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ORIGINAL RESEARCH

Triangular Numerical Semigroups

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Abstract We define triangular numerical semigroup as the numerical semigroups generated by all the triangular numbers greater than or equal to a given triangular number and we study the minimal set of generators and the embedding dimension of these numerical semigroups.

Keywords Numerical semigroup · Embedding dimension · Triangular number

Mathematics Subject Classification 11D07 · 20M14

A numerical semigroup S is an additive submonoid of $\mathbb{N}$ such that $\mathbb{N}\backslash S$ is finite. If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $\mathbb{N}$ generated by A. We say that A is a minimal system of generators of $\langle A\rangle$ and $\langle A\rangle$ is a numerical semigroup if and only if the greatest common divisor of the elements of A is one [5, Lemma 2.1].

Every numerical semigroup S is finitely generated and admits a unique minimal system of generators [5, Theorem 2.7]. The *embedding dimension* of a numerical semigroup S, denoted by $e(S)$, is the cardinality of the minimal system of generators. The smallest nonzero element of S, denoted by $m(S)$, is the *multiplicity* of S and we have $e(S) \leq m(S)$ [5, Proposition 2.10].

In [4] the authors, using the Mersenne numbers, defined and studied a new class of numerical semigroups, Mersenne numerical semigroups. In this paper our aim is to use the triangular numbers to define the triangular numerical semigroups and to study the minimal set of generators for these semigroups.

Very recently, in [3], Robles-Pérez and Rosales used the tringular numbers to define certain numerical semigroups generated by 3 consecutive triangular numbers and they found the Frobenius number for those semigroups ([3, Proposition 6]).

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A natural number is a *triangular number*, if is the sum of the first n natural numbers, for some $n \in \mathbb{N}$, and, in this case, we denote it by t_n:

$$t_n := \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Given a natural number n, consider the additive submonoid of $\mathbb{N}$ generated by all the triangular numbers greater than or equal to t_n:

$$S =< t_{n+k}, k \in \mathbb{N}_0 > .$$

Note that if $d \in \mathbb{N}$ and $d|t_{n+k}$ for every $k \in \mathbb{N}_0$, then $d|\,(t_{n+1} - t_n) = n + 1$ and $d|\,(t_{n+2} - t_{n+1}) = n + 2$. Therefore $d|\big((n + 2) - (n + 1)\big) = 1$ and we conclude that S is a numerical semigroup. From now on we will denote this numerical semigroup by S_n:

$$S_n :=< t_{n+k}, k \in \mathbb{N}_0 >$$

and we say that S_n is a *triangular numerical semigroup* associated to n.

In this paper we study the minimal set of generators and, consequently, the embedding dimension of the triangular numerical semigroups. The minimal set of generators of S_n will be denoted by G_n.

From the properties of numerical semigroups we have

$$e(S_n) \le m(S_n) = t_n = \frac{n(n+1)}{2}.$$

Furthermore, for every natural r, we can write that

$$t_{2(n+r)} = 3t_{n+r} + t_{(n+r)-1}$$

and

$$t_{2(n+r)+1} = 3t_{n+r} + t_{(n+r)+1}.$$

Then

$$t_{2n+k} \in \langle t_n, t_{n+1}, \dots, t_{2n} \rangle, \tag{1}$$

for $k = 2r$ or $k = 2r + 1$ and $r \in \mathbb{N} \cap [0, n]$. In fact, we can prove that (1) is satisfied for every $k \in \mathbb{N}$:

Proposition 1 *Let n be a positive integer and let S_n be the triangular numerical semigroup associated to n. Then*

$$G_n \subseteq \{t_k,\ k \in \{n, n+1, \dots, 2n\}\}.$$

Proof For each $i \in \mathbb{N}$, let $r \in \mathbb{N} \cap [(i-1)n, in]$. For all $k = 2r$ or $k = 2r + 1$, our aim is to prove that t_{2n+k} is a linear combination of $t_n, t_{n+1}, \dots, t_{2n}$. The proof is by induction in i.

For $i = 1$, the result is proved.

Consider $i > 1$ and suppose that the result is true for all the natural $w \le i$.

Let $r \in \mathbb{N} \cap [in, (i+1)n[$. As $r = in + j$, for some $j \in \{0, 1, \dots, n-1\}$ we can write that

$$t_{2n+k} = \begin{cases} 3t_{(i+1)n+j} + t_{(i+1)n+j+1} & \text{if } k = 2r + 1 \\ t_{(i+1)n+j-1} + 3t_{(i+1)n+j} & \text{if } k = 2r. \end{cases}$$

Writting $(i+1)n = 2n + (i-1)n$ we have

$$t_{(i+1)n+a} = t_{2n+(i-1)n+a}$$

and, so, we can write t_{2n+k} as linear combination of $t_{2n+(i-1)n+a}$, for some $a \in \{j-1, j, j+1\}$. As $j-1, j, j+1 \leq n$,

$$(i-1)n + a \leq (i-1)n + n = in.$$

Now, if in is even we have $in = 2r$, where $r = \frac{in}{2}$ and, if in is odd, we write $in = 2r + 1$, with $r = \frac{in-1}{2}$. In both cases we have $r \in \mathbb{N} \cap [(w-1)n, wn[$, for some $w < i$. By induction we may conclude that all the triangular numbers $t_{(i+1)n+a} = t_{2n+(i-1)n+a}$ can be written as linear combination of $t_n, t_{n+1}, \ldots, t_{2n}$ and as consequence we have the desired result. □

Corollary 2 *Let n be a positive integer and let S_n be the triangular numerical semigroup associated to n. Then*

$$e(S_n) \leq n + 1.$$

By the last two results we can state that, for example,

$$S_3 = \langle t_3, t_4, t_5, t_6 \rangle \text{ and } e(S_3) \leq 4.$$

However, as $t_6 = t_3 + t_5$, we conclude that $e(S_3) \leq 3$. Also, it is easy to see that,

- $t_8 = t_5 + t_6$ and then $e(S_4) \leq 4$ and $e(S_5) \leq 5$;
- $t_{11} = t_6 + t_9$, so $e(S_6) \leq 6$;
- $t_{13} = t_8 + t_{10}$ and conclude that $e(S_7) \leq 7$ and $e(S_8) \leq 8$.

So, in general, can we prove that $e(S_n) \leq n$?

In order to do that, we start by proving that triangular numbers of the form t_{10k+j} with $j \in \{1, 3, 6, 8\}$, can be written as sum of two triangular numbers. We have

$$t_6 = t_3 + t_5 \text{ and } t_{16} = t_9 + t_{13} = t_{6+3} + t_{8+5},$$

and, in general,

$$\begin{aligned} t_{10k+6} &= \frac{(10k+6)(10k+7)}{2} = \frac{100k^2 + 130k + 42}{2} = \frac{36k^2 + 42k + 12}{2} + \frac{64k^2 + 88k + 30}{2} = \\ &= \frac{(6k+3)(6k+4)}{2} + \frac{(8k+5)(8k+6)}{2} = t_{6k+3} + t_{8k+5}. \end{aligned}$$

We also have

$$\begin{aligned} t_{10k+8} &= t_{10k+6} + (10k+7) + (10k+8) = t_{6k+3} + t_{8k+5} + (6k+4) + (6k+5) + (8k+6) = \\ &= t_{6k+5} + t_{8k+6} \end{aligned}$$

and, with the same type of arguments we may conclude that, for all $k \in \mathbb{N}$,

$$t_{10k+1} = t_{6k} + t_{8k+1} \quad \text{and that} \quad t_{10k+3} = t_{6k+2} + t_{8k+2}.$$

Now we write $n = 6k + i$, $i \in \{0, \ldots, 5\}$ and, as by Proposition 1 the minimal set of generators of S_{6k+i} is a subset of the set $\{t_{6k+i}, t_{6k+i+1}, \ldots, t_{2(6k+i)}\}$, we want to know which triangular numbers t_{10k+j}, $j \geq 0$, with $j(mod_{10}) \in \{1, 3, 6, 8\}$, are in this last set and which can be written as linear combinations of some of the others.

In order to be an element of the set we need to assure that $10k + j \leq 2(6k + i)$, and so

$$j \leq 2(k + i).$$

On the other hand, to write t_{10k+1} as linear combination we will use t_{6k} and so we can say that if $i \geq 1$ it is not possible to write t_{10k+1}; using a similar argument we can note that in order to write t_{10k+3} we will use t_{6k+2} and so if $i \geq 3$ we can not write or t_{10k+1}, or t_{10k+3}. The same idea can be showed to t_{10k+6} (it is necessary to use t_{6k+3}) and t_{10k+8} (we will need t_{6k+5}). So, we can say that

- $i = 0 \Rightarrow 1 \leq j \leq 2k$;
- $i = 1 \Rightarrow 3 \leq j \leq 2k + 2$;
- $i = 2 \Rightarrow 3 \leq j \leq 2k + 4$;
- $i = 3 \Rightarrow 6 \leq j \leq 2k + 6$;
- $i = 4 \Rightarrow 8 \leq j \leq 2k + 8$;
- $i = 5 \Rightarrow 8 \leq j \leq 2k + 10$.

Keeping in mind that j can only take values in the set $\{1, 3, 6, 8\}$ and defining a_i as

$$a_i = \begin{cases} i & \text{if } i = 0, 1, 2, 3 \\ i + 2 & \text{if } i = 4, 5 \end{cases}$$

the lower bounds for j of the above inequalities can be re-written, for all $i \in \{0, \ldots 5\}$, as

$$a_i < j.$$

Consider now the set

$$A_{k,i} = \{10k + j : \ j(mod_{10}) \in \{1, 3, 6, 8\}, \ a_i < j \leq 2(k + i)\}.$$

For example, for $k \in \{1, 2, 3, 4\}$:

k	$A_{k,0}$	$A_{k,1}$	$A_{k,2}$	$A_{k,3}$	$A_{k,4}$	$A_{k,5}$
1	$\{11\}$	$\{13\}$	$\{13, 16\}$	$\{16, 18\}$	$\{18\}$	$\{18, 21\}$
2	$\{21, 23\}$	$\{23, 26\}$	$\{23, 26, 28\}$	$\{26, 28\}$	$\{28, 31\}$	$\{28, 31, 33\}$
3	$\{31, 33, 36\}$	$\{33, 36, 38\}$	$\{33, 36, 38\}$	$\{36, 38, 41\}$	$\{38, 41, 43\}$	$\{38, 41, 43, 46\}$
4	$\{41, 43, 46, 48\}$	$\{43, 46, 48\}$	$\{43, 46, 48, 51\}$	$\{46, 48, 51, 53\}$	$\{48, 51, 53, 56\}$	$\{48, 51, 53, 56, 58\}$

From these observations and calculations, we can, easilly, conclude the result:

Proposition 3 *For any natural number n written as $n = 6k + i$ for some $k \in \mathbb{N}$ and $i \in \{0, \ldots 5\}$ let $T_{k,i}$ be the following subset*

$$T_{k,i} = \{t_w : w \in A_{k,i}\}.$$

The minimal set of generators of S_n is a subset of

$$\{t_r : \ r \in \{n, n + 1, \ldots, 2n\}\} \backslash T_{k,i}.$$

In particular, we have

Corollary 4

$$e(S_{6k+i}) \leq (6k + i + 1) - \#A_{k,i},$$

being

$$\#A_{k,i} = w_i + 4\left(\left\lfloor \frac{2(k+i)}{10} \right\rfloor - 1\right) + \#\left\{j \in \{1, 3, 6, 8\} : \ j + 10\left\lfloor \frac{2(k+i)}{10} \right\rfloor \leq 2(k+i)\right\},$$

$$\textit{where } w_i = \begin{cases} 4 & \textit{if} \ \ i = 0 \\ 3 & \textit{if} \ \ i = 1, 2 \\ 2 & \textit{if} \ \ i = 3 \\ 1 & \textit{if} \ \ i = 4, 5. \end{cases}$$

Example 5 In the following table we list the elements of the set $\left\{t_{6k+i}, t_{6k+i+1}, \ldots, t_{2(6k+i)}\right\}$ that do not belong to the minimal set of generators of S_{6k+i}:

k	S_{6k}	S_{6k+1}	S_{6k+2}
0	–	2	–
1	**11**, 12	**13**	**13**, **16**
2	20, **21**, **23**	20, **23**, **26**	20, **23**, **26**, 27, **28**
3	**31**, **33**, 34, 35, **36**	**33**, 35, **36**, 37, **38**	**33**, 35, **36**, 37, **38**
4	37, **41**, **43**, 45, **46**, 47, **48**	37, **43**, **46**, **48**, 49	**43**, **46**, **48**, 49, **51**
5	49, **51**, **53**, **56**, **58**, 59, 60	49, **53**, **56**, **58**, 59, 60, **61**	49, **53**, **56**, **58**, 59, 60, **61**, **63**, 64
6	59, **61**, **63**, **66**, 67, **68**, **71**	59, **63**, **66**, 67, **68**, **71**, **73**	59, **63**, **66**, **68**, **71**, **73**, 75, **76**
7	66, **71**, **73**, **76**, **78**, 79, **81**, 82, **83**	66, **73**, **76**, **78**, **81**, 82, **83**, **86**	66, **73**, **76**, **78**, **81**, 82, **83**, **86**, 87, **88**

k	S_{6k+3}	S_{6k+4}	S_{6k+5}
0	**6**	**8**	**8**, 9
1	**16**, **18**	**18**, 20	**18**, 20, **21**
2	**26**, **28**	**28**, **31**	**28**, **31**, **33**, 34
3	37, **36**, **38**, **41**, 42	37, **38**, **41**, 42, **43**	37, **38**, **41**, 42, **43**, 45, **46**
4	**46**, **48**, 49, **51**, **53**	**48**, 49, **51**, **53**, **56**	**48**, 49, **51**, **53**, **56**, **58**
5	49, **56**, **58**, 59, **61**, **63**, 64, **66**, 67, **68**	49, **58**, 59, **61**, **63**, 64, **66**, 67, **68**	**58**, 59, **61**, **63**, 64, **66**, 67, **68**, 70
6	59, **66**, **68**, **71**, **73**, **76**, **78**	66, **68**, **71**, **73**, 75, **76**, **78**, 79	66, **68**, **71**, **73**, 75, **76**, **78**, 79, **81**, 82
7	66, **76**, **78**, **81**, **83**, **86**, 87, **88**, 89	**78**, **81**, **83**, **86**, 87, **88**, 89, **91**, 92	**78**, **81**, **83**, **86**, 87, **88**, 89, **91**, 92, **93**, 94

The bold elements are those that we knew, from our previous calculus, that can be written as linear combinations of others and for that reason will not be part of the minimal set of generators. For the other ones we could not find a general formula.

Now we will study the elements of $\{t_k : k \in \{n, n+1, \ldots, 2n\}\}$ that must necessarily belong to the minimal set of generators of S_n. Our aim is to determine which elements are smaller than $2t_n$ and for that reason can not be written as linear combinations of others.

Consider $i \in \{1, \ldots, n\}$. We have

$$\begin{aligned} t_{n+i} < 2t_n &\Leftrightarrow \frac{1}{2}(n+i)(n+i+1) < n(n+1) \\ &\Leftrightarrow i^2 + (2n+1)i - n(n+1) < 0 \end{aligned}$$

As $i > 0$ we have $i^2 + (2n+1)i - n(n+1) = 0$ if and only if $i = \frac{-(2n+1)+\sqrt{2(2n+1)^2-1}}{2}$ and so

$$t_{n+i} < 2t_n \quad \Leftrightarrow \quad 0 < i < \frac{-(2n+1) + \sqrt{2(2n+1)^2 - 1}}{2}.$$

Note that

$$\frac{-(2n+1) + \sqrt{2(2n+1)^2 - 1}}{2} \approx \frac{-(2n+1) + \sqrt{2(2n+1)^2}}{2} = \frac{(\sqrt{2} - 1)(2n+1)}{2}$$

and that $\sqrt{2} - 1 \approx 0,4142$. So, from our previous calculations we can state that:

$$\left(i \leq \frac{2n}{5} \text{ and } i \in \mathbb{N}\right) \Rightarrow t_{n+i} < 2t_n, \quad \forall n \in \mathbb{N}.$$

If $\frac{2}{5}n \notin \mathbb{N}$ then we must have $\frac{2n+r}{5} \in \mathbb{N}$, for some $r \in \{1, 2, 3, 4\}$. As

$$\begin{aligned} \frac{2n+4}{5} < \frac{-(2n+1) + \sqrt{2(2n+1)^2 - 1}}{2} &\Leftrightarrow (2n+1)^2 - 84(2n+1) - 61 > 0 \\ &\Leftrightarrow (2n+1) > 42 + 5\sqrt{73} \\ &\Leftrightarrow n > 20,5 + 21,36 \approx 41,86 \end{aligned}$$

and $\left\lceil \frac{2n}{5} \right\rceil \leq \frac{2n+4}{5}$, we conclude:

Proposition 6 *If n is a natural number such that $n \geq 42$, then*

$$\left\{t_k \ : \ k \in \left\{n, \ldots, n + \left\lceil \frac{2n}{5} \right\rceil\right\}\right\} \subseteq G_n.$$

Remark 7 A detailed analysis shows that $\left\lceil \frac{2n}{5} \right\rceil < \frac{-(2n+1) + \sqrt{2(2n+1)^2 - 1}}{2}$ if and only if

- $n = 5k$: $0 < k^2 + 3k$, that is true for all $k \geq 1$;
- $n = 5k + 1$: $0 < k^2 - 5k - 2$, that is true for all $k \geq 6$;
- $n = 5k + 2$: $0 < k^2 + k$, that is true for all $k \geq 1$;
- $n = 5k + 3$: $0 < k^2 - 7k - 6$, that is true for all $k \geq 8$;
- $n = 5k + 4$: $0 < k^2 - k - 2$, that is true for all $k \geq 3$;

Remark 8 Note that, by Proposition 6 and observing the table of Example 5, we can conclude that for every $\mathbb{N}\backslash\{14\}$ we have

$$\left\{t_k \ : \ k \in \left\{n, \ldots, n + \left\lceil \frac{2n}{5} \right\rceil\right\}\right\} \subseteq G_n.$$

For $n = 14$, $n + \left\lceil \frac{2n}{5} \right\rceil = 20$ and, in spite of $t_{20} \notin G_{14}$ (note that $t_{20} = 2t_{14}$) we still have

$$\{t_k \ : \ k \in \{14, 15, \ldots, 19\}\} \subseteq G_{14}$$

and, for example, $t_{21} \in G_{14}$.

So, for every natural number n, we get a lower bound for the embedding dimension of S_n:

Corollary 9 *For $n \in \mathbb{N}$,*

$$e(S_n) \geq \left\lceil \frac{2n}{5} \right\rceil + 1.$$

In conclusion, as a result of Corollary 4 and Corollary 9 we obtain a lower and an upper bound for the value of $e(S_n)$.

Our initial goal was, following the type of ideas presented in the articles [1–4] to be able to describe the Apery set for some of the generators in such a way that, from there, we could find the Frobenius number of each S_k. However, after several attempts, we realized that, contrary to what happens with some of the other families of numbers, it is not easy for triangular numbers to find a formula that allows us to construct the Apery set for our semigroups. For this reason, our study then focused on the study of the embeding dimension, having achieved the results described above.

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