r, \frac{1}{L_c^n(f)-a}\right) \\ &\quad + N\left(r, \frac{1}{L_c^n(f)-b}\right) + m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) + N\left(r, \frac{1}{f-a}\right) \\ &\quad + N\left(r, \frac{1}{f-b}\right) \\ &\leq m\left(r, \frac{1}{L_c^n(f)}\right) + m\left(r, \frac{1}{L_c^n(f)-a}\right) \\ &\quad + m\left(r, \frac{1}{L_c^n(f)-b}\right) + N\left(r, \frac{1}{L_c^n(f)-b}\right) + T(r, f) + S(r, f) \\ &\leq m\left(r, \frac{1}{\left(L_c^n(f)\right)'}\right) + T\left(r, \frac{1}{L_c^n(f)-b}\right) + T(r, f) + S(r, f). \end{aligned} \tag{4.14}$$

Since $N(r, f) = S(r, f)$, it is easy to see that

$$\begin{aligned} T\left(r, \frac{1}{L_c^n(f) - b}\right) &= m(r, \left(L_c^n(f)\right)') + N\left(r, \left(L_c^n(f)\right)'\right) + O(1) \\ &= m\left(r, L_c^n(f)\right) + m\left(r, \frac{\left(L_c^n(f)\right)'}{L_c^n(f)}\right) + N\left(r, L_c^n(f)\right) \\ &\quad + \overline{N}\left(r, L_c^n(f)\right) + S(r, f) \\ &\leq m\left(r, L_c^n(f)\right) + S(r, f) \leq T(r, L_c^n(f)) + S(r, f). \end{aligned} \tag{4.15}$$

Thus, from (4.5), (4.14) and (4.15), by *First Fundamental Theorem of Nevanlinna*, we obtain

$$\begin{aligned} 2T(r, L_c^n(f)) + 2T(r, f) &= T\left(r, \frac{1}{L_c^n(f) - a}\right) + T\left(r, \frac{1}{L_c^n(f) - b}\right) \\ &\quad + T\left(r, \frac{1}{f - a}\right) + T\left(r, \frac{1}{f - b}\right) \\ &\leq T\left(r, L_c^n(f)\right) + T\left(r, \frac{1}{L_c^n(f) - b}\right) + T(r, f) + S(r, f) \\ &\leq 2T(r, L_c^n(f)) + T(r, f) + S(r, f) \end{aligned}$$

which shows that $T(r, f) \leq S(r, f)$, a contradiction. Therefore, we must have $\Psi \equiv 0$, that is

$$\frac{\left(L_c^n(f)\right)'}{L_c^n(f) - a} = \frac{f'}{f - a}. \tag{4.16}$$

By integrating (4.16), we obtain

$$\frac{L_c^n(f) - a}{f - a} = B_1, \tag{4.17}$$

where B_1 is a non-zero constant. On the other hand, since $L_c^n(f)$ and f share the value b CM, proceeding in a similar way, it can be shown that

$$\frac{L_c^n(f) - b}{f - b} = B_2, \tag{4.18}$$

where B_2 is a non-zero constant.

We now discuss the following two cases:

Case 1. If $B_1 = 1$ or $B_2 = 1$, then from (4.17) and (4.18), we easily obtain $L_c^n(f) \equiv f$. This can be written as

$$a_n f(z + nc) + \cdots + a_1 f(z + c) + (a_0 - 1) f(z) = 0, \tag{4.19}$$

where $a_n \neq 0$ and $a_0 \neq 1$. This equation, being a linear difference equation with constant coefficients, is always explicitly solvable. To show this, we prove here a formula similar to the Leibnitz rule. Henceforth, we put $f(z) = w(z)x(z)$. Then by the principle of Mathematical Induction, it can be shown that

$$f(z + mc) = w(z + mc) \sum_{k=0}^{m} \binom{m}{k} \Delta^k x(z), \ (m = 1.2, \ldots) \tag{4.20}$$

In order to find the explicit solution of (4.19), we define

$$b_k = \begin{cases} a_k, \ \text{if } k \neq 0 \\ a_0 - 1, \ \text{if } k = 0. \end{cases}$$

Then (4.19) can be written as

$$b_n f(z + nc) + \cdots + b_1 f(z + c) + b_0 f(z) = 0, \tag{4.21}$$

where $b_n \neq 0$ and $b_0 \neq 0$. We put $f(z) = \rho^{z/c}x(z)$. Substituting this into (4.21), and taking account of (4.20), we obtain

$$\begin{aligned}\sum_{m=0}^{n} b_m f(z+mc) &= \sum_{m=0}^{n} b_m \left(\rho^{(z+m)/c}\sum_{k=0}^{m}\binom{m}{k}\Delta^k x(z)\right)\\ &= \sum_{k=0}^{n}\Delta^k x(z)\left(\sum_{m=k}^{n}\binom{m}{k}b_m\rho^{(z+m)/c}\right)\\ &= \sum_{k=0}^{n}\frac{\rho^{(z+k)/c}}{k!}\Delta^k x(z)\left(\sum_{m=k}^{n}\prod_{j=0}^{k-1}(m-j)b_m\rho^{m-k}/c\right)\\ &= \sum_{k=0}^{n}\frac{\rho^{(z+k)/c}}{k!}\Delta^k x(z) f^{(k)}(\rho) = 0,\end{aligned}$$

where $f^{(k)}(\rho)$ denotes the k-th derivative

$$f(\rho) = b_n\rho^n + b_{n-1}\rho^{n-1} + \cdots + b_1\rho + b_0. \tag{4.22}$$

We call the equation $f(\rho) = 0$, the characteristic equation. Let ρ be a root of the equation (4.22) and take $x(z) = 1$. Then it is easy to see that $f(z) = \rho^{z/c}$ becomes a solution of (4.21).

Case A. Let the characteristic equation (4.22) has n distinct roots, say ρ_j $(j = 1, 2, \ldots, n)$. Then we have the following n particular solutions

$$\rho_1^{z/c}, \rho_2^{z/c}, \ldots, \rho_n^{z/c},$$

of (4.21) which form a fundamental set of solutions of (4.21). One can compute the Casoratian with respect to them as

$$C(\rho_1, \rho_2, \ldots, \rho_n) = (\rho_1\rho_2\cdots\rho_n)^{z/c}\, V(\rho_1, \rho_2, \ldots, \rho_n),$$

where $V(\rho_1, \rho_2, \ldots, \rho_n)$ is the Vandermonde determinant

$$V(\rho_1, \rho_2, \ldots, \rho_n) = \prod_{1\leq k<j\leq n}\left(\rho_j - \rho_k\right) \neq 0.$$

Therefore, the general solution of (4.21) can be expressed as

$$f(z) = \rho_1^{z/c}\pi_1(z) + \rho_2^{z/c}\pi_2(z) + \cdots + \rho_n^{z/c}\pi_n(z),$$

where $\pi_j(z)$ $(j = 1, 2, \ldots, n)$ are periodic functions with period c.

Case B. Let the characteristic equation (4.22) has multiple roots, say $\sigma_1, \sigma_2, \ldots, \sigma_q$ with respective multiplicities $N_1, N_2, \ldots, N_q$. Then, as solutions corresponding to σ_j $(j = 1, 2, \ldots, q)$, we obtain N_j linearly independent solution

$$\sigma_j^{z/c}, z\sigma_j^{z/c}, \ldots, z^{N_j-1}\sigma_j^{z/c}.$$

As a matter of course, we must have $N_1 + N_2 + \ldots + N_q = n$. Then we have q-sets of solutions of the form

$$S_j = \left\{\sigma_j^{z/c}, z\sigma_j^{z/c}, \ldots, z^{N_j-1}\sigma_j^{z/c}\right\},\ (j = 1, 2, \ldots, q) \tag{4.23}$$

and a routine computation shows that (4.23) forms a fundamental set of solutions of (4.21). Thus the general solution is expressed by a linear combination of them with c-periodic functions as its coefficients. Hence, the general solution is

$$f(z)=\left(\sum_{m_1=1}^{N_1-1} z^{m_1}\pi_{m_1}(z)\right)\sigma_1^{z/c}+\cdots+\left(\sum_{m_q=1}^{N_q-1} z^{m_q}\pi_{m_q}(z)\right)\sigma_q^{z/c},$$

where π_j's are periodic functions with period c.

Case 2. If $B_1 \neq 1$ and $B_2 \neq 1$, then it is easy to see from (4.17) and (4.18) that

$$\big(L_c^n(f)-a\big)-\big(L_c^n(f)-b\big)=(B_1f-B_1a)-(B_2f-B_2b)$$

which can be written as

$$(B_1-B_2)f=(b-a)+(B_1a-B_2b). \tag{4.24}$$

If $B_1 \neq B_2$, then it is easy to see from (4.24) that f is a constant, which is a contradiction. Therefore, we must have $B_1=B_2$, hence, it follows from (4.24) that $B_1=1=B_2$, which leads to a contradiction. This completes the proof. □

Proof of corollary 2.4 Let $a_j=\binom{n}{j}(-1)^{n-j}$, where $j=0,1,2,\ldots,n$. Then it is easy to see that $L_c^n(f)=\Delta_c^n f$ with $\sum_{j=0}^{n} a_j=0$. By the assumption, $\Delta_c^n f$ and f share the values a and b CM, hence following exactly the proof of Theorem 2.1, it can be easily shown that $\Delta_c^n f \equiv f$. This linear difference equation can be written as

$$\Delta_c^n f(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(z+kc)=f(z). \tag{4.25}$$

We can solve (4.25) in terms of exponential functions with coefficients in $\mathcal{M}_c$. Since, the distinct roots of $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\lambda^k=1$ are $\lambda_k=1+e^{2k\pi i/n}$, where $k=0,1,\ldots,n-1$ (see [8]), a routine computation then shows that

$$\mathcal{S}_\lambda^* := \{\lambda_1,\lambda_2,\ldots,\lambda_n\}$$

forms a fundamental set of solutions to the difference equation $\Delta_c^n f \equiv f$. Therefore, the general solution of (4.25) is

$$f(z)=\sum_{k=0}^{n}\lambda_k^{z/c}\pi_k(z),\ \text{where } \pi_k\in\mathcal{M}_c.$$

This completes the proof. □

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Declarations

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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