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ORIGINAL RESEARCH

Secure domination of some graph operators

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Abstract A set $S \subseteq V(G)$ is said to be **secure** if the security condition, for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$ holds. Now, a set $S \subseteq V(G)$ is **secure dominating** if it is both secure and dominating. The secure domination number of G is the minimum cardinality of a secure dominating set in G. In this paper, we have obtained results regarding the secure domination number of some graph operators.

Keywords Secure set · Secure domination number · Generalized Mycielskian · Double graph · Corona

Mathematics Subject Classification 05C69 · 05C76

1 Introduction

All graphs considered here are finite, undirected and simple. For standard terminology and notations related to graph theory we follow Balakrishnan and Ranganathan [1]. Let G be a graph and $v \in V(G)$. The number of edges incident at v in G is called the degree of the vertex v in G, denoted by $d_G(v)$. The minimum degree of the vertices of a graph G is denoted by $\delta(G)$. For $x \in V(G)$, the open neighborhood of x is the set $N_G(x) = \{v \in V(G)/uv \in E(G)\}$. The closed neighborhood of $x \in V(G)$ is the set $N_G[x] = \{x\} \cup N_G(x)$. For a set $S \subseteq V(G)$, $N(S) = \cup_{x \in S} N[x]$ and $N[S] = S \cup N(S)$. If $N[S] = V(G)$ then S is a dominating set of G. The **domination number** of a graph, $\gamma(G)$ is the cardinality of a minimum dominating set in G.

The study of alliances in graphs was first investigated by Kristiansen et al. [11] by defining different types of alliances that have been extensively studied in the last decade. This was motivated by the alliance of nations in war for mutual support. Essentially, Kristiansen et al. [11] studied the mathematical properties of defensive alliances in graphs. In [3], Brigham et al. modelled this concept and they coined the term Security instead of 'alliance'. Thus Security in graphs was first studied by Brigham et al. in 2007 [3].

While studying the security of S, we can consider vertices of $N[x] - S$ as attackers of x and vertices in $N[x] \cap S$ as defenders of x, where $x \in S$. A set $S \subseteq V(G)$ is secure if for any subset $X \subseteq S$, any attack on vertices of X can be prevented. It is observed in [3], a set S is secure if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$. The cardinality of a minimum secure set in G is called the **security number** of G, $s(G)$. In [3], they obtained some necessary and sufficient conditions for a set to be secure. In [7], exact value

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of the security number for some families of graphs were studied. In [8], Dutton et al. studied some bounds on a graph's security number.

If $S \subseteq V(G)$ is both dominating and secure then it is a secure dominating set. The minimum cardinality of a secure dominating set in G is the **secure domination number** of G and is denoted as $\gamma_s(G)$. In [2], they found secure dominating sets in cylindrical and toroidal grid. In [4], we have studied the security number and secure domination number of Honeycomb Networks. This work is truly motivated by those works.

The concept of secure domination used in this paper is different from the concept of secure domination in graphs that was introduced by Cockayne et al. in 2005 in his paper 'Protection of Graph' [5]. More reference on this can be seen in [6,9,10,15] and [16]. We use the same terminology as it's defined and studied as 'Secure domination' in [14] and [2] and the term 'security' is coined in many celebrated papers such as [3,7], and [8] with the same meaning as in this paper.

The **direct product** of two graphs G and H denoted by $G \times H$ is the graph with vertex set $V(G) \times V(H)$. The adjacency is defined as follows. A vertex (v_1, w_1) will be adjacent to (v_2, w_2) in $G \times H$ if and only if v_1 adjacent to v_2 in G and w_1 adjacent to w_2 in H. In [13], the **double graph** $D(G)$ of a simple graph G is introduced and it is the direct product of G with the graph obtained by the complete graph K_2 adding a loop to each vertex. Thus, if $V(G) = \{u_1, u_2, \dots, u_n\}$ then the vertices of $D(G)$ will be (u_i, v_j), where $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2\}$. More over the vertices (u_i, v_j) and (u_x, v_y) in $D(G)$ are adjacent whenever u_i is adjacent to u_x in G.

The **corona** $G \circ H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H. Moreover join each vertex of the ith copy of H to the i^{th} vertex of G, where $1 \leq i \leq |V(G)|$. If $V(G) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(H) = \{v_1, v_2, \dots, v_{n_2}\}$ then $V(G \circ H) = \{u_1, u_2, \dots, u_{n_1}, (v_i, u_j)/\, i \in \{1, 2, \dots, n_2\}$ and $j \in \{1, 2, \dots, n_1\}$ such that each (v_i, u_j) is adjacent to u_j and u_i's are adjacent as in G.

For a graph $G = (V, E)$, the **Mycielskian of** G is defined in [12] as the graph $\mu(G)$ with the vertex set $V(\mu(G)) = V \cup V' \cup \{z\}$, where $V' = \{u' : u \in V\}$ and the edge set $E(\mu(G)) = E \cup \{uv' : uv \in E\} \cup \{v'z : v' \in V'\}$. The vertex v' is called the twin of the vertex v and vice versa. The vertex z is called the root of $\mu(G)$.

The **generalized Mycielskian** of a graph is defined as follows. Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \dots, v_n^0\}$ and edge set E^0. Given an integer $m \geq 1$, the m- Mycielskian of G, denoted by $\mu_m(G)$ is the graph with vertex set $V^0 \cup V^1 \cup V^2 \cdots V^m \cup \{z\}$, where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the ith distinct copy of V^0 for $i = 1, 2, \dots, m$ and edge set $E^0 \cup \left(\cup_{i=0}^{m-1}\{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\}\right) \cup \{v_j^m z : v_j^m \in V^m\}$. $\mu_0(G)$ is defined to be the graph obtained from G by adding a universal vertex z and the Mycielskian of G is simply $\mu_1(G)$. We call the vertices of V^i as vertices of level i.

In this paper we study the secure domination number of some graph operators.

2 Preliminary results

The following are some known results that we used.

- 2.1: **Security Condition**: A set $S \subseteq V(G)$ is secure in G if and only if for any $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$ [3].
- 2.2: $\gamma_s^c(G) \geq \gamma_s(G) \geq |V(G)|/2$ [2].

In this paper we say that $S \subseteq V(G)$ is **Strongly Secure** if for any $X \subseteq S$, $|N[X] \cap S| > |N[X] - S|$ is satisfied.

3 Generalized Mycielskian

Theorem 3.1 *Let G be any graph with n vertices. Then $s(\mu_m(G)) \leq n$.*

Proof Consider $S = \{u_1^0, u_2^0, \dots, u_n^0\}$ in $\mu_m(G)$. Let X be any subset of S.

Case 1: $X = S$
$\left|N_{\mu_m(G)}[X] \cap S\right| = n$. In $N_{\mu_m(G)}(X) - S$, there exists n vertices $\{u_1^1, u_2^1, \dots, u_n^1\}$. Thus, $\left|N_{\mu_m(G)}(X) - S\right| = n$.

Case 2: $X = \{u_1^0, u_2^0, \ldots, u_k^0 \subseteq S\}$
Let $N_G[X]$ be $\{u_1, u_2, \ldots, u_k, u_i, u_j, \ldots, u_m\}$ and $|N_G[X]| = k + l$. Then, $N_{\mu_m(G)}[X]$ is $\{u_1^0, u_2^0, \ldots, u_k^0, u_i^0, u_j^0, \ldots, u_m^0, u_1^1, u_2^1, \ldots, u_k^1, u_i^1, u_j^1, \ldots, u_m^1\}$. Thus, $N_{\mu_m(G)}[X] \cap S = \{u_1^0, u_2^0, \ldots, u_k^0, u_i^0, u_j^0, \ldots, u_m^0\}$ and $N_{\mu_m(G)}(X) - S = \{u_1^1, u_2^1, \ldots, u_k^1, u_i^1, u_j^1, \ldots, u_m^1\}$, when $\{u_a/a \in \{1, 2, \ldots, k\}\}$ are adjacent in G or $\{u_i^1, u_j^1, \ldots, u_m^1\}$ when $\{u_a/a \in \{1, 2, \ldots, k\}\}$ are not adjacent in G . Then, $\left|N_{\mu_m(G)}[X] \cap S\right| = k+l$ and $|N[X] - S| \le k+l$. Any sub set of S is a secure set in this case too. Hence, the security condition holds in these cases and $s(\mu_m(G)) \le n$.

□

Theorem 3.2 *Let G be any graph with n vertices, then $\gamma_s(\mu(G)) = n + 1$.*

Proof We have $\gamma_s(G) \ge \left\lceil \frac{V(G)}{2} \right\rceil$, by the condition 2.2. Thus, $\gamma_s(\mu(G)) \ge \left\lceil \frac{2n+1}{2} \right\rceil = n + 1$. Consider $S = \{u_1, u_2, \ldots, u_n, u_n{}'\}$ in $\mu(G)$. Clearly S is a dominating set, since the vertices u_i dominates the vertices u_i' and u_n' dominates the vertex z. Now, we have to prove that S is secure in G. Let X be any subset of S.

Case 1: $X = S$
$\left|N_{\mu(G)}[X] \cap S\right| = n + 1$. In $N_{\mu(G)}(X) - S$, there exists n vertices $\{u_1{}', u_2{}', \ldots, u_{n-1}{}', z\}$. Thus, $\left|N_{\mu(G)}[X] \cap S\right| > \left|N_{\mu(G)}(X) - S\right|$.
Case 2: $X = \{u_1, u_2, \ldots, u_k\} \subseteq \{u_1, u_2, \ldots, u_n\}$.
Let $N_G[X]$ be $\{u_1, u_2, \ldots, u_k, u_i, u_j, \ldots, u_m\}$ and $|N_G[X]| = l$.
Case 2 (i): When $X = \{u_a/a \in \{1, 2, \ldots, k\}\}$ are adjacent in G.
$N_{\mu(G)}[X] = \{u_1, u_2, \ldots, u_k, u_i, u_j, \ldots, u_m, u_1', u_2', \ldots, u_k', u_i', u_j', \ldots, u_m'\}$. Thus, $N_{\mu(G)}[X] \cap S = \{u_1, u_2, \ldots, u_k, u_i, \ldots, u_m\}$
$N_{\mu(G)}(X) - S = \{u_1', u_2', \ldots, u_k', u_i', \ldots, u_m'\}$. Then, $\left|N_{\mu(G)}[X] \cap S\right| = l$ and $\left|N_{\mu(G)}(X) - S\right| = l$.
Case 2 (ii): When $X = \{u_a/a \in \{1, 2, \ldots, k\}\}$ are not adjacent in G.
Then, $N_{\mu(G)}[X] = \{u_1, u_2, \ldots, u_k, u_i, u_j, \ldots, u_m, u_i', u_j', \ldots, u_m'\}$. Thus, $N_{\mu(G)}[X] \cap S = \{u_1, u_2, \ldots, u_k, u_i, u_j, \ldots, u_m, \}$ and $N_{\mu(G)}(X) - S = \{u_i', u_j', \ldots, u_m'\}$. Then, $\left|N_{\mu(G)}[X] \cap S\right| = l$ and $\left|N_{\mu(G)}(X) - S\right| < l$. Hence the security condition holds in these cases.
Case 2 (iii): When $X = \{u_a, u_b/a, b \in \{1, 2, \ldots, k\}\}$ u_a's are adjacent in G and u_b's are not adjacent in G Proof follows from the above two cases.
Case 3: $X = u_n'$
Let $N_G[u_n]$ be $\{u_i, u_j, \ldots, u_s, u_n\}$, then $N_{\mu(G)}[u_n'] = \{u_i, u_j, \ldots, u_s, u_n', z\}$. Then, $N_{\mu(G)}[X] \cap S = \{u_i, u_j, \ldots, u_s, u_n'\}$ and $\left|N_{\mu(G)}[X] - S\right| = 1$. Then, $\left|N_{\mu(G)}(X) \cap S\right| = deg(u_n) + 1$. Hence the security condition holds. Thus $\gamma_s(\mu(G)) \le n + 1$. Hence, $\gamma_s(\mu(G)) = n + 1$.

□

Theorem 3.3 *Let G be any graph, then $\left\lceil \frac{n(m+1)+1}{2} \right\rceil \le \gamma_s(\mu_m(G)) \le n \left\lceil \frac{(m+1)}{2} \right\rceil + n$.*

Proof We have $\gamma_s(G) \ge \left\lceil \frac{V(G)}{2} \right\rceil$, by the condition 2.2. Thus, $\gamma_s(\mu_m(G)) \ge \left\lceil \frac{n(m+1)+1}{2} \right\rceil$.

Case I : When m is odd We have to prove that $\gamma_s(\mu_m(G)) \le n \left\lceil \frac{(m+1)}{2} \right\rceil + n$.

Case (1) : When $\frac{m-1}{2}$ is even
Consider $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{1, 2, 5, 6, \ldots, m-4, m-3, m-1, m\}$ in $\mu_m(G)$. [When $m = 5$, $i \in \{1, 2, 4, 5\}$]. Clearly S is a dominating set, since the vertices $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ dominates the vertices $\{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{0, 3, 4, \ldots, m-6, m-5, m-2\}$. Now, we have to prove that S is secure in G. Let X be any subset of S.
Case 1(i): $X = S$
In $\mu_m(G)$, there are $m+1$ layers, which is even. Thus, $\left|N_{\mu_m(G)}[X] \cap S\right| = n\left(\frac{m+1}{2}\right) + n$. In $N_{\mu_m(G)}(X) - S$, there exists $n\frac{(m+1)}{2} + 1 - n$ vertices $\{u_1^i, u_2^i, \ldots, u_n^i, z\}$ where $i \in \{0, 3, 4, \ldots, m-5, m-2\}$. Thus, $\left|N_{\mu_m(G)}[X] \cap S\right| > \left|N_{\mu_m(G)}(X) - S\right|$.
Case 1(ii): $X = \{u_1^i, u_2^i, \ldots, u_n^i\}$, where $i \in \{1, 2, 5, 6, \ldots, m-4, m-3, m-1\}$
In this case, $N_{\mu_m(G)}[X] = \{u_1^{i-1}, u_2^{i-1}, \ldots, u_n^{i-1}, u_1^i, u_2^i, \ldots, u_n^i, u_1^{i+1}, u_2^{i+1}, \ldots, u_n^{i+1}\}$.

$N_{\mu_m(G)}[X] \cap S = \{u_1^i, u_2^i, \ldots, u_n^i, u_1^{i+1}, u_2^{i+1}, \ldots, u_n^{i+1}\}$ or $\{u_1^{i-1}, \ldots, u_n^{i-1}, u_1^i, u_2^i, \ldots, u_n^i\}$ and $N_{\mu_m(G)}[X] - S = \{u_1^{i-1}, u_2^{i-1}, \ldots, u_n^{i-1}\}$ or $\{u_1^{i+1}, u_2^{i+1}, \ldots, u_n^{i+1}\}$. Hence, $\left|N_{\mu_m(G)}[X] \cap S\right| = 2n$ and $\left|N_{\mu_m(G)}(X) - S\right| = n$. Any sub set of S is a secure set in this case too.

Case 1(iii): $X = \{u_1^i, u_2^i, \ldots, u_k^i\}$, where $k < n$ and $i \in \{1, 2, 5, 6, \ldots, m-4, m-3, m-1\}$.

Case a : When $X = \{u_a/a \in \{1, 2, \ldots, k\}\}$ are adjacent in G

Let $N_G[X]$ be $\{u_1, u_2, \ldots, u_k, u_a, u_b, \ldots, u_z\}$ and $|N_G[X]| = k + l$. Then $N_{\mu_m(G)}[X] = \{u_1^{i-1}, u_2^{i-1}, \ldots, u_k^{i-1}, u_a^{i-1}, u_b^{i-1}, \ldots, u_z^{i-1}, u_1^i, u_2^i, \ldots, u_k^i, u_1^{i+1}, \ldots, u_k^{i+1}, u_a^{i+1}, u_b^{i+1}, \ldots, u_z^{i+1}\}$. Thus, $N_{\mu_m(G)}[X] \cap S = \{u_1^i, u_2^i, \ldots, u_k^i, u_1^{i+1}, u_2^{i+1}, \ldots, u_k^{i+1}, u_a^{i+1}, u_b^{i+1}, \ldots, u_z^{i+1}\}$ or $\{u_1^{i-1}, u_2^{i-1}, \ldots, u_k^{i-1}, u_a^{i-1}, u_b^{i-1}, \ldots, u_z^{i-1}, u_1^i, u_2^i, \ldots, u_k^i\}$ and $N_{\mu_m(G)}(X) - S = \{u_1^{i-1}, u_2^{i-1}, \ldots, u_k^{i-1}, u_a^{i-1}, u_b^{i-1}, \ldots, u_z^{i-1}\}$ or $\{u_1^{i+1}, u_2^{i+1}, \ldots, u_k^{i+1}, u_a^{i+1}, u_b^{i+1}, \ldots, u_z^{i+1}\}$. Hence, $|N_{\mu_m(G)}[X] \cap S| \leq |X| + (k+l) = k + (k+l)$ and $\left|N_{\mu_m(G)}(X) - S\right| \leq k + l$. Any sub set of S is a secure set in this case too.

Case b :When $X = \{u_a/a \in \{1, 2, \ldots, k\}\}$ are not adjacent in G

Let $N_G[X]$ be $\{u_1, u_2, \ldots, u_k, u_a, u_b, \ldots, u_z\}$ and $|N_G[X]| = k + l$. Then, $N_{\mu_m(G)}[X] = \{u_a^{i-1}, u_b^{i-1}, \ldots, u_z^{i-1}, u_1^i, u_2^i, \ldots, u_k^i, u_a^{i+1}, u_b^{i+1}, \ldots, u_z^{i+1}\}$. Thus, $N_{\mu_m(G)}[X] \cap S = \{u_1^i, u_2^i, \ldots, u_k^i, u_a^{i+1}, u_b^{i+1}, \ldots, u_z^{i+1}\}$ or $\{u_a^{i-1}, u_b^{i-1}, \ldots, u_z^{i-1}, u_1^i, u_2^i, \ldots, u_k^i\}$ and $N_{\mu_m(G)}(X) - S = \{u_a^{i-1}, u_b^{i-1}, \ldots, u_z^{i-1}\}$ or $\{u_a^{i+1}, u_b^{i+1}, \ldots, u_z^{i+1}\}$. Hence, $|N_{\mu_m(G)}[X] \cap S| = |X| + l = k + l$ and $|N_{\mu_m(G)}(X) - S| = l$. Any sub set of S is a secure set in this case too.

Case c: When $X = \{u_a, u_b/a, b \in \{1, 2, \ldots, k\}\}$ u_a's are adjacent in G and u_b's are not adjacent in G

Proof follows from the above Case(a) and Case(b).

Case 1(iv): $X \subseteq \{u_1^m, u_2^m, \ldots, u_n^m\}$.

Let $N_G[X]$ be $\{u_1, u_2, \ldots, u_k, u_a, u_b, \ldots, u_z\}$ and $|N_G[X]| = k + l$. Let $N_{\mu_m(G)}[X] = \{u_1^{m-1}, u_2^{m-1}, \ldots, u_k^{m-1}, u_a^{m-1}, u_b^{m-1}, \ldots, u_z^{m-1}, u_1^m, u_2^m, \ldots, u_k^m, z\}$. Then, $N_{\mu_m(G)}[X] \cap S = \{u_1^{m-1}, u_2^{m-1}, \ldots, u_k^{m-1}, u_a^{m-1}, u_b^{m-1}, \ldots, u_z^{m-1}, u_1^m, u_2^m, \ldots, u_k^m\}$.

Hence, $\left|N_{\mu_m(G)}[X] \cap S\right| \leq k + (k+l)$ and $\left|N_{\mu_m(G)}(X) - S\right| = 1$. In any other case, it will be a combination of the above four cases and hence the security condition will hold.

Case(2) : When $\frac{m-1}{2}$ is odd

Consider $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{1, 2, 5, 6, \ldots, m-6, m-5, m-2, m-1, m\}$ in $\mu_m(G)$. Clearly S is a dominating set, since the vertices $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ dominates the vertices $\{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{0, 3, 4, \ldots, m-8, m-7, m-4, m-3\}$.

Now, we have to prove that S is secure in G. Let X be any subset of S.

Case 2(i): $X = S$

Proof follows same as in Case 1(i).

Case 2(ii): $X = \{u_1^i, u_2^i, \ldots, u_n^i\}$, where $i \in \{1, 2, 5, 6, \ldots, m-6, m-5, m-2\}$

Proof follows same as in Case 1(ii).

Case 2(iii): $X = \{u_1^i, u_2^i, \ldots, u_k^i\}$, where $k < n$ and $i \in \{1, 2, 5, 6, \ldots, m-6, m-5, m-2\}$.

Proof follows same as in Case 1(iii).

Case 2(iv): $X \subseteq \{u_1^{m-1}, u_2^{m-1}, \ldots, u_n^{m-1}\}$

$N_{\mu_m(G)}[X] = \{u_1^{m-2}, u_2^{m-2}, \ldots, u_k^{m-2}, u_a^{m-2}, u_b^{m-2}, \ldots, u_z^{m-2}, u_1^{m-1}, \ldots, u_n^{m-1}, u_1^m, \ldots, u_k^m, u_a^m, \ldots, u_z^m\}$. Then, $N_{\mu_m(G)}[X] \cap S = \{u_1^{m-2}, u_2^{m-2}, \ldots, u_k^{m-2}, u_a^{m-2}, u_b^{m-2}, \ldots, u_z^{m-2}, u_1^{m-1}, u_2^{m-1}, \ldots, u_n^{m-1}, u_1^m, u_2^m, \ldots, u_k^m, u_a^m, u_b^m, \ldots, u_z^m\}$. Hence, $\left|N_{\mu_m(G)}[X] \cap S\right| \leq k + 2(k+l)$ and $\left|N_{\mu_m(G)}(X) - S\right| = 0$.

Case 2(v): $X \subseteq \{u_1^m, u_2^m, \ldots, u_n^m\}$

$N_{\mu_m(G)}[X] = \{u_1^{m-1}, u_2^{m-1}, \ldots, u_n^{m-1}, u_a^{m-1}, u_b^{m-1}, \ldots, u_z^{m-1}, u_1^m, u_2^m, \ldots, u_k^m, z\}$. Then, $N_{\mu_m(G)}[X] \cap S = \{u_1^{m-1}, u_2^{m-1}, \ldots, u_n^{m-1}, u_a^{m-1}, u_b^{m-1}, \ldots, u_z^{m-1}, u_1^m, u_2^m, \ldots, u_k^m\}$. Hence, $\left|N_{\mu_m(G)}[X] \cap S\right| \leq k + (k+l)$ and $\left|N_{\mu_m(G)}(X) - S\right| = 1$.

In any other case, it will be a combination of the above five cases and hence the security condition will hold. Hence, $\gamma_s(\mu_m(G)) \leq n \left\lceil \frac{(m+1)}{2} \right\rceil + n$.

Case II : When m is even

We have to prove that $\gamma_s(\mu_m(G)) \leq n \left\lceil \frac{(m+1)}{2} \right\rceil + n$.

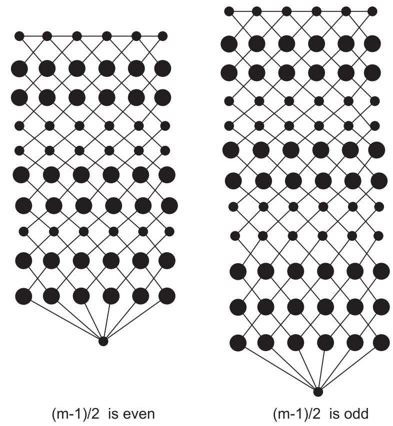

Fig. 1 .

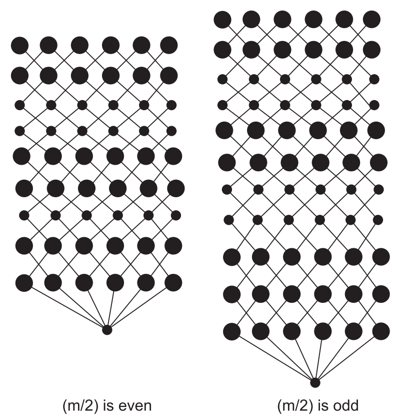

Fig. 2 .

Case(1) : When $\frac{m}{2}$ is even
Consider $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{0, 1, 4, 5, \ldots, m-4, m-3, m-1, m\}$ in $\mu_m(G)$. [When $m = 4$, $i \in \{0, 1, 3, 4\}$]. Clearly S is a dominating set, since the vertices $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ dominates the vertices $\{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{2, 3, 6, 7, \ldots, m-6, m-5, m-2\}$. Now, we have to prove that S is secure in $\mu_m(G)$. Let X be any subset of S. Proof follows same as in Case I (1).
Case(2) : When $\frac{m}{2}$ is odd
Consider $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{0, 1, 4, 5, \ldots, m-6, m-5, m-2, m-1, m\}$ in $\mu_m(G)$. Clearly S is a dominating set, since the vertices $S = \{u_1^i, u_2^i, \ldots, u_n^i\}$ dominates the vertices $\{u_1^i, u_2^i, \ldots, u_n^i\}$ where $i \in \{2, 3, 6, 7, \ldots, m-8, m-7, m-4, m-3\}$.
Now, we have to prove that S is secure in G. Proof follows same as in Case I(2). Thus, the security condition holds in these cases and $\gamma_s(\mu_m(G)) \le n \left\lceil \frac{(m+1)}{2} \right\rceil + n$. Hence the result.

□

4 Double graphs

The following result gives an upper bound and a lower bound for the security number of double graph of any graph.

Lemma 4.1 *If G is a connected graph with n vertices in which the secure set is strongly secure, then* $2 \le s(D(G)) \le min\{n, 2s(G)\}$.

Proof We know that $s(G) = 1$ if and only if $\delta(G) \le 1$, [3]. From the definition of Double graphs, it's clear that if G is a connected graph then $\delta(D(G)) \ge 2$. Thus $2 \le s(D(G))$.

Let G be a connected graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$. Consider $S_1 = \{(u_i, v_1)/i \in \{1, 2, \ldots, n\}\}$ in $D(G)$.

(i) For any $(u_a, v_1) \in S_1$, $N_{D(G)}[(u_a, v_1)] \cap S_1 = deg_G(u_a) + 1$, where $N_{D(G)}((u_a, v_1)) - S_1 = deg_G(u_a)$. Thus, the security condition will hold for any subset X of S_1.

(ii) $X = \{(u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_k, v_1)\} \subseteq S_1$ in $D(G)$. Let $N_G[X]$ be $\{u_1, u_2, \ldots, u_k, u_i, u_j, \ldots, u_m\}$ and $|N_G[X]| = k+l$. Then $\left|N_{D(G)}[X] \cap S_1\right| = (k+l)$ and $N_{D(G)}(X) - S_1 = \{(u_1, v_2), (u_2, v_2)(u_3, v_2), \ldots, (u_k, v_2),$
$(u_i, v_2), (u_j, v_2), \ldots, (u_m, v_2)\}$, when $\{u_a/a \in \{1, 2, \ldots, k\}\}$ are adjacent in G
or $\{(u_i, v_2), (u_j, v_2), \ldots, (u_m, v_2)\}$ when $\{u_a/a \in \{1, 2, \ldots, k\}\}$ are not adjacent in G. Thus, $\left|N_{D(G)}[X] \cap S_1\right| = (k+l)$ and $|N[X] - S_1| \leq k + l$. Hence, the security condition will hold for any subset X of S_1. Thus, S_1 is a secure set in $D(G)$ and hence $s(D(G)) \leq n$.

Let $S = \{u_{a_1}, u_{a_2}, \ldots, u_{a_m}\}$, $a_i \in \{1, 2, \ldots, n\}$ be a strongly secure set in G. Consider $S_2 = \{(u_{a_1}, v_1), (u_{a_2}, v_1), \ldots, (u_{a_m}, v_1), (u_{a_1}, v_2), (u_{a_2}, v_2), \ldots, (u_{a_m}, v_2)\}$ which is a subset of $V(D(G))$. For any $X \subseteq S$, $|N[X] \cap S| > |N[X] - S|$ as S is a strongly secure set in G. Let $X' \subseteq S_2$ in $D(G)$. Corresponding to a vertex $(u_i, v_j) \in X'$, $i \in \{a_1, a_2, \ldots, a_m\}$ and $j \in \{1, 2\}$, there corresponds $u_i \in S$. Then $\left|N[X'] \cap S_2\right| \geq 2\,|N[X] \cap S| > 2\,|N[X] - S| = \left|N[X'] - S_2\right|$. □

Corollary 4.2 *If G is a connected graph with n vertices, then $s(D(G)) \leq n$.*

Corollary 4.3 *For any connected graph G, $s(D(G)) = 2$ if and only if $G = K_2$.*

Proof If G is K_2, $D(G) = C_4$ and hence $s(D(G)) = 2$. In the previous lemma, its proved that $s(D(G)) > 1$. It's known that $s(G) = 2$ if and only if $\delta(G) \geq 2$ and G has a set $S = \{u, v\}$ where u and v are adjacent and $|N[S] - S| \leq 2$ [3]. This condition is satisfied in $D(G)$ only if $G = K_2$. □

Theorem 4.4 *If G is a connected graph with n vertices, then $\gamma_s(D(G)) = n$.*

Proof If G is a connected graph with n vertices then $D(G)$ is a connected graph with $2n$ vertices. Thus, $\gamma_s(D(G)) \geq n$, by 2.2.

Let G be a graph with n vertices $\{u_1, u_2, \ldots, u_n\}$. Consider the set $S = \{(u_i, v_1)/i \in \{1, 2, \ldots, n\}\}$ in $D(G)$. This is clearly a dominating set in $D(G)$. Let $X \subseteq S$. Let $X' = \{u_i/i \in \{1, 2, \ldots, n\}$ where $(u_i, v_1) \in X\}$. Let $N_G(X') = k$. Then $\left|N_{D(G)}[X] \cap S\right| = k$. By the definition of $D(G)$, if u_x is adjacent to u_y in G then (u_x, v_1) is adjacent to (u_y, v_1) as well as (u_y, v_2). Thus $\left|N_{D(G)}(X) - S\right| = k$. Hence for any subset X of S, $\left|N_{D(G)}[X] \cap S\right| = \left|N_{D(G)}(X) - S\right|$. Thus S is a secure set in $D(G)$ and hence the result follows. □

5 Corona of graphs

Lemma 5.1 *For any two connected graphs $G = (n_1, m_1)$ and $H = (n_2, m_2)$, $2 \leq s(G \circ H) \leq n_2$.*

Proof As noted, $s(G) = 1$ if and only if $\delta(G) \leq 1$, [3]. For any two connected graphs G and H, $\delta(G \circ H) > 1$ and hence $s(G \circ H) \geq 2$. Consider $S = \{(v_1, u_1), (v_2, u_1), \ldots, (v_{n_2}, u_1)\}$. In $G \circ H$, for any $X \subseteq S$, $|N[X] - S| = 1 < |N[X] \cap S|$. □

Corollary 5.2 *If $H = P_n$, path on n-vertices, then $s(G \circ H) = 2$.*

Lemma 5.3 *For any two connected graphs $G = (n_1, m_1)$ and $H = (n_2, m_2)$ in which the secure set in H is strongly secure, then $s(G \circ H) \leq s(H)$.*

Proof Consider $S = \{(v_1, u_1), (v_2, u_1), \ldots, (v_k, u_1)\}$ in $G \circ H$ where $S' = \{v_1, v_2, \ldots, v_k\}$ is a secure set in H. For $X \subseteq S$, $|N_{G \circ H}[X] \cap S| = N_H[X] \cap S'$ and $|N_{G \circ H}(X) - S| = 1 + \left|N_H(X) - S'\right|$. So the security condition will hold for S in $G \circ H$. □

Theorem 5.4 *For any two connected graphs $G = (n_1, m_1)$ and $H = (n_2, m_2)$ with $\gamma_s(H) = \lceil n_2/2 \rceil$, $\gamma_s(G \circ H) \leq n_1 \lceil (1 + n_2/2) \rceil$.*

Proof Let $S_1 = \{v_1, v_2, \ldots, v_{\lceil n_2/2 \rceil}\}$ be any secure set of vertices in H. Then for any $X_1 \subseteq S_1$, $|N_H[X_1] \cap S_1| \geq |N_H(X_1) - S_1|$, by the security condition. Let $\{u_1, u_2, \ldots, u_{n_1}\}$ be the vertices of G. Let $\{(v_1, u_i), (v_2, u_i), \ldots, (v_{n_2}, u_i)\}$ be the vertices in the ith copy of H all vertices of which are attached to u_i in $G \circ H$. Consider a set of vertices $S = \{u_1, u_2, \ldots, u_{n_1}, (v_1, u_i), (v_2, u_i), \ldots, (v_{\lceil n_2/2 \rceil}, u_i)\}$, $i \in \{1, 2, \ldots, n_1\}$ in $G \circ H$. Because of the n_1 vertices of G in S, S is a dominating set in $G \circ H$.
Let X be any subset of S. Case 1: $X = S$

Clearly in $N_{G\circ H}(X) - S$, there exists $n_1 \lfloor n_2/2 \rfloor$ vertices which is clearly less than $|N_{G\circ H}[X] \cap S| = n_1 \lceil (1 + n_2/2) \rceil$.
Case 2: $X \subseteq \{u_1, u_2, \ldots, u_{n_1}\}$
Then $|N_{G\circ H}(X) - S| = |X| \lfloor n_2/2 \rfloor$ where as $|N_{G\circ H}[X] \cap S|$ is at least $|X| + |X| \lceil n_2/2 \rceil$. Thus any sub set of S is a secure set in this case too.
Case 3: $X \subseteq \{(v_1, u_i), (v_2, u_i), \ldots, (v_{\lceil n_2/2 \rceil}, u_i)\}, i \in \{1, 2, \ldots, n_1\}$
Since $\gamma_s(H) = \lceil n_2/2 \rceil$ and we have chosen those secure vertices from each copy of H, the security condition will hold in this case too. In any other case, it will be a combination of the above three cases and hence the security condition will hold. □

Corollary 5.5 *For any two connected graphs $G = (n_1, m_1)$ and $H = (n_2, m_2)$ with $\gamma_s(H) = \lceil n_2/2 \rceil$, where n_2 is odd and $|N_H[X] \cap S| > |N_H(X) - S|$ for $X \subseteq S \subseteq H$, then $\gamma_s(G \circ H) \leq n_1 \lceil n_2/2 \rceil$.*

Proof Let $S_1 = \{v_1, v_2, \ldots, v_{\lceil n_2/2 \rceil}\}$ be any secure dominating set of vertices in H. Consider any subset X_1 of S_1 in H. Let $X_1 = \{v_1, v_2, \ldots, v_k\}$ and $N_G[X_1] = \{v_1, v_2, \ldots, v_k, v_a, v_b, \ldots, v_m\}$. Then $|N_G[X_1] \cap S_1| = l$ and $|N_G(X_1) - S_1| < l$, by the assumption. ....(1)

Consider $S = \{(v_1, u_i), (v_2, u_i), \ldots, (v_{\lceil n_2/2 \rceil}, u_i)\}, i \in \{1, 2, \ldots, n_1\}$ in $G \circ H$. Clearly S is a dominating set.

Let X be any subset of S.

Case 1: $X = S$
$|N_{G\circ H}[X] \cap S| = n_1 \left\lceil \frac{n_2}{2} \right\rceil$ and $|N_{G\circ H}(X) - S| = n_1 + n_1 \left\lfloor \frac{n_2}{2} \right\rfloor = n_1(1 + \left\lfloor \frac{n_2}{2} \right\rfloor = n_1 \left\lceil \frac{n_2}{2} \right\rceil$. Thus, $|N_{G\circ H}[X] \cap S| = |N_{G\circ H}(X) - S|$.
Case 2: $X \subseteq \{(v_1, u_i), (v_2, u_i), \ldots, (v_{\lceil n_2/2 \rceil}, u_i)\}$
Let $|N_{G\circ H}[X] \cap S| = l$, then by eqn(1), $N_{G\circ H}(X) - S$ has a maximum of $(l - 1)$ vertices from H and a vertex u_i from G. Thus, $|N_{G\circ H}(X) - S| \leq (l - 1) + 1 \leq l$. Hence, any sub set of S is a secure set in this case too. In any other case, it will be a combination of the above two cases and hence the security condition will hold.

□

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