



Operator inequalities associated with the Kantorovich type inequalities for s -convex functions

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Abstract In this paper, we prove some operator inequalities associated with an extension of the Kantorovich type inequality for s -convex functions. We also give an application to the order preserving power inequality of three variables and find a better lower bound for the numerical radius of a positive operator under some conditions.

Keywords s -convex function · Kantorovich inequality · Operator inequality · Hölder-McCarthy inequality

Mathematics Subject Classification 47A63 · 46N10 · 47A60 · 26D15

1 Introduction

The class of s -convex functions in the second sense is defined in [10]. Note that all s -convex functions in the second sense are nonnegative [10]. There is an identity between the class of 1-convex functions and the class of ordinary convex functions. Indeed, the s -convexity means just the convexity when $s = 1$, no matter in the first sense or in the second sense. Moreover, when $s \rightarrow 0$ we reach another class of functions the so-called P -class functions. We refer to see some inequalities for P -class functions in [14]. Some properties of s -convex functions in both senses are considered in [10] where various examples and counterexamples are given. For many research results and applications related to the s -convex functions see [2–4, 15–17] and so on.

A function $f : [0, \infty) \rightarrow \mathbb{R}$ is s -convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \quad (1)$$

holds for all $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is s -convex in the second sense whenever

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y) \quad (2)$$

holds for all $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Lemma 1 [15] Let $0 < s \leq r < 1$. Then, the function $f(t) = t^r$, $t \geq 0$ is s -convex in the second sense.

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Lemma 2 [15] *Let X be a normed space and $0 < s \leq r < 1$. Then, the function $f : X \rightarrow \mathbb{R}$ defined by $f(t) = \|t\|^r$ is s -convex in the second sense.*

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. An operator A in $B(\mathcal{H})$ is positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and denoted by $A \geq 0$. By $Sp(A)$ we denote the spectrum of an operator $A \in B(\mathcal{H})$.

The celebrated Kantorovich inequality asserts that if A is an operator on $\mathcal{H}$ such that $M \geq A \geq m > 0$, then

$$\langle A^{-1}x, x \rangle \langle Ax, x \rangle \leq \frac{(m+M)^2}{4mM} \quad (3)$$

holds for every unit vector x in $\mathcal{H}$. We call the constant $\frac{(m+M)^2}{4mM}$ Kantorovich constant. The inequality (3) is closely related to properties of convex functions, and many authors have given many results and comments [6, 8, 11, 12, 18]. By replacing x with $\frac{A^{\frac{1}{2}}x}{\|A^{\frac{1}{2}}x\|}$ in (3) it is well known that the inequality (3) is equivalent to the following inequality

$$\langle A^2x, x \rangle \leq \frac{(m+M)^2}{4mM} \langle Ax, x \rangle^2$$

for every unit vector x in $\mathcal{H}$.

We proved in [15] some inequalities for self-adjoint operators on a Hilbert space including an operator version of the Jensen's inequality for s -convex functions as follows.

Theorem 1 *Let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a continuous s -convex function on $[m, M]$, then*

$$f(\langle Ax, x \rangle) \leq 2^{1-s} \langle f(A)x, x \rangle \quad (4)$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$.

Furuta [5] gave some operator inequalities associated with an extension of the Kantorovich inequality for convex function. In Section 2, we prove some operator inequalities associated with an extension of the Kantorovich type inequality for s -convex functions. We also give an application to the order preserving power inequality of three variables and provide a better lower bound for the numerical radius of a positive operator under some conditions. In Section 3, we show a difference version of Furuta and Giga's result for s -convex functions and provide some applications.

We remark that in the special case when $s = 1$, our results are valid for the convex function. Throughout this paper, we consider s -convex functions in the second sense for some fixed $s \in (0, 1]$.

2 Operator inequalities associated with the Kantorovich type inequality

The following lemma was proved in [5] for the case $q > 1$ or $q < 0$.

Lemma 3 [5, Lemma 1.3] *Let $h(t)$ be defined by*

$$h(t) = \frac{1}{t^q} \left(k + \frac{K-k}{M-m}(t-m) \right) \quad (5)$$

on $[m, M]$, ($M > m > 0$), and any real numbers q, K , and k . Then the function $h(t)$ has the following upper bound on $[m, M]$:

$$\frac{mK - Mk}{(q-1)(M-m)} \left(\frac{(q-1)(K-k)}{q(mK - Mk)} \right)^q, \quad (6)$$

where m, M, k, K and q satisfy any one (i) or (ii) respectively:

- (i) $K > k$, $\frac{K}{M} > \frac{k}{m}$, and $\frac{k}{m}q \leq \frac{K-k}{M-m} \leq \frac{K}{M}q$ holds for a real number $q > 1$,
- (ii) $K < k$, $\frac{K}{M} < \frac{k}{m}$, and $\frac{k}{m}q \leq \frac{K-k}{M-m} \leq \frac{K}{M}q$ holds for a real number $q < 0$.



We extend this lemma for the case $0 < q < 1$ as follows.

Lemma 4 Let $h(t)$ be defined by (5) on $[m, M]$, ($M > m > 0$). Then the function $h(t)$ has the lower bound (6) on $[m, M]$, where m, M, k, K and q satisfy the following conditions:

$$K > k, \frac{K}{M} < \frac{k}{m}, \text{ and } \frac{K}{M}q \leq \frac{K-k}{M-m} \leq \frac{k}{m}q \text{ holds for a real number } 0 < q < 1.$$

Moreover, in this case the upper bound occurs at the beginning or end of the interval $[m, M]$ and the upper bound value is $\max\{\frac{k}{m^q}, \frac{K}{M^q}\}$.

Proof By an easy differential calculus, $h'(t_1) = 0$ when

$$t_1 = \frac{q}{q-1} \frac{mK - Mk}{K - k}.$$

On the other hand, $h''(t_1) = \frac{q(Mk - mK)}{(M-m)t_1^{q+2}} > 0$ and so t_1 gives the lower bound (6) of $h(t)$ on $[m, M]$. Moreover, t_1 is the global minimum point of h and m and M are the local maximum points of h with the maximum values $\frac{k}{m^q}$ and $\frac{K}{M^q}$, respectively. $\square$

We now give some operator inequalities associated with an extension of the Kantorovich inequality. One target of the current paper is to extend the domain of q to values inside the interval $(0, 1)$. We deal with it in Theorem 3.

Theorem 2 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Also let $f(t)$ be a real valued continuous s -convex function on $[m, M]$. Then the following inequality holds for every unit vector x and for any real number q depending on (i) or (ii):

$$\langle f(A)x, x \rangle \leq 2^{1-s} K_f(m, M, q) \langle Ax, x \rangle^q,$$

where

$$K_f(m, M, q) = \frac{mf(M) - Mf(m)}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{qmf(M) - Mf(m)} \right)^q$$

under any one of the following conditions (i) or (ii) respectively:

- (i) $f(M) > f(m)$, $\frac{f(M)}{M} > \frac{f(m)}{m}$ and $\frac{f(m)}{m}q \leq \frac{f(M)-f(m)}{M-m} \leq \frac{f(M)}{M}q$ holds for a real number $q > 1$,
- (ii) $f(M) < f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ and $\frac{f(m)}{m}q \leq \frac{f(M)-f(m)}{M-m} \leq \frac{f(M)}{M}q$ holds for a real number $q < 0$.

Proof Consider $D = \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}$ and $x = \begin{pmatrix} \sqrt{\frac{M-t}{M-m}} \\ \sqrt{\frac{t-m}{M-m}} \end{pmatrix}$. According to Theorem 1, one has

$$\begin{aligned} f(t) &= f(\langle Dx, x \rangle) \\ &\leq 2^{1-s} \langle f(D)x, x \rangle \\ &= 2^{1-s} \left(\frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) \right) \\ &= 2^{1-s} \left(f(m) + \frac{f(M) - f(m)}{M-m} (t-m) \right) \end{aligned} \quad (7)$$

for every $t \in [m, M]$. So,

$$\langle f(A)x, x \rangle \leq 2^{1-s} \left(f(m) + \frac{f(M) - f(m)}{M-m} (\langle Ax, x \rangle - m) \right) \quad (8)$$

for every unit vector x . Multiplying $\langle Ax, x \rangle^{-q}$ on both sides of (8), one can deduce

$$\langle Ax, x \rangle^{-q} \langle f(A)x, x \rangle \leq 2^{1-s} h(\langle Ax, x \rangle), \quad (9)$$



where

$$h(t) = t^{-q} \left(f(m) + \frac{f(M) - f(m)}{M - m} (t - m) \right) \quad (10)$$

for every $t \in [m, M]$. Consequently, it follows from (9) that

$$\langle f(A)x, x \rangle \leq 2^{1-s} \left(\max_{m \leq t \leq M} h(t) \right) \langle Ax, x \rangle^q. \quad (11)$$

Putting $K = f(M)$ and $k = f(m)$ in Lemma 3, then the conditions in Theorem 2 corresponds to the conditions in Lemma 3, so the proof is complete by (11). $\square$

In the special case when we consider $s = 1$, Theorem 2 is valid for the convex functions and recovers [5, Theorem 1.1].

Lemma 5 *The function $f(t) = \log(t)$ on $[m, M]$ with $m \geq 1$ is s -convex.*

Proof Let $x, y \in [m, M]$, $1 < x < y$ and $0 < \alpha < 1$. We realize two cases:

(I) If $\alpha \leq 1 - \alpha$, then by weighted AM-GM inequality [1] one has

$$x^\alpha y^\alpha \leq x^\alpha y^{1-\alpha} \leq \alpha x + (1 - \alpha)y.$$

On the other hand, one has

$$\alpha x + (1 - \alpha)y \leq y < xy.$$

By combining these inequalities one can get

$$x^\alpha y^\alpha \leq \alpha x + (1 - \alpha)y \leq y < xy.$$

We have

$$\alpha \log(xy) \leq \log(\alpha x + (1 - \alpha)y) \leq \log(y) < \log(xy)$$

so that

$$\alpha \leq \frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)} \leq \frac{\log(y)}{\log(xy)} < 1.$$

It follows that

$$\log(\alpha) \leq \log \left(\frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)} \right) \leq \log \left(\frac{\log(y)}{\log(xy)} \right) < 0,$$

which entails

$$1 \geq \frac{\log \left(\frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)} \right)}{\log(\alpha)} \geq \frac{\log \left(\frac{\log(y)}{\log(xy)} \right)}{\log(\alpha)} > 0.$$

Define

$$\theta = \min_{x, y \in [m, M], \alpha \in [0, \frac{1}{2}]} \frac{\log \left(\frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)} \right)}{\log(\alpha)}.$$

Choose $s > 0$ such that $s \leq \theta$. This ensures that

$$s \leq \frac{\log \left(\frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)} \right)}{\log(\alpha)}$$



and so

$$\alpha^s \geq \frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)}.$$

Hence,

$$\log(x^{\alpha^s} y^{\alpha^s}) \geq \log(\alpha x + (1 - \alpha)y).$$

From (I), we deduce

$$\log(x^{\alpha^s} y^{(1-\alpha)^s}) \geq \log(x^{\alpha^s} y^{\alpha^s}),$$

which implies

$$\alpha^s \log(x) + (1 - \alpha)^s \log(y) = \log(x^{\alpha^s} y^{(1-\alpha)^s}) \geq \log(\alpha x + (1 - \alpha)y),$$

(II) If $\alpha \geq 1 - \alpha$, then by weighted AM-GM inequality we get

$$x^{1-\alpha} y^{1-\alpha} \leq x^\alpha y^{1-\alpha} \leq \alpha x + (1 - \alpha)y.$$

We follow a similar path to the proof of the case (I). So,

$$(1 - \alpha) \log(xy) \leq \log(\alpha x + (1 - \alpha)y) \leq \log(y) < \log(xy).$$

Define

$$\theta = \min_{x, y \in [m, M], \alpha \in [\frac{1}{2}, 1]} \frac{\log \left(\frac{\log(\alpha x + (1 - \alpha)y)}{\log(xy)} \right)}{\log(1 - \alpha)}$$

and choose $s > 0$ such that $s \leq \theta$ to deduce the result. $\square$

Corollary 1 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $1 < A \leq M$. Then the following inequality holds for every unit vector x , $q \geq \frac{M}{M-1}$, and for some $0 < s < 1$:

$$\langle \log(A)x, x \rangle \leq \frac{2^{1-s} \log(M)}{(q-1)(M-1)} \left(\frac{q-1}{q} \right)^q \langle Ax, x \rangle^q. \quad (12)$$

Proof Put $f(t) = \log(t)$ for $1 < t \leq M$ in Theorem 2(i). By Lemma 5 the function $f(t)$ is a real valued s -convex function for some $0 < s < 1$ on $(1, M]$. The function f satisfies the conditions of Theorem 2(i) on $(1, M]$. Indeed, since f is increasing, $\log(M) > \log(1) = 0$. So, $\frac{\log(M)}{M} > 0$ and by assumption since $q \geq \frac{M}{M-1}$, one has

$$0 \leq \frac{\log(M)}{M-1} \leq \frac{\log(M)}{M} q.$$

According to Theorem 2(i), we find that

$$\langle \log(A)x, x \rangle \leq 2^{1-s} K_{\log}(1, M, q) \langle Ax, x \rangle^q. \quad (13)$$

Note that $K_{\log}(1, M, q) = \frac{\log(M)}{(q-1)(M-1)} \left(\frac{q-1}{q} \right)^q$. Whence the proof is complete by Theorem 2(i). $\square$



Remark 1 We remark that Corollary 1 is not a trivial result. The following inequality is due to Mond–Pečarić [11]

$$\langle \log(A)x, x \rangle \leq \log(\langle Ax, x \rangle) \quad (14)$$

for every positive operator A and every unit vector x . By a numerical computation we compare two upper bounds provided in (13) and (14). We show that there is no ordering between the upper bounds given in the inequalities (13) and (14) and so Corollary 1 is not a trivial result.

Consider $A = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.1 \end{pmatrix}$, $M = 10$, $q = 2$, and $x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Let $\log t$ be the decimal or common logarithm.

So, one can see

$$\begin{aligned} \langle Ax, x \rangle &= 1.05, \\ \log(\langle Ax, x \rangle) &= 0.021189, \\ 2^{1-s} K_{\log}(1, M, q) \langle Ax, x \rangle^q &= 2^{1-s} (0.030625) > 0.021189. \end{aligned}$$

This entails

$$2^{1-s} K_{\log}(1, M, q) \langle Ax, x \rangle^q > \log(\langle Ax, x \rangle), \quad (15)$$

while for $q = 8$ a numerical computation shows

$$2^{1-s} K_{\log}(1, M, q) \langle Ax, x \rangle^q < (2)(0.00806) < 0.021189$$

and so the reverse inequality holds in (15).

We provide a complementary result for Theorem 2 in the case $0 < q < 1$.

Theorem 3 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Also let $f(t)$ be a real valued continuous s -convex function on $[m, M]$. Then the following inequality holds for every unit vector x :

$$\langle f(A)x, x \rangle \leq 2^{1-s} \max \left\{ \frac{f(m)}{m^q}, \frac{f(M)}{M^q} \right\} \langle Ax, x \rangle^q,$$

where $f(M) > f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ and the condition $\frac{f(M)}{M} q \leq \frac{f(M)-f(m)}{M-m} \leq \frac{f(m)}{m} q$ holds for a real number q such that $0 < q < 1$.

Proof As in the proof of Theorem 2, one has

$$\langle f(A)x, x \rangle \leq 2^{1-s} \left(\max_{m \leq t \leq M} h(t) \right) \langle Ax, x \rangle^q. \quad (16)$$

Putting $K = f(M)$ and $k = f(m)$ in Lemma 4, then the conditions in Theorem 3 corresponds to the conditions in Lemma 4, so the proof is complete by (16). $\square$

Corollary 2 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x :

$$\langle A^p x, x \rangle \leq 2^{1-s} \max\{m^{p-q}, M^{p-q}\} \langle Ax, x \rangle^q, \quad (17)$$

where $0 < s < p < 1$, $0 < q < 1$, and $M^{p-1} q \leq \frac{M^p - m^p}{M - m} \leq m^{p-1} q$.

Proof Put $f(t) = t^p$ for $0 < s < p < 1$ in Theorem 3. By Lemma 1 the function $f(t)$ is a real valued continuous s -convex function on $[m, M]$. On the other hand, since f is increasing, $M^p > m^p$. For $g(t) = t^{p-1}$, $0 < s < p < 1$, the function $g(t)$ is decreasing on $[m, M]$ and so $M^{p-1} < m^{p-1}$ holds for $0 < s < p < 1$, that is, $f(M) = M^p > m^p = f(m)$ and $\frac{f(M)}{M} = M^{p-1} < m^{p-1} = \frac{f(m)}{m}$. Whence the proof is complete by Theorem 3. $\square$



Corollary 3 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x :

$$\|A\|^p \leq 2^{1-s} \max\{m^{p-q}, M^{p-q}\} \langle Ax, x \rangle^q, \quad (18)$$

where $0 < s < p < 1$, $0 < q < 1$, and $M^{p-1}q \leq \frac{M^p-m^p}{M-m} \leq m^{p-1}q$.

Proof Consider $f(t) = \|t\|^p$ for $0 < s < p < 1$ in Theorem 3. The function $f(t)$ is a real valued continuous s -convex function on $[m, M]$ by Lemma 2. The rest of the proof is similar to that of Corollary 2. $\square$

Corollary 4 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x :

$$\frac{1}{2^{\frac{1-s}{p}}} \|A\| \leq \langle Ax, x \rangle \leq \|A\|, \quad (19)$$

where $0 < s < p < 1$ and $M^{p-1}p \leq \frac{M^p-m^p}{M-m} \leq m^{p-1}p$.

Proof The first inequality comes from Corollary 3 by letting $q = p$. The second inequality follows from the Cauchy-Schwartz inequality, i.e.,

$$\langle Ax, x \rangle \leq \|Ax\| \|x\| \leq \|A\| \|x\|^2 = \|A\|$$

for every unit vector x and every positive operator A . $\square$

Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$. The numerical range $W(A)$ is defined by

$$W(A) := \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The largest absolute value of the numbers in the numerical range is called the numerical radius, i.e.

$$w(A) := \sup\{|\lambda| : \lambda \in W(A)\} = \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

It is known that $W(A)$ lies in the closed disc of radius $\|A\|$ centered at the origin and so $w(A) \leq \|A\|$. The following corollary is a simple consequence of Corollary 3.

Corollary 5 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Then the numerical range of A is contained in the closed interval $\left[\frac{1}{2^{\frac{1-s}{p}}} \|A\|, \|A\| \right]$ and

$$\frac{1}{2^{\frac{1-s}{p}}} \|A\| \leq w(A) \leq \|A\|,$$

where $0 < s < p < 1$, $1 < p + s$, and $M^{p-1}p \leq \frac{M^p-m^p}{M-m} \leq m^{p-1}p$.

The advantages of this corollary is that it gives a better lower bound for the numerical radius of the positive operator A under some conditions than the lower bound presented in [9, Theorem 1.2]. Indeed, it has been proved in [9, Theorem 1.2] that $\frac{1}{2} \|A\| \leq w(A)$, whence under the conditions of this corollary one has

$$\frac{1}{2} \|A\| < 2^{\frac{s-1}{p}} \|A\| \leq w(A).$$

We refined in [15, Corollary 5] the Hölder-McCarthy inequality by providing an upper bound as follows. We apply this corollary in our results.

Corollary 6 Let A be a positive operator on a Hilbert space $\mathcal{H}$. Then,

(i) for all $0 < s \leq q < 1$ and $x \in \mathcal{H}$ with $\|x\| = 1$,

$$\langle A^q x, x \rangle \leq \langle Ax, x \rangle^q \leq 2^{1-s} \langle A^q x, x \rangle, \quad (20)$$



(ii) for all $\frac{1}{s} \geq q > 1$ and $x \in \mathcal{H}$ with $\|x\| = 1$,

$$\langle Ax, x \rangle^q \leq \langle A^q x, x \rangle \leq 2^{(1-s)q} \langle Ax, x \rangle^q. \quad (21)$$

As another application of our results, we provide a complementary result of an order preserving inequality of three variables given by Mićić–Pečarić–Seo [13]. The following result associated with the Hölder-McCarthy inequality.

Corollary 7 Let A and B be positive operators on a Hilbert space $\mathcal{H}$ satisfying $m \leq A \leq M$ with $0 < m < M$ and $A \leq B$. Then

$$A^p \leq 2^{2(1-s)} \max\{m^{p-q}, M^{p-q}\} B^q, \quad (22)$$

where $0 < s < p < 1$, $0 < s \leq q < 1$, and $M^{p-1}q \leq \frac{M^p - m^p}{M - m} \leq m^{p-1}q$.

Proof By using Corollary 2 and applying the refined Hölder-McCarthy inequality (20) one can see that

$$\begin{aligned} \langle A^p x, x \rangle &\leq 2^{1-s} \max\{m^{p-q}, M^{p-q}\} \langle Ax, x \rangle^q \\ &\leq 2^{1-s} \max\{m^{p-q}, M^{p-q}\} \langle Bx, x \rangle^q \\ &\leq 2^{2(1-s)} \max\{m^{p-q}, M^{p-q}\} \langle B^q x, x \rangle \end{aligned}$$

for every unit vector x and so the result follows. $\square$

3 A difference version of the Kantorovich type inequalities

Recently, Furuta and Giga [7] gave the complementary result of the Kantorovich type order preserving inequalities by Mićić–Pečarić–Seo. In this section, we show a difference version of Furuta and Giga's result for s -convex functions. We first prove the following lemma which is a difference version of Lemma 4.

Lemma 6 Let $u(t)$ be defined by

$$u(t) = k + \frac{K - k}{M - m}(t - m) - t^q \quad (23)$$

on $[m, M]$, ($M > m > 0$), and any real number q such that $0 < q < 1$ and any real numbers K and k . Then the function $u(t)$ has the following lower bound on $[m, M]$:

$$k + \frac{K - k}{M - m} \left(\left(\frac{K - k}{q(M - m)} \right)^{\frac{1}{q-1}} - m \right) - \left(\frac{K - k}{q(M - m)} \right)^{\frac{q}{q-1}} \quad (24)$$

where $K > k$ and $qm^{q-1} \geq \frac{K-k}{M-m} \geq qM^{q-1}$. Moreover, in this case the upper bound occurs at the beginning or end of the interval $[m, M]$ and the upper bound value is $\max\{k - m^q, K - M^q\}$.

Proof By an easy differential calculus, we know that $u'(t_1) = 0$ when

$$t_1 = \left(\frac{K - k}{q(M - m)} \right)^{\frac{1}{q-1}}.$$

On the other hand, $u''(t_1) > 0$ and so t_1 gives the lower bound (24) of $u(t)$ on $[m, M]$. Moreover, t_1 is the global minimum point of u and m and M are the local maximum point of u with the maximum values $k - m^q$ and $K - M^q$, respectively. $\square$

Lemma 7 Let $u(t)$ be defined by (23) on $[m, M]$, ($M > m > 0$), and any real number q and any real numbers K and k . Then the function $u(t)$ has the upper bound (24) on $[m, M]$, where m, M, k, K and q satisfy any one (i) or (ii) respectively:

- (i) $K > k$ and $qm^{q-1} \leq \frac{K-k}{M-m} \leq qM^{q-1}$ holds for a real number $q > 1$,
- (ii) $K < k$ and $qM^{q-1} \leq \frac{K-k}{M-m} \leq qm^{q-1}$ holds for a real number $q < 0$.



Proof The results follow from the fact that in these cases $u''(t_1) < 0$. $\square$

Theorem 4 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Also let $f(t)$ be a real valued continuous s -convex function on $[m, M]$. Then the following inequality holds for every unit vector x :

$$\langle f(A)x, x \rangle \leq 2^{1-s} \left(\max\{f(m) - m^q, f(M) - M^q\} + \langle Ax, x \rangle^q \right),$$

where $f(M) > f(m)$ and the condition $qm^{q-1} \geq \frac{K-k}{M-m} \geq qM^{q-1}$ holds for a real number q such that $0 < q < 1$.

Proof Subtracting $2^{1-s} \langle Ax, x \rangle^q$ from both sides of (8), we get

$$\langle f(A)x, x \rangle - 2^{1-s} \langle Ax, x \rangle^q \leq 2^{1-s} u(\langle Ax, x \rangle), \quad (25)$$

where

$$u(t) = f(m) + \frac{f(M) - f(m)}{M - m} (t - m) - t^q \quad (26)$$

for every $t \in [m, M]$. So, one can deduce from (25) that

$$\langle f(A)x, x \rangle - 2^{1-s} \langle Ax, x \rangle^q \leq 2^{1-s} \left(\max_{m \leq t \leq M} u(t) \right). \quad (27)$$

Putting $K = f(M)$ and $k = f(m)$ in Lemma 6, then the conditions in Theorem 4 corresponds to the conditions in Lemma 6, so the result follows from (27). $\square$

Theorem 5 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Also let $f(t)$ be a real valued continuous s -convex function on $[m, M]$. Then the following inequality holds for every unit vector x and for any real number q depending on (i) or (ii):

$$\langle f(A)x, x \rangle \leq 2^{1-s} (K_f^d(m, M, q) + \langle Ax, x \rangle^q),$$

where

$$K_f^d(m, M, q) = f(m) + \frac{f(M) - f(m)}{M - m} \left(\left(\frac{f(M) - f(m)}{q(M - m)} \right)^{\frac{1}{q-1}} - m \right) - \left(\frac{f(M) - f(m)}{q(M - m)} \right)^{\frac{q}{q-1}}$$

under any one of the following conditions (i) or (ii) respectively:

- (i) $f(M) > f(m)$ and $qm^{q-1} \leq \frac{f(M)-f(m)}{M-m} \leq qM^{q-1}$ holds for a real number $q > 1$,
- (ii) $f(M) < f(m)$ and $qM^{q-1} \leq \frac{f(M)-f(m)}{M-m} \leq qm^{q-1}$ holds for a real number $q < 0$.

Proof By the same approach as in the proof of Theorem 4 and using Lemma 7 one can reach the result. $\square$

Corollary 8 Let A be a positive operator on a Hilbert space $\mathcal{H}$ satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x :

$$\langle A^p x, x \rangle \leq 2^{1-s} \left(\max\{m^{p-q}, M^{p-q}\} + \langle Ax, x \rangle^q \right), \quad (28)$$

where $0 < s < p < 1$, $0 < q < 1$, and $M^{p-1}q \leq \frac{M^p-m^p}{M-m} \leq m^{p-1}q$.

Proof Put $f(t) = t^p$ for $0 < s < p < 1$ in Theorem 4. Like what was mentioned in Corollary 2, the function f satisfies the conditions of Theorem 4. Whence the proof is complete by Theorem 4. $\square$

As an application of this result, we give a complementary result of an order preserving inequality of three variables.



Corollary 9 Let A and B be positive operators on a Hilbert space $\mathcal{H}$ satisfying $m \leq A \leq M$ with $0 < m < M$ and $A \leq B$. Then

$$A^p \leq 2^{1-s} \left(\max\{m^{p-q}, M^{p-q}\} + 2^{1-s} B^q \right), \quad (29)$$

where $0 < s \leq p < 1$, $0 < s \leq q < 1$, and $M^{q-1}q \leq \frac{M^p-m^p}{M-m} \leq m^{q-1}q$.

Proof By using Corollary 8 and the refined Hölder-McCarthy inequality (20) one can get the result, like what was done in Corollary 7. $\square$

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Declarations

Conflict of interest We remark that the potential conflicts of interest.

Data Availability Data sharing not applicable to this article and no data sets were generated during the current study.

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