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ORIGINAL RESEARCH

Arithmetic properties of 3-regular 6-tuple partitions

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Abstract The objective of this paper is primarily on the study of various properties of the infinite family of congruences and divisibility for $BS_3(n)$ with the assistance of Hecke eigenforms and certain properties of modular forms which are generally arithmetic in nature. For n being a positive integer, $BS_3(n)$ represents its 3-regular 6-tuple partitions.

Keywords Partition · Congruence · Modular forms · Hecke eigenforms · Eta quotients

Mathematics Subject Classification 11P83 · 05A17

1 Introduction

In this paper, the basic assumption which is being made is that $| q | < 1$. The conventional notation for any positive integer k is denoted as

$$f_k := \prod_{i=0}^{\infty}(1 - q^{ki}).$$

Consider $\lambda_1, \lambda_2, \ldots, \lambda_k$ be a sequence of integers which are positive. This particular sequence is called a partition for any positive integer n, provided it is finite, non-increasing and most importantly $\sum_{i=1}^{k} \lambda_i = n$. Now, any partition of n can be referred as l-regular provided for integers such as $l \geq 2$, no parts of the partition is exactly divisible by l. For any positive integer n, the number of l-regular partitions is represented as $b_l(n)$ and its generating function can be formulated as

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}.$$

Various properties of l-regular partitions $b_l(n)$ had been explicitly mentioned and discussed by several mathematicians [1,2,6,8,10–12,16,23,26,27].

For any positive integer n, the l-regular partition triples is denoted by $T_l(n)$ and its subsequent generating function is given as

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$$\sum_{n=0}^{\infty} T_l(n)q^n = \frac{f_l^3}{f_1^3}.$$

Gireesh and Naika [13] investigated an arithmetic properties of congruences for $T_3(n)$, Saikia and Boruah [25] established for $l \in \{2, 3, 4, 5\}$, certain properties of the infinite families of congruences for $T_l(n)$ modulo 2, 4, 5, and 9. Recently, Chern et al. [9] obtained for $T_7(n)$ the infinite families of congruences modulo powers of 7. Lately, Baruah and Das [4] obtained the congruence properties modulo powers of 7 and the generating functions for $T_9(n)$ and $T_{27}(n)$.

Now, for a positive integer n, $BS_3(n)$ represents the number of 3-regular 6-tuple partitions and the subsequent generating function is defined as

$$\sum_{n=0}^{\infty} BS_3(n)q^n = \frac{f_3^6}{f_1^6}. \tag{1.1}$$

Certain congruence properties of $BS_3(n)$ with aid of the theory of Hecke eigenforms and modular forms are discussed in this paper. Also, we investigated divisibility properties of $BS_3(n)$. The below mentioned statements form our main findings.

Theorem 1.1 *Let $k, n \geq 0$ and j are integers. $p_i > 2$ is prime such that $p_i \equiv 3 \pmod 4$ for $i \in \{1, 2, \ldots, k+1\}$. Then, for $j \not\equiv 0 \pmod{p_{k+1}}$, the following result is obtained*

$$BS_3\left(6p_1^2 \ldots p_{k+1}^2 n + \frac{3p_1^2 \ldots p_k^2 p_{k+1}(p_{k+1} + 4j) - 1}{2}\right) \equiv 0 \pmod 4. \tag{1.2}$$

Corollary 1.1 *Let $p > 2$ be a prime with $p \equiv 3 \pmod 4$. Then, for $j \not\equiv 0 \pmod p$, where j is an integer, a subsequent result follows*

$$BS_3\left(6p^{2k+2}n + 6p^{2k+1}j + \frac{3p^{2k+2} - 1}{2}\right) \equiv 0 \pmod 4.$$

Theorem 1.2 *Let $d_a(n)$ points the number of divisors d of n which satisfies $d \equiv a \pmod 4$, then we have*

$$BS_3(6n + 1) \equiv 2[d_1(4n + 1) - d_3(4n + 1)] \pmod 4 \tag{1.3}$$

Theorem 1.3 *Let $p \equiv 1 \pmod 4$ be prime. For all $k > 0$ and $n \geq 0$*
if $BS_3\left(\frac{3p-1}{2}\right) \equiv 0 \pmod 4$, then we have

$$BS_3\left(6p^{2k}n + \frac{3p^{2k} - 1}{2}\right) \equiv BS_3(6n + 1) \pmod 4, \tag{1.4}$$

if $BS_3\left(\frac{3p-1}{2}\right) \equiv 2 \pmod 4$, then we have

$$BS_3\left(6p^{3k}n + \frac{3p^{3k} - 1}{2}\right) \equiv BS_3(6n + 1) \pmod 4. \tag{1.5}$$

Theorem 1.4 *Let $k, n \geq 0$ and j are integers. $p_i > 3$ be a prime such that $p \not\equiv 1 \pmod 6$ for $i \in \{1, 2, \ldots, k+1\}$. Then, for $j \not\equiv 0 \pmod{p_{k+1}}$, the following result is obtained*

$$BS_3\left(3p_1^2 \ldots p_{k+1}^2 n + \frac{p_1^2 \ldots p_k^2 p_{k+1}(p_{k+1} + 6j) - 1}{2}\right) \equiv 0 \pmod 3. \tag{1.6}$$

Corollary 1.2 *Let $p > 3$ be a prime with $p \not\equiv 1 \pmod 6$ and j being any integer satisfying $j \not\equiv 0 \pmod p$, then*

$$BS_3\left(3p^{2k+2}n + 3p^{2k+1}j + \frac{p^{2k+2} - 1}{2}\right) \equiv 0 \pmod 3.$$

Theorem 1.5 *p be a prime with* $p \equiv 1 \pmod 6$ *and if* $BS_3\left(\frac{p-1}{2}\right) \equiv 0 \pmod 3$, *then for any* $k > 0$ *and* $n \geq 0$, *we have*

$$BS_3\left(p^{4k}n + \frac{p^{4k}-1}{2}\right) \equiv BS_3(3n) \pmod 3. \tag{1.7}$$

Gordon and Ono [14] obtained that for almost every n, the number of partitions of n into different parts is exactly divisible by 2^k, for k being a fixed positive integer. Lot of authors [3,7,24] focused on the analogous studies for some other forms of partitions. Here, we focused on the divisibility of the partition $BS_3(n)$ by 3^k.

Theorem 1.6 *Let* $k > 0$ *be a fixed integer, then* $BS_3(n)$ *is almost always divisible by* 3^k,

$$\lim_{X\to\infty} \frac{\#\left\{n \leq X : BS_3(n) \equiv 0 \pmod{3^k}\right\}}{X} = 1. \tag{1.8}$$

Theorem 1.7 *Let n be a non-negative integer, then* $BS_3(4n)$ *is almost always divisible by* 2,

$$\lim_{X\to\infty} \frac{\#\{n \leq X : BS_3(4n) \equiv 0 \pmod 2\}}{X} = 1. \tag{1.9}$$

The generalization of the work of Ono [21] was done by Jameson and Wieczorek [17] where important results regarding the parity of coefficients of the weakly holomorphic modular forms are discussed. In addition, the generalized Frobenius partition functions were also studied. In this paper, a similar study is discussed for $BS_3(n)$.

Theorem 1.8 $BS_3(N)$ *is even for infinitely many integers* $N \equiv r \pmod t$, *where* $r \pmod t$ *be any arithmetic progression.*

Theorem 1.9 *For an arithmetic progression* $r \pmod t$, *there are infinitely many integers* $M \equiv r \pmod t$ *for which* $BS_3(M)$ *is odd, provided there is one such M. Moreover, if* $BS_3(M)$ *is odd for an integer* $M \equiv r \pmod t$, *then the smallest such M is less than* $C_{r,t}$ *for*

$$C_{r,t} := \frac{2^{j+9}\cdot 3^2 t^6}{d^2}\prod_{p|6t}\left(1-\frac{1}{p^2}\right) - 2^j,$$

where $d := \gcd(2r-1, t)$ *and* $2^j > \frac{8t}{3}$.

2 Preliminaries

Herewith, certain definitions and related results on modular forms are discussed. Elaborate discussions are mentioned in [18,22].
Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the upper half of the complex plane. The congruence subgroup of $SL_2(\mathbb{Z})$ of level N can be defined as follows

$$\Gamma_0(N) := \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N\right\} \text{ and}$$
$$\Gamma_1(N) := \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N\right\},$$

where N is a positive integer. $M_k(\Gamma_1(N))$ (resp.$S_k(\Gamma_1(N))$) represents the complex vector space of modular forms (resp. cusp forms) of weight k, where k be a positive integer, with respect to a congruence subgroup $\Gamma_1(N)$.

Definition 1 [22, Definition 1.15] Let χ be a Dirichlet character modulo N. Then a modular form $f \in M_k(\Gamma_1(N))$ (resp.$S_k(\Gamma_1(N))$) has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms (resp. cusp forms) is denoted by $f \in M_k(\Gamma_0(N), \chi)$ $\big(\text{resp. } S_k(\Gamma_0(N), \chi)\big)$.

We know that Dedekind's η-function $\eta(z)$ is formulated as

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

in which $q = e^{2\pi iz}$ and $z \in \mathbb{H}$. A function $f(z)$ is defined as an eta-quotient provided it satisfies

$$f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta},$$

in which N is a positive integer and $r_\delta \in \mathbb{Z}$.

Theorem 2.1 [22, Theorem 1.64 and 1.65] *If* $f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta}$ *represents an* η*-quotient with* $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,

$$\sum_{\delta|N}\delta r_\delta \equiv 0 \pmod{24} \quad \textit{and}$$
$$\sum_{\delta|N}\frac{N}{\delta}r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *The character* χ *is formulated as* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, *in which* $s := \prod_{\delta|N}\delta^{r_\delta}$. *Moreover, for* c, d, *and* N *being positive integers with* $d \mid N$ *and* $\gcd(c,d) = 1$, *hence the order of disappearance of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is* $\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}$.

Let f be an eta-quotient fulfilling the conditions of the above mentioned Theorem 2.1. In case, f is also holomorphic for all the cusps of $\Gamma_0(N)$, then $f \in M_k(\Gamma_0(N), \chi)$.

We recall the definition of Hecke operators which is plays a critical role of our proofs.

Definition 2 For $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ and m being a positive integer, the action of Hecke operator T_m on $f(z)$ can be formulated as

$$f(z) \mid T_m := \sum_{n=0}^{\infty}\left(\sum_{d|\gcd(n,m)}\chi(d)d^{\ell-1}a\left(\frac{nm}{d^2}\right)\right)q^n.$$

Precisely, if $m = p$ is prime, then

$$f(z) \mid T_p := \sum_{n=0}^{\infty}\left(a(pn) + \chi(p)p^{\ell-1}a\left(\frac{n}{p}\right)\right)q^n. \tag{2.1}$$

It is to be noted that $a(n) = 0$ unless n is a non-negative integer.

Definition 3 A modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ can be referred as Hecke eigenform if for every $m \geq 2$, a complex number $\lambda(m)$ can be found, such that

$$f(z) \mid T_m = \lambda(m)f(z). \tag{2.2}$$

The following theorem of Serre [22, Theorem 2.65] will be essential in proving the divisibility properties of $BS_3(n)$.

Theorem 2.2 *If* $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ *has a Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]],$$

then for any positive integer m, a constant $\alpha > 0$ *exists, such that*

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod{m}\} = \mathcal{O}\left(\frac{X}{(logX)^{\alpha}}\right).$$

Definition 4 If f is a holomorphic function on $\mathbb{H}$ that satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$ and that are meromorphic at all cusps, then f is a weakly holomorphic modular form of level N_0 and weight k. We denote The space of all weakly holomorphic modular forms with Nebentypus χ by $M_k^!(\Gamma_0(N_0), \chi)$.

Theorem 2.3 [17, Proposition 1] *For* N_0, α, β, t *being integers where* N_0, α *and* t *are positive, satisfies*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in M_k^!(\Gamma_0(N_0), \chi),$$

in which $c(n)$ *are algebraic integers in certain number field. Then, for large values of* j*, we have*

$$f_t(z) := \Delta^{2^j}(\alpha t z) \sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta}$$

is a cusp form in $S_{12\cdot 2^j+k}(\Gamma_0(N), \chi)$*, in which* $\Delta(z) := \eta^{24}(z)$ *and* $N := lcm(\alpha t, N_0)$.

Theorem 2.4 [17, Theorem 5] *For* N_0, α, β, t *being integers with* N_0, α *positive and* $t > 1$*, satisfies*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in M_k^!(\Gamma_0(N_0), \chi),$$

in which $c(n)$ *are algebraic integers in certain number field. Let* $r \pmod{t}$ *be an arithmetic progression, then there are infinitely many integers* $M \equiv r \pmod{t}$ *for which* $c(M)$ *is even.*

Theorem 2.5 [17, Theorem 6] *For* N_0, α, β, t *be integers with* N_0, α *positive and* $t > 1$*, satisfies*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in M_k^!(\Gamma_0(N_0), \chi),$$

in which $c(n)$ *are algebraic integers in certain number field. Let* $r \pmod{t}$ *be an arithmetic progression, then there are infinitely many integers* $M \equiv r \pmod{t}$ *for which* $c(M)$ *is odd, provided there is one such* M*. Moreover, if there does exist an* $M \equiv r \pmod{t}$ *for which* $c(M)$ *is odd, then the smallest such* M *is less than* $C_{r,t}$ *for*

$$C_{r,t} := \frac{2^j \cdot 12 + k}{12\alpha}\left[\frac{N\alpha^2 t^2}{d}\right]^2 \prod_{p|N\alpha t}\left(1 - \frac{1}{p^2}\right) - 2^j,$$

in which $N := lcm(\alpha t, N_0)$*,* $d := \gcd(\alpha r + \beta, t)$ *and* j *is any sufficiently large integer* (*as in Theorem* 2.3).

3 Proof of Theorems 1.1–1.3

Proof of Theorem 1.1 Due to Hirschhorn [15, Identity 30.10.4], we have

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{3.1}$$

Employing binomial theorem in (1.1), we obtain

$$\sum_{n=0}^{\infty} BS_3(n)q^n \equiv \frac{f_3^2 f_6^2}{f_1^2 f_2^2} \pmod 4.$$

Invoking (3.1) in the above identity and drawing out the common terms of q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} BS_3(2n+1)q^n \equiv 2f_3^6 \pmod 4.$$

Drawing out the common terms of q^{3n} from above identity , we obtain

$$\sum_{n=0}^{\infty} BS_3(6n+1)q^n \equiv 2f_1^6 \pmod 4. \tag{3.2}$$

Which implies

$$\sum_{n=0}^{\infty} BS_3(6n+1)q^{4n+1} \equiv 2\eta^6(4z) \pmod 4.$$

Let $\eta^6(4z) = \sum_{n=0}^{\infty} c(n)q^n$. Note that $c(n) = 0$ if $n \not\equiv 1 \pmod 4$ for all $n \geq 0$ and

$$BS_3(6n+1) \equiv c(4n+1) \pmod 4. \tag{3.3}$$

By Theorem 2.1, we have $\eta^6(4z) \in S_3\big(\Gamma_0(16), \big(\frac{-1}{\bullet}\big)\big)$. Since $\eta^6(4z)$ is Hecke eigenform (see, [19]),for an odd prime p (2.1) and (2.2) yield

$$\eta^6(4z) \mid T_p = \sum_{n=1}^{\infty}\left[c(pn) + p^2\left(\frac{-1}{p}\right)c\left(\frac{n}{p}\right)\right]q^n = \lambda(p)\sum_{n=1}^{\infty} c(n)q^n,$$

which suggests

$$c(pn) + p^2\left(\frac{-1}{p}\right)c\left(\frac{n}{p}\right) = \lambda(p)c(n). \tag{3.4}$$

For $c(1) = 1$, $c\big(\frac{1}{p}\big) = 0$ and by mentioning $n = 1$ in (3.4), we obtain $c(p) = \lambda(p)$. As $c(p) = 0$ for all $p \not\equiv 1 \pmod 4$, we obtain $\lambda(p) = 0$. Thus (3.4) gives

$$c(pn) + p^2\left(\frac{-1}{p}\right)c\left(\frac{n}{p}\right) = 0, \tag{3.5}$$

which is true for all primes $p \equiv 3 \pmod 4$.
Substituting $pn + r$ instead of n in (3.5), we found that for all $n \geq 0$ and $p \nmid r$

$$c(p^2n + pr) = 0. \tag{3.6}$$

In the same way, substituting pn instead of n in (3.5), we derive that for all $n \geq 0$ and $p \nmid r$

$$c(p^2n) = -p^2c(n) \equiv c(n) \pmod 2. \tag{3.7}$$

Substituting $4n - pr + 1$ instead of n in (3.6) and along with (3.3), we obtain

$$BS_3\left(6\left(p^2n + \frac{(p^2-1)}{4} + pr\frac{(1-p^2)}{4}\right) + 1\right) \equiv 0 \pmod 4. \tag{3.8}$$

In the same way, substituting $4n + 1$ instead of n in (3.7) and together with (3.3), we obtain

$$BS_3\left(6p^2n + \frac{(3p^2-1)}{2}\right) \equiv BS_3(6n+1) \pmod 4. \tag{3.9}$$

Since $p > 2$ is prime, so $4 \mid (1 - p^2)$ and $\gcd(\frac{1-p^2}{4}, p) = 1$. Therefore, when r runs over a residue system excluding the multiples of p, so does $\frac{1-p^2}{4}r$. Therefore, (3.8) can be mentioned as

$$BS_3\left(6p^2n + \frac{(3p^2-1)}{2} + 6pj\right) \equiv 0 \pmod 4, \tag{3.10}$$

in which $p \nmid j$. For p_i being odd primes such that $p_i \equiv 3 \pmod 4$, the following holds

$$p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{4} = p_1^2\left(p_2^2 \dots p_k^2 n + \frac{p_2^2 \dots p_k^2 - 1}{4}\right) + \frac{p_1^2 - 1}{4},$$

applying (3.9) continuously, we have

$$BS_3\left(6p_1^2 \dots p_k^2 n + \frac{(3p_1^2 \dots p_k^2 - 1)}{2}\right) \equiv BS_3(6n+1) \pmod 4. \tag{3.11}$$

For $j \not\equiv 0 \pmod{p_{k+1}}$ and fixing $n = p_{k+1}^2 n + \frac{p_{k+1}^2-1}{4} + p_{k+1}j$ in (3.11) and employing (3.10), the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 From (3.2), we obtain

$$\sum_{n=0}^{\infty} BS_3(6n+1)q^n \equiv 2\frac{f_2^4}{f_1^2} \pmod 4.$$

For $t_k(n)$ which signifies the number of representations of n as a sum of k triangular numbers, we have

$$\psi^k(q) = \frac{f_2^{2k}}{f_1^k} = \sum_{n=0}^{\infty} t_k(n)q^n.$$

Setting $k = 2$, we obtain

$$\sum_{n=0}^{\infty} BS_3(6n+1)q^n \equiv 2\sum_{n=0}^{\infty} t_2(n)q^n \pmod 4. \tag{3.12}$$

For any positive integer n, $t_2(n)$ can be defined as [5, p.71, Eqn. 3.6.4]

$$t_2(n) = d_1(4n+1) - d_3(4n+1). \tag{3.13}$$

Identity (1.3) can be deduced from (3.12) and (3.13). Hence, the proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3 From (3.2), we have

$$\sum_{n=0}^{\infty} BS_3(6n+1)q^n \equiv 2f_1^6 \pmod 4.$$

Define $\sum_{n=0}^{\infty} c(n)q^n = f_1^6$. From this, it is observed that for all non-negative integer n,

$$BS_3(6n+1) \equiv 2c(n) \pmod 4. \tag{3.14}$$

Due to Newmann [20], we have

$$c\left(pn+\frac{p-1}{4}\right) \equiv c\left(\frac{p-1}{4}\right)c(n) - p^2 c\left(\frac{n-\left(\frac{p-1}{4}\right)}{p}\right) \pmod 2.$$

Substituting n by $pn+\frac{p-1}{4}$ from above identity, we obtain

$$c\left(p^2n+\frac{p^2-1}{4}\right) \equiv c\left(\frac{p-1}{4}\right)c\left(pn+\frac{p-1}{4}\right) - c(n) \pmod 2. \tag{3.15}$$

Substituting n by $pn+\frac{p-1}{4}$ in (3.15), we obtain the following identity

$$c\left(p^3n+\frac{p^3-1}{4}\right) \equiv c\left(\frac{p-1}{4}\right)c\left(p^2n+\frac{p^2-1}{4}\right) - c\left(pn+\frac{p-1}{4}\right) \pmod 2. \tag{3.16}$$

For p being an odd prime with $p \equiv 1 \pmod 4$ such that $BS_3\left(\frac{3p-1}{2}\right) \equiv 0 \pmod 4$ then using the relation (3.14), we have $c\left(\frac{p-1}{4}\right) \equiv 0 \pmod 2$. Thus (3.15) gives

$$c\left(p^2n+\frac{p^2-1}{4}\right) \equiv c(n) \pmod 2. \tag{3.17}$$

Once again, for p being an odd prime with $p \equiv 1 \pmod 4$ such that $BS_3\left(\frac{3p-1}{2}\right) \equiv 2 \pmod 4$ then using the relation in (3.14), we have $c\left(\frac{p-1}{4}\right) \equiv 1 \pmod 2$. Thus (3.16) gives

$$c\left(p^3n+\frac{p^3-1}{4}\right) \equiv c(n) \pmod 2. \tag{3.18}$$

Identities (1.4) and (1.5) follows form (3.17) and (3.18) by using the relation (3.14) with the aid of mathematical induction on k. Hence, the proof of Theorem 1.3 is completed. □

4 Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4 Employing binomial theorem in (1.1), we obtain

$$\sum_{n=0}^{\infty} BS_3(3n)q^n \equiv f_1^4 \pmod 3.$$

Which denotes

$$\sum_{n=0}^{\infty} BS_3(3n)q^{6n+1} \equiv \eta^4(6z) \pmod 3.$$

Let $\eta^4(6z) = \sum_{n=0}^{\infty} b(n)q^n$. Note that $b(n) = 0$ if $n \not\equiv 1 \pmod 6$ for all $n \geq 0$ and

$$BS_3(3n) \equiv b(6n+1) \pmod 3. \tag{4.1}$$

From Theorem 2.1, we obtain $\eta^4(6z) \in S_2\big(\Gamma_0(36)\big)$. Since $\eta^4(6z)$ is Hecke eigenform (see, [19]), (2.1) and (2.2) results

$$\eta^4(6z) \mid T_p = \sum_{n=1}^{\infty}\left[b(pn) + pb\left(\frac{n}{p}\right)\right]q^n = \lambda(p)\sum_{n=1}^{\infty} b(n)q^n,$$

which signifies

$$b(pn) + pb\left(\frac{n}{p}\right) = \lambda(p)b(n). \tag{4.2}$$

As $b(1) = 1$ and $b\big(\frac{1}{p}\big) = 0$ by mentioning $n = 1$ in (4.2), we easily obtain that $b(p) = \lambda(p)$. As $b(p) = 0$ for all $p \not\equiv 1 \pmod 6$, so that $\lambda(p) = 0$. Thus (4.2) gives

$$b(pn) + pb\left(\frac{n}{p}\right) = 0, \tag{4.3}$$

which is true for all primes $p \not\equiv 1 \pmod 6$.
Substituting n by $pn + r$ in (4.3), we obtain that for all $n \geq 0$ and $p \nmid r$

$$b(p^2n + pr) = 0. \tag{4.4}$$

Similarly, substituting n by pn in (4.3), we derive that for all $n \geq 0$ and $p \nmid r$

$$b(p^2n) \equiv 2p \cdot b(n) \pmod 3 \tag{4.5}$$

Substituting n by $6n - pr + 1$ in (4.4) and along with (4.1), results the following identity

$$BS_3\left(3\left(p^2n + \frac{(p^2-1)}{6} + pr\frac{(1-p^2)}{6}\right)\right) \equiv 0 \pmod 3. \tag{4.6}$$

Similarly, substituting n by $6n + 1$ in (4.5) and jointly with (4.1), we obtain

$$BS_3\left(3\left(p^2n + \frac{(p^2-1)}{6}\right)\right) \equiv 2p \cdot BS_3(3n) \pmod 3. \tag{4.7}$$

As $p > 3$ is prime, so $6 \mid (1 - p^2)$ and $\gcd(\frac{1-p^2}{6}, p) = 1$. Therefore, when r takes values over a residue system excluding the multiples of p, so does $\frac{1-p^2}{6}r$. Hence, (4.6) is reformulated as

$$BS_3\left(3p^2n + \frac{(p^2-1)}{2} + 3pj\right) \equiv 0 \pmod 3, \tag{4.8}$$

in which $p \nmid j$. $p_i > 3$ are primes such that $p \not\equiv 1 \pmod 6$, the subsequent identity follows

$$p_1^2 \ldots p_k^2 n + \frac{p_1^2 \ldots p_k^2 - 1}{6} = p_1^2\left(p_2^2 \ldots p_k^2 n + \frac{p_2^2 \ldots p_k^2 - 1}{6}\right) + \frac{p_1^2 - 1}{6},$$

using (4.7) continuously we have

$$BS_3\left(3p_1^2 \ldots p_k^2 n + \frac{(p_1^2 \ldots p_k^2 - 1)}{2}\right) \equiv 2p \cdot BS_3(3n) \pmod 3. \tag{4.9}$$

For $j \not\equiv 0 \pmod{p_{k+1}}$, then fixing $n = p_{k+1}^2 n + \frac{p_{k+1}^2 - 1}{6} + p_{k+1}j$ in (4.9) and employing (4.8). This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5 Employing binomial theorem in (1.1), we obtain

$$\sum_{n=0}^{\infty} BS_3(3n)q^n \equiv f_1^4 \pmod 3.$$

Define $\sum_{n=0}^{\infty} b(n)q^n = f_1^4$. Then it is observed that for all $n \geq 0$,

$$BS_3(3n) \equiv b(n) \pmod 3. \tag{4.10}$$

Due to Newmann [20], the following congruence relation holds for all primes $p \equiv 1 \pmod 6$

$$b\left(pn + \frac{p-1}{6}\right) \equiv b\left(\frac{p-1}{6}\right)b(n) - pb\left(\frac{n - \left(\frac{p-1}{6}\right)}{p}\right) \pmod 3.$$

Substituting n by $pn + \frac{p-1}{6}$ from above identity, we obtain

$$b\left(p^2n + \frac{p^2-1}{6}\right) \equiv b\left(\frac{p-1}{6}\right)b\left(pn + \frac{p-1}{6}\right) - pb(n) \pmod 3. \tag{4.11}$$

Substituting n by $pn + \frac{p-1}{6}$ in (4.11), we have

$$b\left(p^3n + \frac{p^3-1}{6}\right) \equiv b\left(\frac{p-1}{6}\right)b\left(p^2n + \frac{p^2-1}{6}\right) - pb\left(pn + \frac{p-1}{6}\right) \pmod 3. \tag{4.12}$$

Substituting n by $pn + \frac{p-1}{6}$ in (4.12) together with (4.11) and (4.12), we obtain

$$\begin{aligned} b\left(p^4n + \frac{p^4-1}{6}\right) \equiv\ & b\left(\frac{p-1}{6}\right)\left(b^2\left(\frac{p-1}{6}\right) - 2p\right)b\left(pn + \frac{p-1}{6}\right) \\ & - p\left(b^2\left(\frac{p-1}{6}\right) - p\right)b(n) \pmod 3. \end{aligned} \tag{4.13}$$

For $p \geq 5$ be prime with $p \equiv 1 \pmod 6$ such that $BS_3\left(\frac{p-1}{2}\right) \equiv 0 \pmod 3$ then using the relation (4.10), we have $b\left(\frac{p-1}{6}\right) \equiv 0 \pmod 3$. Thus (4.13) gives

$$b\left(p^4n + \frac{p^4-1}{6}\right) \equiv p^2b(n) \pmod 3. \tag{4.14}$$

Identity (1.7) follows form (4.14) by using the relation (4.10) with the aid of mathematical induction on k, completes the proof of Theorem 1.5. □

5 Proof of Theorems 1.6 and 1.7

Proof of Theorem 1.6 Rewriting (1.1) using the η-quotients, we have

$$\sum_{n=0}^{\infty} BS_3(n)q^{24n+12} = \frac{\eta^6(72z)}{\eta^6(24z)}. \tag{5.1}$$

Let $B_k(z) = \frac{\eta^{3k}(24z)}{\eta^k(72z)}$. On invoking binomial theorem we obtain

$$B_{3^k}(z) = \frac{\eta^{3^{k+1}}(24z)}{\eta^{3^k}(72z)} \equiv 1 \pmod{3^{k+1}}.$$

Define $C_k(z)$ by

$$C_k(z) := \frac{\eta^6(72z)}{\eta^6(24z)} B^{3^k}(z) = \frac{\eta^{3^{k+1}-6}(24z)}{\eta^{3^k-6}(72z)}.$$

Modulo 3^{k+1}, due to (5.1) we have

$$C_k(z) \equiv \sum_{n=0}^{\infty} BS_3(n) q^{24n+12} \pmod{3^{k+1}}. \tag{5.2}$$

The cusp of $\Gamma_0(144)$ can be written as fractions $\frac{c}{d}$, where $d \mid 144$ and $\gcd(c,d) = 1$. Using Theorem 2.1, we observe that $C_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24}\left(3^{k+1}-6\right) - \frac{\gcd(d,72)^2}{72}\left(3^k-6\right) \geq 0.$$

Similarly,

$$\begin{aligned}
&\gcd(d,24)^2\left(3^{k+1}-6\right) + \gcd(d,72)^2\left(2-3^{k-1}\right)\\
&= \gcd(d,72)^2\left(\frac{\gcd(d,24)^2}{\gcd(d,72)^2}\left(3^{k+1}-6\right) + \left(2-3^{k-1}\right)\right)\\
&\geq \frac{1}{9}\left(3^{k+1}-6\right) + \left(2-3^{k-1}\right)\\
&> 0.
\end{aligned}$$

Hence by Theorem 2.1, $C_k(z) \in M_{3^k}\left(\Gamma_0(144), \chi_1\right)$ where $\chi_1 = \left(\frac{(-1)^{3^k}\cdot 3^{3^k-6}\cdot 2^{2\cdot 3^k-6}}{\bullet}\right)$. Now, using Serre's Theorem 2.2, the Fourier coefficients of $C_k(z)$ are almost always divisible by $m = 3^k$, for a positive integer k. Using (5.2) the proof of Theorem 1.6 is completed. □

Proof of Theorem 1.7 Employing binomial theorem in (1.1), we obtain

$$\sum_{n=0}^{\infty} BS_3(2n) q^n \equiv \frac{f_3^3}{f_1 f_2} \pmod{2}. \tag{5.3}$$

We know that 2-dissection of f_3^3/f_1 from [15, Identity 22.1.13]

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}.$$

Using the above identity in (5.3) and drawing out common terms of q^{2n}, we obtain

$$\sum_{n=0}^{\infty} BS_3(4n) q^n \equiv \frac{f_2^2}{f_1} \pmod{2}.$$

Which implies

$$\sum_{n=0}^{\infty} BS_3(4n) q^{24n+3} \equiv \frac{\eta^2(48z)}{\eta(24z)} \pmod{2}. \tag{5.4}$$

Let $B(z) = \frac{\eta^2(24z)}{\eta(48z)}$. Then, $B(z) \equiv 1 \pmod{2}$. Define $C(z)$ by

$$C(z) := \frac{\eta^2(48z)}{\eta(24z)} B(z) \equiv \eta(24z)\eta(48z) \pmod{2}.$$

The cusp of $\Gamma_0(384)$ can be written as fractions $\frac{c}{d}$, where $d \mid 384$ and $\gcd(c,d) = 1$. Using Theorem 2.1, we observe that $C(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48} \geq 0.$$

Equivalently,

$$\begin{aligned}&\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48}\\ &= \frac{\gcd(d,48)^2}{48}\left(2\frac{\gcd(d,24)^2}{\gcd(d,48)^2} + 1\right)\\ &> 0.\end{aligned}$$

Hence by Theorem 2.1, $C(z) \in M_1\left(\Gamma_0(384), \left(\frac{9}{\bullet}\right)\right)$. Now, using Serre's Theorem 2.2, the the Fourier coefficients of $C(z)$ are almost always divisible by 2. Hence, the proof of Theorem 1.7 is completed. □

6 Proof of Theorems 1.8 and 1.9

Proof Employing binomial theorem in (1.1), we have

$$\sum_{n=0}^{\infty} BS_3(n)q^n \equiv \frac{f_3^2 f_6^2}{f_1^2 f_2^2} \pmod 2.$$

Which implies

$$\sum_{n=0}^{\infty} BS_3(n)q^{2n+1} \equiv \frac{\eta^2(6z)\eta^2(12)}{\eta^2(2z)\eta^2(4z)} \pmod 2.$$

Hence, by Theorem 2.1, we can see that $\frac{\eta^2(6z)\eta^2(12)}{\eta^2(2z)\eta^2(4z)} \in M_0^!\left(\Gamma_0(24)\right)$.

Let $f_t(z) := \frac{\eta^2(6z)\eta^2(12)}{\eta^2(2z)\eta^2(4z)} \Delta^{2^j}(2tz)$, where $\Delta(z) := \eta^{24}(z)$. The cusp of $\Gamma_0(24t)$ can be written as fractions $\frac{c}{d}$, where $d \mid 24t$ and $\gcd(c,d) = 1$. $f_t(z)$ vanishes at the cusp $\frac{c}{d}$ if and only if

$$2\frac{\gcd(d,6)^2}{6} + 2\frac{\gcd(d,12)^2}{12} - 2\frac{\gcd(d,2)^2}{2} - 2\frac{\gcd(d,4)^2}{4} + 24\cdot 2^j\frac{\gcd(d,2t)^2}{2t} > 0.$$

Equivalently, we obtain

$$\begin{aligned}&2\frac{\gcd(d,6)^2}{6} + 2\frac{\gcd(d,12)^2}{12} - 2\frac{\gcd(d,2)^2}{2}\\ &- 2\frac{\gcd(d,4)^2}{4} + 24\cdot 2^j\frac{\gcd(d,2t)^2}{2t} \geq 2^j\frac{3}{t} - 8.\end{aligned}$$

Hence by Theorem 2.3, $f_t(z) \in S_{12\cdot 2^j}\left(\Gamma_0(24t)\right)$ in case j is an integer which satisfies $2^j > \frac{8t}{3}$. With the aid of Theorems 2.4 and 2.5, the proof of Theorems 1.8 and 1.9 are completed. □

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