



Exact solutions for a Solow-Swan model with non-constant returns to scale

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Abstract The Solow-Swan model is shortly reviewed from a mathematical point of view. By considering non-constant returns to scale, we obtain a general solution strategy. We then compute the exact solution for the Cobb-Douglas production function, for both the classical model and the von Bertalanffy model. Numerical results are eventually provided.

Keywords Solow-Swan model · Cobb-Douglas production function · Returns to scale · von Bertalanffy model

Mathematics Subject Classification 34A05 (Primary) · 91B02 (Secondary) · 91B55 (Secondary)

1 Introduction

The Solow-Swan model plays an important role in neoclassical economics. Even though more than 60 years have passed since it was developed, independently, by Robert Solow [36] and Trevor Swan [37] in 1956, the model is still being analyzed and generalized, as evidenced by a huge literature, which involves many fields of studies [2, 4, 12, 14, 16, 17, 21, 24, 25, 27, 29, 32, 34]. The Solow-Swan model represented an important development of its precursor, the Keynesian Harrod-Domar model [35], which presented some criticalities regarding the stability of its solutions. For a more detailed explanation of the relation between these two models, see for instance [19].

This work is devoted to provide a useful mathematical perspective of the Solow-Swan model. In particular, we are interested in investigating the classical model by weakening the conditions on the returns to scale contrary to the usual ones (see, e.g., [17]). Thus, we are going to relax the hypothesis of constant returns to scale, which in the classical model allows to rewrite the production function as a function of the output per effective unit of labour; instead, we let the production function to have increasing or decreasing returns to scale. We obtain a non-autonomous first order differential equation, for which we provide the exact solution. This choice can be justified by both historical studies in the economical framework [8, 9] and more recent analysis based on real data [11]. Moreover, we underline that our work could find a natural application in connection with the concept of replication [23].

The manuscript is organized as follows. In Section 2, we present the Solow-Swan model, with a focus on the Cobb-Douglas production function; moreover, we study the non-constant returns to scale case, obtaining the exact solution for the model. In Section 3, we explore a different model, namely the von Bertalanffy model, by

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using the same techniques. Section 4 is devoted to presenting some numerical results, to better understand the behaviour of the solutions obtained. Finally, in Section 5 we suggest some insights for future research.

2 The classical model

As highlighted in the introduction, Solow-Swan models have a key role in neoclassical growth theory. Let us denote by $C^2(\mathbb{R}^2)$ the class of twice continuously differentiable functions $F : \mathbb{R}^2 \rightarrow \mathbb{R}$. We may restrict the study to the only economically relevant subset in this setting. In particular, we are going to consider only differentiable functions $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. Moreover, the mathematics of the model is based on the hypothesis that a production function $F(x_1, x_2)$ satisfies the following conditions:

$$\frac{\partial F}{\partial x_i} > 0, \quad \frac{\partial^2 F}{\partial x_i \partial x_j} < 0, \quad \lim_{x_i \rightarrow 0} \frac{\partial F}{\partial x_i} = +\infty, \quad \lim_{x_i \rightarrow \infty} \frac{\partial F}{\partial x_i} = 0, \quad (1)$$

for $i, j = 1, 2$.

In literature, conditions (1), which are aimed at ensuring the existence of an unique stable steady state in a neoclassical growth model, are called *Inada conditons*. For further details and properties about the Inada conditions, we refer to [3, 22, 28, 39, 40].

Classically, the variables x_1 and x_2 are denoted with K and L , respectively. We switch to this notation for the remainder of the article.

Moreover, the quite stringent assumption that F has constant returns to scale is rather frequent in many studies. In fact, thanks to such a hypothesis, it is not hard to obtain an exact solution for the ODE describing the model, at least in its most famous *autonomous* form. Since it is a useful step towards our more general construction, we briefly recall this strategy. Let us suppose that the rate of change of K is proportional to F , and the labor force grows exponentially; such a setting can be described by the following system:

$$\begin{aligned} \frac{dK}{dt} &= sF(K, L), \\ \frac{dL}{dt} &= \gamma L, \end{aligned}$$

with $s, \gamma > 0$ as constants. Thus, since the equation for L is autonomous and easily solved, we focus our attention on the ODE describing the evolution in time of K , which is, explicitly:

$$\frac{dK}{dt} = sF(K, L). \quad (2)$$

We notice that the constant return to scale hypothesis implies that

$$F(\gamma K, \gamma L) = \gamma F(K, L). \quad (3)$$

Dividing both sides of (2) by L , the equation becomes

$$\frac{1}{L} \frac{dK}{dt} = sF\left(\frac{K}{L}, 1\right). \quad (4)$$

Let us now consider the following derivative:

$$\frac{d}{dt} \left(\frac{K}{L} \right) = \frac{1}{L} \frac{dK}{dt} - K \frac{dL}{dt} \frac{1}{L^2} = \frac{1}{L} \frac{dK}{dt} - \gamma \frac{K}{L}. \quad (5)$$

Combining (4) and (5), and introducing the variable $k := \frac{K}{L}$, i.e. the capital-labor ratio, and the notation $f(k) := F(k, 1)$, we are finally ready to write the classic Solow-Swan model:

$$\frac{dk}{dt} = sf(k) - \gamma k. \quad (6)$$



However, in this paper we shall present a different approach to the Solow-Swan model compared to the one given in [17], which is

$$\dot{k} = sf(k) - (\delta + \gamma(t))k, \quad (7)$$

where k is the capital-labor ratio, s is the fraction of output which is saved, δ is the depreciation rate, f is a production function and $\gamma(t)$ is the ratio $\dot{L}/L$; $\dot{k}$ indicates the derivative of k with respect to the time variable t , i.e. $\frac{d}{dt}k(t)$. In fact, in [17], the author assumed f to have constant return to scale (as in the original model), and γ to be variable in time. Conversely, we assume γ to be constant, from which we obtain

$$\frac{\dot{L}}{L} = \gamma \implies L(t) = L_0 e^{\gamma t}, \quad (8)$$

where $L_0 := L(0)$, for ease of notation. However, we do *not* assume our production function f to have constant return to scale. Instead, we choose a generic homogeneous production function, namely

$$F(\gamma K, \gamma L) = \gamma^n F(K, L). \quad (9)$$

This means that, if $n = 1$, the function has constant return to scale, if $n < 1$ ($n > 1$) the function has decreasing (increasing) returns to scale. In particular, we notice that

$$F(K/L, 1) = F(L^{-1}K, L^{-1}L) = L^{-n}F(K, L). \quad (10)$$

Starting from the usual equations

$$\dot{K} = sF(K, L), \quad (11a)$$

$$\dot{L} = \gamma L, \quad (11b)$$

we can derive a non-autonomous equation for the capital-labor (K/L) ratio k , as stated in the following proposition.

Proposition 1 *The ratio k evolves in time obeying the ODE*

$$\dot{k} = sL^{n-1}(t)f(k) - \gamma k, \quad (12)$$

where $f(k) := F(k, 1)$ and $L(t) = L_0 e^{\gamma t}$; recall (8) and (11b).

Proof By direct computation, we notice that

$$\frac{d}{dt} \left(\frac{K}{L} \right) = \frac{1}{L} \frac{dK}{dt} - \frac{K}{L^2} \frac{dL}{dt} = \frac{1}{L} \frac{dK}{dt} - \gamma \frac{K}{L}.$$

Now, combining (11a) and (10), we notice that

$$\frac{1}{L} \frac{dK}{dt} = sL^{n-1}F(K/L, 1).$$

Recalling the definitions of $k = K/L$ and $f(k) := F(k, 1)$, we conclude the proof. $\square$

Remark 1 There are many standard properties of the following Cauchy problem:

$$\begin{cases} \dot{k} = sL^{n-1}(t)f(k) - \gamma k, \\ k(0) = k_0, \end{cases} \quad (13)$$

that one can easily obtain by simple observations or by using to use the so-called *Comparison theorems* (for results in that direction see [17, Sec. 3] and [15]).

Remark 2 The results obtained so far are valid for a wide class of production functions; however, in order to proceed with the analysis of the Cauchy problem (13) one needs to specify a production function.



Our investigation now proceeds with a very natural choice for the production function $f(k)$, i.e. the *Cobb-Douglas production function* [7]. For the standard Cobb-Douglas production function (in which we fixed, without loss of generality for this level of the analysis, the total-factor productivity coefficient equal to 1)

$$F(K, L) = K^\alpha L^\beta, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad \alpha + \beta = n,$$

it is easy to compute the law of the capital-labor ratio $k(t)$:

$$\dot{k} = sL_0^{n-1}e^{(n-1)\gamma t}k^\alpha - \gamma k. \quad (14)$$

The following theorem provides the exact solution for (14).

Theorem 1 *Let $k(t)$ be a solution of (14). Then if $n \neq 1$ and $\alpha \neq 1$*

$$k(t) = \left(e^{(\alpha-1)\gamma t} \left[s(1-\alpha)L_0^{n-1} \frac{(e^{\gamma\beta t} - 1)}{\gamma\beta} + k_0^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}, \quad (15)$$

where we denote $k_0 := k(0)$.

Proof Consider (14), which is clearly a Bernoulli differential equation [20]. We divide both sides by k^α , and apply the substitution $v = k^{1-\alpha}$. Then, after some algebraic steps, (14) becomes

$$\frac{1}{1-\alpha} \dot{v} + \gamma v = sL_0^{n-1}e^{(n-1)\gamma t},$$

We multiply both sides by $(1-\alpha)$, which brings the equation to a standard form

$$\dot{v} + (1-\alpha)\gamma v = sL_0^{n-1}(1-\alpha)e^{(n-1)\gamma t}.$$

Recalling $\beta = n - \alpha$, we apply the well-known formula to solve this first order linear ODE, obtaining (15). $\square$

Remark 3 If $n = \alpha + \beta = 1$, i.e. if the Cobb-Douglas function has constant return to scale, we recover Thm. 10 of [17].

Remark 4 It is clear that for $\alpha = 1$ Eq. (14) is a linear differential equation. The solution of such equation is computable by standard methods and, for $k(0) = k_0$, we have

$$k(t) = k_0 \exp \left(\frac{sL_0^{n-1}(e^{(n-1)t} - 1)}{n-1} - \gamma t \right).$$

3 The von Bertalanffy model

In this section, our analysis move on a different model, in which the labor force follows a well-known von Bertalanffy law [41]:

$$\begin{cases} \dot{L} = r(L_\infty - L), \\ L(0) = 0, \end{cases} \quad (16)$$

where

$$L_\infty = \lim_{t \rightarrow \infty} L(t), \quad (17)$$

is a theoretical maximum asymptote size of the labor force, and $r > 0$ determines the speed at which the labor force approaches the asymptote. As in the previous section, for ease of notation, we fix $L_0 := L(0)$. The model was exhaustively studied by Brida and Limas in [5], where the authors present many important results for the constant returns to scale case. As in Sect. 2, we are going to relax this hypothesis, considering also increasing (and decreasing) returns to scale, and we present the exact solution for the model.

Remark 5 The von Bertalanffy equation was widely studied by many authors from different fields. See, for instance, [6, 26, 30]. Moreover, in the last decades there has been a growing interest in the field of economics, as well [1, 18].



The first step is to compute the law of the ratio k .

Proposition 2 *The ratio k evolves in time obeying the ODE*

$$\dot{k} = sL^{n-1}(t)f(k) - rk(L_\infty - L(t)), \quad (18)$$

where $f(k) := F(k, 1)$ and, from (16), $L(t) = L_\infty - (L_\infty - L_0)e^{-rt}$.

Proof The proof is analogous to the proof of Prop. 1. $\square$

We consider the Cobb-Douglas production function to proceed with our investigation, thus obtaining the following Cauchy problem:

$$\begin{cases} \dot{k} = s(L_\infty - (L_\infty - L_0)e^{-rt})^{n-1}k^\alpha - r(L_\infty - L_0)e^{-rt}k, \\ k(0) = k_0. \end{cases} \quad (19)$$

The solution of the Cauchy problem (19) is given by the following theorem.

Theorem 2 *Let $k(t)$ be a solution of (19). Then if $n \neq 1$ and $\alpha \neq 1$*

$$k(t) = \left(e^{-(\alpha-1)(L_\infty-L_0)e^{-rt}} \left(k_0^{1-\alpha} e^{(\alpha-1)(L_\infty-L_0)} - (\alpha-1) \int_0^t \mathcal{L}(\tau) d\tau \right) \right)^{\frac{1}{1-\alpha}}, \quad (20)$$

where

$$\mathcal{L}(\tau) := \frac{s(L_\infty - (L_\infty - L_0)e^{-r\tau})^n \cdot \exp[r\tau + (\alpha-1)(L_\infty - L_0)e^{-r\tau}]}{L_\infty(e^{r\tau} - 1) + L_0}.$$

Proof The proof is analogous to the proof of Thm. 1. In fact, by the same substitution $v = k^{1-\alpha}$, we get the following linear differential equation:

$$\dot{v} = (1-\alpha)s(L_\infty - (L_\infty - L_0)e^{-rt})^{n-1} + (1-\alpha)r(L_\infty - L_0)e^{-rt}v.$$

Thus, applying the classical formula, we compute the solution. $\square$

Remark 6 A comment analogous to Rmk. 4 can be made. For $\alpha = 1$, we have the following solution (which involves hypergeometric functions ${}_2F_1$), where we introduce, for ease of notation, $L_* := L_\infty - L_0$:

$$k(t) = k_0 \cdot \exp \left[(e^{-rt} - 1)L_* + \frac{sL_0^{n-1} \left(-\frac{L_0}{L_*} \right)^{1-n} {}_2F_1 \left(1-n, 1-n; 2-n; \frac{L_\infty}{L_*} \right)}{(n-1)r} - \frac{s \left(\frac{L_\infty e^{rt}}{-L_*} + 1 \right)^{1-n} (-L_* e^{-rt} + L_\infty)^{n-1} {}_2F_1 \left(1-n, 1-n; 2-n; \frac{e^{rt}L_\infty}{L_*} \right)}{(n-1)r} \right].$$

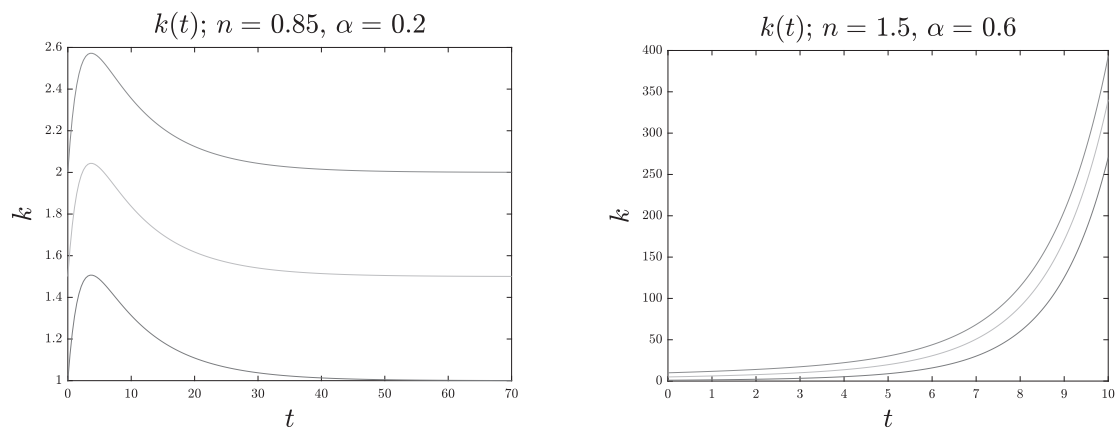
For further details on the use of the hypergeometric function in this context see, for instance, [30].

4 Numerical Results

In this section, we propose some numerical results¹ for both the classical and the von Bertalanffy model, for the specific choice of Cobb-Douglas for our production function $f(k)$. The results of the classical case clearly agree with the expectations, consistently with neoclassical growth theory with a convergence toward the initial conditions for the decreasing returns to scale and an exponential growth for increasing returns to scale as one can see in Figure 1. A very interesting output comes from the study of the von Bertalanffy model, as displayed in Figure 2. The latter seems to level out the differences between the two cases, namely increasing returns to scale and decreasing returns to scale. The following graphs show the behaviour of the capital-labor ratio.

¹ The software used to performed the computations is MATLAB 2020b developed by MathWorks.

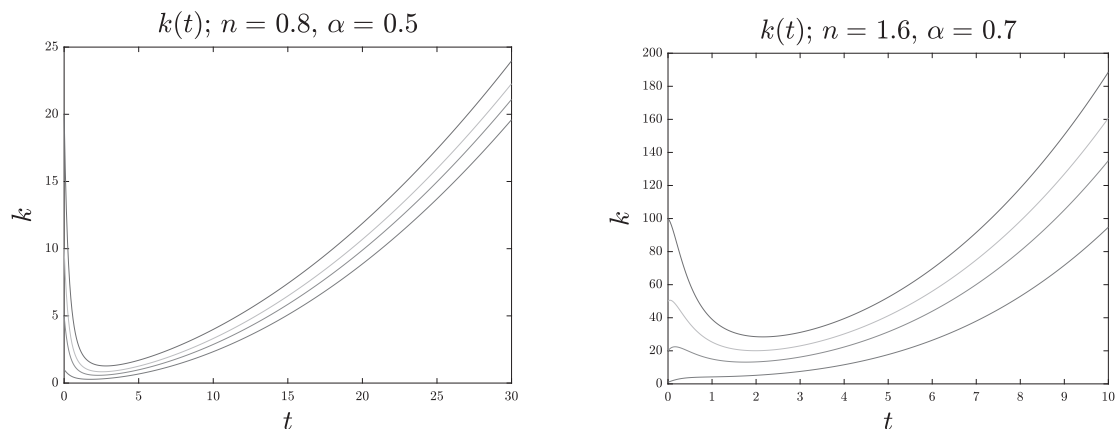




(a) For $n < 1$, we observe a rapid growth, followed by a convergence towards the initial conditions $k_0 = 1, 1.5, 2$.

(b) For $n > 1$, we observe an exponential growth, independent on the initial conditions $k_0 = 1, 5, 10$.

Fig. 1 Numerical computations of (15) for (a) $n < 1$ (b) $n > 1$. The value of α is displayed in the titles of each figure. The other values of the parameters are $\beta = n - \alpha$, $\gamma = 0.7$, $s = 0.4$, $L_0 = 1$.



(a) For $n < 1$, we observe a rapid and short decrease, dependent on initial conditions, followed by a exponential growth starting for all initial conditions $k_0 = 1, 5, 10, 20$.

(b) For $n > 1$, we observe a dependence on initial condition for the first part of the dynamics, followed by exponential growth, with initial conditions $k_0 = 1, 20, 50, 100$.

Fig. 2 Numerical computations of (20) for (a) $n < 1$ (b) $n > 1$. The value of α is displayed in the titles of each figure. The other values of the parameters are $L_0 = 1$, $L_\infty = 5$, $s = 0.4$, $r = 0.9$.

Remark 7 Despite the abstraction of our argument, the real world offers many interesting examples in which the economic data underline the emergence of the necessity of a model with non-constant returns to scale. In particular, several publications provide important support in the direction of *increasing* returns to scale (see [33] for the economic growth of East Asia, and [13] for the French business sector). These kind of works are also analyzed in [31] with an observability-based approach. A very peculiar case study concerns the *decreasing* returns to scale in a health capital model, which was thoroughly studied in [38]. Other fascinating examples can be found for instance in [10, 42]. The plan for a further work is to closely compare our theoretical result with the behaviour emerging from this sort of data sets.



5 Conclusions

The analysis of the Solow-Swan type models presents several stimulating mathematical challenges, which might be explored. In this work, we dwell on the case of non-constant returns to scale, providing an exact solution for the model that arise for the Cobb-Douglas production function. A more complicated case, namely the von Bertalanffy model, is also studied with similar results. We remark that this model has not been applied yet to a real case, although it seems quite promising. As one could expect, numerical computations support the economical idea under the behaviour of the capital-labor ratio. Our aim is to provide a different perspective on the Solow-Swan model, by starting an investigation based on the non-constantness of the returns to scale. This approach seems supported by different economical studies and can be seen as a first step in a new and more realistic direction. We believe this insight can help economists and modellers in the research of a suitable and realistic model for their purposes. One of the future step that we want to develop is a more concrete approach, starting from the real-life economic data, which already suggests a non-constant behaviour for the returns to scale, as we remarked above. Many other issues still remain open. At first, one can consider as non-constant also the total-factor productivity coefficient (which in this work is fixed equal to 1). Moreover, a very fascinating challenge seems also the study of this model for the CES production function (which actually does not satisfy the Inada conditions), by considering as this work non-constant returns to scale or trying other, more exotic, production functions. We plan to explore these possibilities in the near future.

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