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ORIGINAL RESEARCH

The minimum principle on the sequential complete local convex spaces

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Abstract The minimum principle establishes cases when the infimum of a family of functions is plurisubharmonic. In [18], the author has otained this principle on the space that is the special inductive limits of Banach spaces. In this paper, we will establish the minimum principle on some classes of Hausdorff sequential complete local convex spaces.

Keywords The minimum principle · Plurisubharmonic functions · The complete locally convex topological vector space

Mathematics Subject Classification Primary 32U05 ; Secondary 32U15

1 Introduction

In this section, we will review some definitions and the motivation to obtain the main result of the paper. In this paper by U we always mean the unit disk in $\mathbb{C}$, $\overline{U}$ is the closure of U and S is the unit circle. And with X is any set, we denote $H(X)$ the set of all holomorphic mappings of a neighborhood of $\overline{U}$ into X, $\overline{H}(X)$ the set of all mappings of unit disk continuous on $\overline{U}$ and holomorphic on U into X

Let W be an open set in a topological vector space Z. We denote $\mathrm{PSH}(W)$ is the set of the plurisubharmonic (psh) functions on W, that is $u \in \mathrm{PSH}(W)$ if anh only if it is an upper semicontinuous (usc) functions and satisfy the subaveraging inequality follow

$$u(f(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} u(f(e^{i\theta}))d\theta, \forall f \in \overline{H}(W). \tag{1.1}$$

Let $P : Z \to Y$ be a projection from Z onto a subspace $Y \subset Z$ and $\phi \in \mathrm{PSH}(W)$. Then the subenvelope of ϕ is defined as follow

$$I_P\phi(z) = \inf\{\phi(w) : w \in W, P(w) = z\}, \forall z \in W_0 = P(W). \tag{1.2}$$

From (1.1) we see that the supremum of any family of psh functions on W is also psh, provided it is usc. But the minimum of two psh functions on W need not to be psh. So the subenvelope $I_P\phi$ of ϕ is not psh in general.

In [10], C. Kiselman has been proved that under some conditions, the subenvelope is psh that he called "the minimum principle", as follow

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Theorem 1.1 *Suppose that W is a pseudoconvex open set in $\mathbb{C}^2$ and ϕ is a psh function on W such that both W and ϕ are S-invariant, i.e., if $(z, w) \in W$, then $(z, e^{i\alpha}w) \in W$ and $\phi(z, e^{i\alpha}w) = \phi(z, w)$ for every real number α. Also suppose that for every $(z, w) \in W$ the fiber $\{x|(z, x) \in W\}$ are connected. If $P(z, w) = z$, then the function $I_P\phi(z) = \inf_{(z,w)\in W} \phi(z, w)$ is subharmonic.*

We can see in the Theorem 1.1 that all the sufficient conditions such as "pseudoconvex", "S-invariant", "psh" and "connected fibers" are very close to the necessary. Indeed, first clearly ϕ must be psh. If W consists of two disjoint components, then the function $I_P\phi$ is the minimum of two psh functions and it is easy to make the minimum not psh. In [18], the author has given two example to show that the remain conditions are important. We cite here for convenient.

Example 1.2 Let W be the unit bidisk in $\mathbb{C}^2$ and $\phi(z, w) = \log |2zw - 1|$. Then $I_P\phi(z) = -\infty$ when $|z| \geq \frac{1}{2}$ and $I_P\phi(0) = 0$ when $z = 0$. This means that $I_P\phi$ is not subharmonic.

Example 1.3 Let $W = \{(z, w) \in \mathbb{C}^2 : |z| < 2, -|z|^2 + 4 < |w|^2\}$ and $\phi(z, w) = |w|^2$. Then both W and ϕ are S-invariant but W is not pseudoconvex. Now $I_P\phi(0) = 4$ and $I_P\phi(z) = 3$ when $|z| = 1$. Again we see that $I_P\phi$ is not subharmonic.

In [16] the author has been investigated the following situation: Let $Y = \mathbb{C}^n$, let W_0 be an open set in Y, and let $Z = H(\mathbb{C}^n)$. We define $W = \{f \in Z : f(\overline{U}) \subset W_0\}$. Let $Pf = f(0)$ be the canonical projection of Z onto Y. It was proved in [16] that, under mild conditions on a function ϕ on W, its subenvelope is psh. Here, we can see that the main results in [10] and [16] are very different each other because their conditions such as, by [16], Z was infinitely dimensional, ϕ was not psh or even usc, and W was not psuedoconvex.

In [18], the author has been given the definition P-psh function. Let us call an absolutely measurable function (see section 3) ϕ on W be P-psh if for every holomorphic mapping $f \in H(Z)$ such that $f(S) \subset W$ and Pf maps $\overline{U}$ into $W_0 = P(W)$ we have

$$I_P\phi(Pf(0)) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))d\theta. \tag{1.3}$$

Comparing the inequality (1.3) with the subaveraging inequality (1.1) we see that a bigger choice of values of ϕ in the left side is compensated by also a bigger choice of mappings f in the right side. So the inequality (1.3) does not follow immediately from the plurisubharmonicity of ϕ. It will if we need to verify (1.3) only for $f \in H(W)$. In this case, the function ϕ is called a weakly P-psh function.

By considering the P-psh function ϕ on a open subset in Z that is the special inductive limits of Banach spaces Z_i, in [18], the author has been improved the Kiselman's framework in [10] to be a new result that include the results from [16]. In this paper, we use the same technics similar to [18] to prove the minimum principle on the Hausdorff sequential complete local convex spaces.

2 Holomorphic $\overline{U}-$ actions on the local convex space

First we review some basic properties in functional analysis. Let X be a compact metric space and Z be Hausdorff sequential complete local convex space. Let $F : X \to Z$ be continuous mapping. We denote the set A is the closure of the convex hull of $F(X)$ and p_A is Minkowski functional of the set A. If we set $Z_A = span(A)$ then (Z_A, p_A) is a Banach space. Moreover, if Z' is the topological dual space of Z and $z^* \in Z'$ then $z^* \in (Z_A, p_A)'$.

Lemma 2.1 *Let Z be a Hausdorff sequential complete local convex space. If f is a continuous mapping of $\overline{U}^n$ into Z holomorphic in U^n then it is holomorphic in U^n as a mapping into (Z_A, p_A) with $A = \overline{conv(f(\overline{U}^n))}$.*

Proof Since f is continuous as a mapping from U^n into (Z_A, p_A) so if g is a function is defined as follow

$$g(\zeta) = \frac{1}{(2\pi i)^n}\int_S \cdots \int_S \frac{f(\xi_1, \ldots, \xi_n)}{\prod_{j=1}^n(\xi_j - \zeta_j)}d\xi_1 \ldots d\xi_n,\ \forall\zeta = (\zeta_1, \ldots, \zeta_n) \in U^n,$$

then g maps U^n holomorphically into (Z_A, p_A). Let $z^* \in Z'$. If $h(\zeta) = z^*(f(\zeta)), \forall\zeta \in U^n$ then h is holomorphic on U^n. By Cauchy integral formula we have

$$h(\zeta) = \frac{1}{(2\pi i)^n}\int_S \cdots \int_S \frac{h(\xi_1, \ldots, \xi_n)}{\prod_{j=1}^n(\xi_j - \zeta_j)}d\xi_1 \ldots dxi_n.$$

From the definition of function g above we have $z^*(g(\zeta)) = h(\zeta)$. So $z^*(f(\zeta)) = z^*(g(\zeta)), \forall \zeta \in U^n, z^* \in Z'$. This imply that $f = g$ on U^n. ◻

Suppose that $E = \{\zeta_1, \dots, \zeta_n\} \subset U$ and f is a continuous mapping of $\overline{U} \setminus E$ into Z, holomorphic in $U \setminus E$. We say that f has a pole of order at most m_j at ζ_j if

$$\lim_{\zeta \to \zeta_j} (\zeta - \zeta_j)^{m_j+1} f(\zeta) = 0.$$

Corollary 2.2 *Let Z be a Hausdorff sequential complete local convex space. If f is a continuous mapping of $\overline{U} \setminus E$ into Z, holomorphic in $U \setminus E$ with poles of order at most m_j at ζ_j, then there exist $g \in \overline{H}(Z_A)$ such that*

$$f(\zeta) = \frac{g(\zeta)}{\prod_{j=1}^n (\zeta - \zeta_j)^{m_j}}.$$

Proof We set $h(\zeta) = \prod_{j=1}^n (\zeta - \zeta_j)^{m_j+2} f(\zeta)$. Then for each $\zeta_j \in E$, $j = 1, \dots, n$ we have

$$h(\zeta_j) = \lim_{\zeta \to \zeta_j} h(\zeta) = 0$$

So h is a continuous mapping of $\overline{U}$ into Z, holomorphic in U. By Lemma 2.1 then $h \in \overline{H}(Z_A)$.
On the other hand, it is easy to see that $h'(\zeta_j) = 0, \forall 1 \le j \le n$. So by Taylor expansion formula of h at ζ_j we have

$$g(\zeta) := \frac{h(\zeta)}{\prod_{j=1}^n (\zeta - \zeta_j)^2} \in \overline{H}(Z_A)$$

Since $g(\zeta) = \prod_{j=1}^n (\zeta - \zeta_j)^{m_j} f(\zeta) \in \overline{H}(Z_A)$ and the corollary follows. ◻

Let Z be a sequential complete local convex space. A continuous mapping $A(\xi, z)$ of $\overline{U} \times Z$ into Z is called a holomophic $\overline{U}$-action if:

i. A is a holomorphhic mapping of $U \times Z$ into Z
ii. For every $\xi \in \overline{U}$ the mapping $A(\xi, z)$ is linear in z
iii. $A(1, z) = z$
iv. $A(0, A(\xi, z)) \equiv A(0, z)$.

It followsn that $Pz = A(0, z)$ is a projection of Z onto $Y = PZ$ and the orbit $A(\xi, z)$ of a point $z \in Z$ lies in the fiber of P over Pz.
Given a holomorphic $\overline{U}$-action A on Z. Let $W \subset Z$ be open set and ϕ be function on W. The set W is called S-invariant if $A(\xi, z) \in W$ for every $z \in W$ and $\xi \in S$. The function ϕ is called S-invariant if W is S-invariant and $\phi(A(\xi, z)) = \phi(z)$ for every $z \in W$ and $\xi \in S$.

Lemma 2.3 *Let Z be a Hausdorff sequential complete local convex space and A be a holomorphic $\overline{U}$-action on Z. Let $Pz = A(0, z)$ be a projection in Z. Let $E = \{\zeta_1, \dots, \zeta_n\} \subset U$ and let f be a continuous mapping of $\overline{U} \setminus E$ into Z, holomorphic in $U \setminus E$, with poles of order at most m_j at ζ_j, $1 \le j \le n$. We denote by B the Blaschke product as follow*

$$B(\zeta) = \prod_{j=1}^n \left(\frac{\zeta - \zeta_j}{1 - \overline{\zeta_j}\zeta} \right)^{n_j},$$

where $n_j \ge m_j$, $1 \le j \le n$. If the mapping $h = Pf$ is in $\overline{H}(Z)$, then for every $\eta \in S$ the mapping $A(\eta B(\zeta), f(\zeta))$ is also in $\overline{H}(Z)$.

Proof By Corollary 2.2 the mapping $g(\zeta) = Q(\zeta) f(\zeta)$, where $Q(\zeta) = \prod_{j=1}^n (\zeta - \zeta_j)^{m_j}$, is in $\overline{H}(Z_A)$. If we set

$$F(\xi, \zeta) = A(\xi, f(\zeta)) - h(\zeta) = \frac{A(\xi, g(\zeta)) - Q(\zeta)h(\zeta)}{Q(\zeta)},$$

then $F(0, \zeta) \equiv 0$.

Set $G(\xi,\zeta) = A(\xi, g(\zeta)) - Q(\zeta)h(\zeta)$, we have $G(0,\zeta) \equiv 0$. But G is a continuous mapping of $\overline{U}^2$ into Z, holomorphic in U^2. So by Lemma 2.1 we have $G \in \overline{H}(Z_A)$.

By the Taylor expansion formula of G at (0, 0) we can see that $G(\xi,\zeta) = \xi G_1(\xi,\zeta)$, where G_1 is holomorphic in U^2 and continuous on the boundary where $\xi \neq 0$.
Since $G(\eta B(\zeta),\zeta) = \eta B(\zeta) G_1(\eta B(\zeta),\zeta)$, the mapping

$$A(\eta B(\zeta), f(\zeta)) = F(\eta B(\zeta),\zeta) + h(\zeta)$$

is holomorphihc in U. It is also continuous up to the boundary because $|B(\zeta)| \equiv 1$ on S and $|\zeta_j| < 1$, $1 \leq j \leq n$. $\square$

A function ϕ on an open set W in Z is called sequential usc (susc) if for every point $z \in W$ and every sequence of points $z_j \in W$ convergence to z we have

$$\limsup_{j\to\infty} \phi(z_j) \leq \phi(z).$$

Here we note that, if a function is usc then it is also sequential usc and if the space Z is metrizable then the both notions coincide.

A function ϕ on an open set W in Z is called sequential psh (spsh) if it is sequential usc and satisfy the subaveraging inequality.

3 Almost upper semicontinuous functions

Let $\overline{U}^N = \{z = (z_1, z_2, ...) : z_j \in \mathbb{C}, |z_j| \leq 1\}$ be the product of countably many closed unit disk equipped with the product topology and, consequently, a compact topological space. This topology has a countable basis: for every choice of naturals $j_1,\dots,j_n$, positive rationals $r_1,\dots,r_n$, and points $\zeta_1,\dots,\zeta_n$ in $\overline{U}$ with rational coordinates we take the set

$$V = \{z \in \overline{U}^N : |z_{j_k} - \zeta_k| < r_k, k = 1,2,\dots,n\}.$$

The set $\overline{U}^N$ carries a unique Radon measure τ with the following property: if $\{j_1,\dots,j_n\}$ is any set of n indices and ϕ is any continuous function on $\overline{U}^N$ such that $\phi(z) = \phi(z_{j_1},\dots,z_{j_n})$, then

$$\int_{\overline{U}^N} \phi d\tau = \frac{1}{(2\pi i)^n} \int_U \dots \int_U \phi(z_{j_1},\dots,z_{j_n}) dz_{j_1} \wedge d\bar{z}_{j_1} \dots dz_{j_n} \wedge d\bar{z}_{j_n}.$$

This measure is translational invariant, i.e., if $F \subset \overline{U}^N$ is a Borel set and $x \in \overline{U}^N$ such that $F + x \subset \overline{U}^N$, then $\tau(F) = \tau(F + x)$.

Let Z be a sequential compete local convex space. The sequence $E = (x_n)$ in Z is called summable if every continuous semi-norm p on Z, then $\sum_{n=1}^{\infty} p(x_n) < \infty$. The set

$$P_E = \{x \in Z : x = \sum_{j=1}^{\infty} \zeta_j x_j, |\zeta_j| \leq 1, \forall j = 1,2,...\}$$

is called a generalized polydisk. The mapping $F : \overline{U}^N \to P_E$ defined as $F(z) = \sum z_j x_j$ is continuous and therefore defines the measure τ_E on P_E as the pushforward of the measure τ.

If P_E is a generalized polydisk and

$$C = \{x \in Z : x = \sum_{j=1}^{\infty} \zeta_j x_j, |\zeta_j| \leq c_j \leq 1, \forall j = 1,2,...\},$$

then C is the intersection of the decreasing sequence of closed sets

$$C_k = \{x \in Z : x = \sum_{j=1}^{\infty} \zeta_j x_j, |\zeta_j| \le c_j, \forall 1 \le j \le k\}.$$

Therefore $\tau_E(C) = \prod_{j=1}^{\infty} c_j^2$.

A function ϕ on a set $W \subset Z$ is called absolutely measureable if it is measurable with respect to any regular Borel measure with compact support on W. Given a summable sequence $E = \{x_j\}$, an open set $W \subset Z$, and a local bounded above absolutely measurable function ϕ on W, we define the function

$$\phi_E(z) = \int_{P_E} \phi(z + y) d\tau_E(y)$$

on the set $W_E = \{z \in W : z + P_E \subset W\}$.

The linear operator $L_E : \ell^1 \to Z$ defined as

$$L_E(\zeta_1, \zeta_2, ...) = \sum_{j=1}^{\infty} \zeta_j x_j$$

is continuous. We say that a function u on W is usc along E if for every $z \in W$ the function $u(z + L_E\zeta)$ is usc at the origin of the space ℓ^1.

Lemma 3.1 *Let Z be a sequential complete local convex space and ϕ be a local bounded above absolutely measurable on an open set W in Z. Let $E = \{x_j\}$ be a summable sequence in Z. Then the function ϕ_E is usc along E on W_E.*

Proof Let us take a point $z_0 \in W_E$. We will assume that $\phi_E(z_0) > -\infty$. If $\phi_E(z_0) = -\infty$ then the proof is similar.

Since the set $K = z_0 + P_E$ is compact, we can find a continuous seminorm p and a constant $c > 0$ such that $\phi(x) < c$ when $p(x - y) < 1$ for some $y \in K$. We also may assume that $z \in W_E$ when $p(z - z_0) < 1$.

Since the mapping L_E is continuous, there is $\delta_1 > 0$ such that $p(L_E\zeta) < 1$ when $\sum |\zeta_j| < \delta_1$. For such ζ we let

$$F = \{y \in P_E : y = \sum_{j=1}^{\infty} \xi_j x_j, |\xi_j| \le 1 - |\zeta_j|\}.$$

By absolute continuity of the integral, for every $\epsilon > 0$ there exist ϵ' such that

$$\int_A \phi(z_0 + y) d\tau_E(y) > -\frac{\epsilon}{2}$$

when $\tau_E(A) < \epsilon'$.

We have $\tau_E(F) = \prod_{j=1}^{\infty}(1 - |\zeta_j|)^2$. So there is $0 < \delta < \delta_1$ such that $\tau_E(P_E \backslash F) < \frac{\epsilon}{2c}$ when $\sum |\zeta_j| < \delta$.

Let $z_1 = z_0 + \sum \zeta_j x_j$, $\sum |\zeta_j| < \delta$, and let $z = z_1 - z_0$. Then $F_1 = F + z \subset P_E$. Hence

$$\begin{aligned}
\phi_E(z_1) &= \int_{P_E} \phi(z_1 + y) d\tau_E(y) \\
&= \int_F \phi(z_1 + y) d\tau_E(y) + \int_{P_E \backslash F} \phi(z_1 + y) d\tau_E(y) \\
&\le \int_{F+z} \phi(z_0 + y) d\tau_E(y) + \frac{\epsilon}{2} \\
&\le \int_{P_E} \phi(z_0 + y) d\tau_E(y) + \epsilon = \phi_E(z_0) + \epsilon,
\end{aligned}$$

and this means that ϕ_E is usc along E. □

Let Z be a sequential complete local convex space and Y be subspace in Z. A system $\mathcal{E}$ of summable sequences $E \subset Y$ will be called basic in Y if for every summable sequence $F = \{x_k\} \subset Y$ there is a family $E \in \mathcal{E}$ and a sequence of vectors $y_k \in \ell^1$ convergence to 0 such that $L_E y_k = x_k$. Here we note that, in a finite-dimensional space a basic system may consist of only one sequence E that is the basis of the space. In an infinite-dimensional space the system of all summable sequence is basic.

Lemma 3.2 *Let Z be a Hausdorff sequential complete local convex space and Y be a subspace of Z and let $\mathcal{E}$ be a basic system in Y. For every $h \in H(Y)$ there is $E \in \mathcal{E}$ and $g \in H(\ell^1)$ such that $h = L_E \circ g$.*

Proof Since $h \in H(Y)$ there exist $r > 1$ such that h is a continuous mapping of $\overline{U}_r$ into Y, holomorphic in U_r. Where $U_r = \{z \in \mathbb{C} : |z| < r\}$. If $h_r(\zeta) = h(r^{-1}\zeta)$ then $h_r \in \overline{H}(Y)$. By Lemma 2.1 then $h_r \in \overline{H}(Y_A, p_A)$, where $A = \overline{conv(h_r(\overline{U}))} = \overline{conv(h(\overline{U}_r))}$ and p_A is the Minkowski functional of A. And so h can be represented by its Taylor series

$$h(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k, \quad \text{where} \sum_{k=0}^{\infty} p_A(a_k) t^k < \infty,$$

for some $1 < t < r$. This infer that the sequence $F = \{t^k a_k\}$ is summable. Thus we can find a sequence $E \in \mathcal{E}$ and a sequence of vectors $b_k \in \ell^1$ converging to 0 such that $L_E b_k = t^k a_k$. We set

$$g(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k.$$

The series for g converges absolutely in U_t so $g \in H(\ell^1)$ and $L_E g(\zeta) = h(\zeta)$ when $|\zeta| < t$. □

Let $\mathcal{E}$ be a system of summable sequence in Z. A function ϕ defined on an open set W in Z is called almost upper semicontinuous with respect to $\mathcal{E}$ ($\mathcal{E}$-ausc) if it is locally bounded above, absolutely measurable and for every sequence $E'\{x_j\}$ of vectors in Z which is the union of a sequence $E \in \mathcal{E}$ and finitely many vectors $z_1, \ldots, z_n$ in Z we have

$$\limsup_{t \to 0^+} \phi_{tE'}(x) \le \phi(x),$$

where $tE' = \{tx_j\}$. By Fatou lemma, an usc function is also $\mathcal{E}$-ausc.

Lemma 3.3 *Let Z be sequential complete local convex space and P be a projection of Z onto Y. Let $\mathcal{E}$ be a basic system in Y. Let W be an open set in Z and let ϕ be an $\mathcal{E}$-ausc function on W. If the function $u = I_P \phi$ is absolutely measurable on $W_0 = P(W)$, then it is $\mathcal{E}$-ausc.*

Proof We fix a point $w_0 \in W_0$, $E \in \mathcal{E}$, a finite set $y_1, \ldots, y_n$ in Y and $\epsilon > 0$. We find $z_0 \in W$ such that $Pz_0 = w_0$ and $\phi(z_0) < u(z_0) + \epsilon$. If $E' = E \cup F$, then

$$\begin{aligned}\limsup_{t \to 0^+} \int_{P_t E'} u(w_0 + y) d\tau_{tE}(y) &\le \limsup_{t \to 0^+} \int_{P_t E'} \phi(z_0 + y) d\tau_{tE}(y) \\ &\le \phi(z_0) < u(z_0) + \epsilon.\end{aligned}$$

□

4 The main results

Before starting and proving the main results we will give some notions.

Let A be a proper subset of Z. We say that a mapping F of $Z \setminus A$ into another topological vector space Z' is rational with poles in A if for every $g \in H(Z)$ either $g(U) \subset A$ or the set $A_g = g^{-1}(A) \cap U$ is finite and $F \circ g$ has poles of finite order at points of A_g. Also for a holomorphic mapping of $h : U \setminus \{0\} \to Z$ with a pole of finite order at 0, the mapping $F \circ h$ has a pole of finite order at 0 too. In the case of finite dimensional topological vector space this definition produces standard rational mappings. Also linear mappings are rational.

Now we recall a basis result in functional analysis. If (Z, ξ) is a sequential complete local convex space then there is the strongest local convex topology η on Z such that ξ and η have the same bounded sets in Z and (Z, η)

is the inductive limit of family Banach spaces $(Z_i, i \in I)$ with some the directed set I. From this we give the notion of weakly $\mathcal{E}$-ausc function as follow.

Let Z be a sequential complete local convex space that is also the inductive limit of family Banach spaces $(Z_i, i \in I)$ with I is the directed set. Let P be a projection of Z onto Y. Let $\mathcal{E}$ be a basic system in Y. A function ϕ on an open set W in Z is called weakly $\mathcal{E}$-ausc if there is rational mappings $F_i : W_i \to Z$ with poles outside of $W_i \subset Z_i$, projections P_i on Z_i and $\mathcal{E}$-ausc functions ϕ_i on W_i such that:

i. $P(Z) = P_i(Z_i) = Y$, $P_i(W_i) = P(W) = W_0$;
ii. $F_i(W_i) \subset W$, $P \circ F_i(z) = P_i(z)$;
iii. $\phi(F_i(z)) \le \phi_i(z)$;
iv. The function $u_i = I_{P_i}\phi_i$ are absolutely measurable and from a decreasing net such that $\lim_i u_i = I_P\phi$.

The net $\{Z_i, W_i, F_i, P_i, \phi_i\}$ will be called an approximating net. We can see that a $\mathcal{E}$-ausc function is also weakly $\mathcal{E}$-ausc.

The following theorem is the main of the paper.

Theorem 4.1 *Let Z be a Hausdorff sequential complete local convex space with a holomorphic $\overline{U}$-action A. Let $Pz = A(0, z)$ and let $Y = P(Z)$. Let $\mathcal{E}$ be a basic system in Y. Let W be an S-invariant open set in Z and let ϕ be an S-invariant absolutely measurable function on W. If ϕ is weakly $\mathcal{E}$-ausc with an approximating net $\{Z_i, W_i, F_i, P_i, \phi_i\}$ and the fibers $W_{iz} = \{w \in W_i : P_i(w) = z\}$ are path connected, then the function $u = I_P\phi$ is sequential psh if and only if ϕ is P-psh.*

Proof First, we will prove the necessity of the theorem. If $h \in H(W_0)$, $W_0 = P(W)$, and $f : S \to W$ is a continuous mapping such that $P \circ f = h$ on S, then for a psh function u we have

$$u(h(0)) \le \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))d\theta.$$

Thus ϕ is P-psh.

The proof of sufficiency is more complicated and will follow from a sequence of the following lemmas.

Lemma 4.2 *In conditions of Theorem 4.1 the function $u = I_P\phi$ is sequential usc.*

Proof Let z_0 be a point in W_0 and let $\{z_k\}$ be a sequence of points in W_0 converging to 0. We have $H = \{z_0, z_0+z_k\}$ is a compact set in Y. Set $G = \overline{conv(H)}$ and $Y_G = span(G)$ and p_G is a Minkovsky functional of G. Then (Y_G, p_G) is the Banach space.

Switching if necessary to a subsequence, we can assume that the family $F = \{kz_k\}$ is summable in Y_G. So we can find a sequence $E \in \mathcal{E}$ and a sequence of vectors $x_k \in \ell^1$ converging to 0 such that $L_E x_k = z_k$.

Fix some $\epsilon > 0$ and find $i \in I$ and a point $w_0 \in W_0$ such that $P_i w_0 = z_0$ and $\phi_i(w_0) < u(z_0) + \epsilon$. For every $k \ge 1$ we set $w_k = w_0 + z_k$. Let us take $s > 0$ so small that $w_k + P_{sE} \subset W_i$ for all k. Let

$$\psi_t(y) = \int_{P_{tE}} \phi_i(y + x)d\tau_{tE}(x).$$

Since ϕ_i is $\mathcal{E}$-ausc, for any $\epsilon > 0$ we can find $0 < t < s$ such that $\psi_t(w_0) < \phi_i(w_0) + \epsilon$. Since the function ψ_t is usc along tE and $L_{tE}(t^{-1}x_k) = z_k$, there is k_0 such that $\psi_t(w_k) < \phi_i(w_0) + 2\epsilon$ when $k > k_0$.

For every $e^{i\theta} \in S$ we have

$$\int_{P_{tE}} \phi_i(w_k + e^{i\theta}x)d\tau_{tE}(x) = \psi_t(w_k).$$

Hence by Fubini's and the mean theorems there exist $x_0 \in P_{tE}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_i(w_k + e^{i\theta}x_0)d\theta \le \phi_i(y_0) + 2\epsilon, \quad \text{when } k > k_0.$$

Let $f_i(\xi) = w_k + \xi x_0$ be a mapping of the unit disk into W_i and let $f = F_i \circ f_i$. Since $Pf(0) = P_i f_i(0) = z_0 + z_k$ and ϕ is weakly P-psh,

$$u(z_0 + z_k) \le \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \phi_i(f_i(e^{i\theta}))d\theta \le u(z_0) + 2\epsilon.$$

Thus u is sequential usc. □

Now to conclude the proof of the theorem for a mapping $h \in H(W_0)$ such that $h(0) = z_0$, we will establish for every $\epsilon > 0$ we have

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + \epsilon. \tag{4.1}$$

For this we will replace the integrand u in inequality above by a topologically better function u_i. Note that by Lemma 3.3 the function u_i are $\mathcal{E}$-ausc.

Lemma 4.3 *Let Z be a Hausdorff sequential complete local convex space and let P be a projection of Z onto Y. Let $\mathcal{E}$ be a basic system in Y. If a decreasing net of $\mathcal{E}$-ausc functions u_i, $i \in I$, on an open set $W_0 \subset Y$ converges to an usc function u, then for every $h \in H(W_0)$ and $\epsilon > 0$ there are a sequence of vectors $y_n \in Y$ converging to 0 and a sequence of $i_n \in I$ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} u_{i_n}(h_n(e^{i\theta}))d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + \epsilon,$$

where $h_n(\zeta) = h(\zeta) + \zeta y_n$.

Proof By Lemma 3.2 we take $E \in \mathcal{E}$ and a mapping $g \in H(\ell^1)$ such that $L_E g = h$. For $n = 1, 2, ...$ let $W_0^n = \{x \in W_0 : x + n^{-1}E \subset W_0\}$ and

$$u_{in}(x) = \int_{P_{n^{-1}E}} u_i(x + y)d\tau_{n^{-1}E}(y),\ \forall x \in W_0^n.$$

By Lemma 3.1 the functions u_{in} are usc on $h(S)$.

Since the functions u_i are decreasing and locally bounded above, $u_{in} \leq u_{jn}$ when $i \geq j$ and for any $i_0 \in I$ the functions u_{in}, $i \geq i_0$, are uniformly bounded above on $h(S)$. Moreover, since the functions u_i are $\mathcal{E}$-ausc,

$$\limsup_{n\to\infty} u_{in} \leq u_i$$

on $h(S)$. By Lebesgue's monotone convergence theorem for every $n \geq n_0$ there is $i_n \in I$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} u_{i_n n}(h(e^{i\theta}))d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + \epsilon.$$

By the definition of $u_{i_n n}$, Fubini's theorem, and the mean velue theorem we can find a vector $y_n \in P_{n^{-1}E}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} u_{i_n}(h(e^{i\theta}) + e^{i\theta}y_n)d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + 2\epsilon.$$

□

We use Lemma 4.3 to find a mapping $h' \in \overline{H}(W_0)$ with $h'(0) = z_0$ and $i \in I$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} u_i(h'(e^{i\theta}))d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + \epsilon.$$

Now we see that the subaveraging inequality (4.1) will be proved if we establish that

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u_i(h'(e^{i\theta}))d\theta + \epsilon. \tag{4.2}$$

Let us denote h' by h.

Now we claim the existence of a continuous selection minimizing some functional.

Lemma 4.4 *Let P be a projection on a Hausdorff sequential complete local convex space and let $W \subset Z$ be an open with path connected fibers. Let $\mathcal{E}$ be a basic system in $Y = P(Z)$ and $W_0 = P(W)$. If ϕ is an $\mathcal{E}$-ausc function on W, then for every $h \in H(W_0)$ and every $\epsilon > 0$ there are a family $E \in \mathcal{E}$, a finite set $F \subset Z$, a positive number $s < \epsilon$, a vector $z_s \in P_{sE}$, a continuous mapping $q : S \to \ell^1$, and $g \in H(\ell^1)$ such that for $E' = F \cup E$ and $f = L_{E'}q$ we have: $h = L_{E'}g$, $P \circ f(\xi) = h(\xi) + \xi z$ on S, and*

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + \epsilon, \quad \text{where } u = I_P\phi.$$

Proof By Lemma 3.2 we choose for h a system $E \in \mathcal{E}$ and a mapping $g \in H(\ell^1)$ such that $L_E \circ g = h$. For $t > 0$ let $W^t = \{x \in W : x + tP_E \subset W\}$ and let

$$\psi_t(z) = \int_{P_{tE}} \phi(x+y)d\tau_{tE}(y).$$

We may assume that $h(\overline{U}) \subset P(W^t)$ and we let

$$v_t(z) = \inf\{\psi_t(w) : Pw = z, w \in W^t\}.$$

Since ϕ is $\mathcal{E}$-ausc, $\limsup_{t\to 0^+} \psi_t(z) \le \phi(z)$, $\forall z \in W$. Thus $\limsup_{t\to 0^+} v_t(z) \le u(z)$, $\forall z \in W_0$.

Moreover, by Lemma 3.1 the functions $\psi_t(x)$ are usc along E. Therefore the functions v_t are usc along E, and consequently the functions $v_t(h(\xi))$ are usc on S.

Since the function ϕ is locally bounded above, we can find a constant C and open sets $W_1, W_2, \dots, W_n$ in W such that $\phi < C$ on each of W_j and the union of $P(W_j)$ covers $h(S)$. Hence $v_t(h(\xi)) < C$ on S when t is sufficiently small, and by Fatou's lemma for $\epsilon > 0$ we can find s so small that

$$\frac{1}{2\pi}\int_0^{2\pi} v_s(h(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + \epsilon. \tag{4.3}$$

Let us take a continuous function v on S such that $v \ge v_s \circ h$ and

$$\frac{1}{2\pi}\int_0^{2\pi} v(e^{i\theta})d\theta \le \frac{1}{2\pi}\int_0^{2\pi} v_s(h(e^{i\theta}))d\theta + \epsilon. \tag{4.4}$$

For every $\xi \in S$ we take $z_\xi \in W^s$ such that $Pz_\xi = h(\xi)$ and $\psi_s(z_\xi) < v(\xi) + \epsilon$. Since ψ_s is usc along E, there is an open arc $V_\xi \subset S$ containing ξ such that $p_\xi(\zeta) = z_\xi + h(\zeta) - h(\xi) \in W^s$ and $\psi_s(p_\xi(\zeta)) < v(\zeta) + \epsilon$ on V_ξ.

We can find finitely many points $\xi_1, \xi_2, \dots, \xi_n$ such that the arcs $V_j = V_{\xi_j}$ cover S. Let us take closed arcs $V_j' \subset V_j$ that still cover S. We may assume that none of these arcs contains another. We let $\gamma_1 = V_1$ and delete the interior of γ_1 from away all V_j', $j \ge 2$. We continue to denote the obtained arcs by V_j'. Then we throw away all V_j' that are empty or consist of one point. We take one of remaining arcs, denote it by γ_2, and repeat the process deleting from all arcs except γ_1 and γ_2 the interior of γ_2. In at most n steps we will get closed arcs γ_j, $1 \le j \le m$, covering S and with disjoint interiors.

Each of the arcs γ_j was obtained from some arc V_k. Hence the mappings p_{ξ_k} are defined on γ_j. We will denote them by p_j. After a renumbering and rotation we may assume that $\gamma_j = \{e^{i\theta} : \alpha_{j-1} \le \theta \le \alpha_j\}$.

Let us denote by p the mapping of the interiors of the arcs γ_j into W^s equal to p_j on γ_j. By the definition of ψ_s, Fubini's, theorem and the mean value theorem there is a vector $z_s \in P_{sE}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(p(e^{i\theta}) + e^{i\theta}z_s)d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \phi_s(p(e^{i\theta})d\theta + \epsilon.$$

Since $\psi_s(p(\zeta)) < v(\zeta) + \epsilon$, it follows from (4.3) and (4.4) that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(p(e^{i\theta}) + e^{i\theta}z_s)d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(h(e^{i\theta}))d\theta + 3\epsilon. \tag{4.5}$$

Let $p_j'(\zeta) = p_j(\zeta) + \zeta z_s$ and let $p' \equiv p$ on the interior of the arcs γ_j. Since fibers of W are path connected, for every $1 \le j \le m$ there is a piecewise linear continuous mapping ρ_j of $[0, 1]$ into the fiber over $p_j'(e^{i\alpha_j})$

such that $\rho_j(0) = p'_j(e^{i\alpha_j})$ and $\rho_j(1) = p'_{j+1}(e^{i\alpha_j})$. The union of all sets $p'_j(\gamma_j)$ and $\rho_j([0,1])$ is compact, and therefore has a neighborhood $V \subset W$ where the function ϕ is bounded above by some constant, say, $A > 0$.

We take as F the set of all points $z_j = z_{\xi_j}$ and all vertices of the paths ρ_j, $1 \le j \le m$. Let $E' = F \cup E$. Then points z_j lie in the image of $L_{E'}$ and we choose w_j, $1 \le j \le m$, and w_s such that $L_{E'}w_s = z_s$. For every $1 \le j \le m$ we define the mapping q_j of γ_j into ℓ^1 as $q_j(\zeta) = g(\zeta) - g(\xi_j) + w_j + \zeta w_s$ (g was introduced in the beginning of the proof). Note that $L_{E'}q_j = p'_j$. Since all vertices of the paths ρ_j are in F, we can find a piecewise linear continuous mapping σ_j of $[0,1]$ into ℓ^1 such that $\sigma_j(0) = q_j(e^{i\alpha_j})$, $\sigma_j(1) = q_{j+1}(e^{i\alpha_j})$ and $L_{E'}\sigma_j = \rho_j$.

Let us choose points $\alpha_{j-1} < \beta'_j < \alpha_j < \beta''_j < \alpha_{j+1}$ so close to each other that the length of the union G' of the arcs $[\beta'_j, \beta''_j]$ is less than ϵ/A and

$$\frac{1}{2\pi}\int_{G'} \phi(p'(e^{i\theta}))d\theta \le \epsilon.$$

We set $h_1(\zeta) = h(\zeta) + \zeta z_s$ and $g_1(\zeta) = g(\zeta) + \zeta w_s$. Let $m_j(t)$ be a linear function on $[\beta'_j, \beta''_j]$ equal to 0 at the left end and to 1 at the right one. We define a mapping $q : S \to \ell^1$ as $q_j(e^{i\alpha})$ when $\beta''_{j-1} \le \alpha \le \beta'_j$, and for $\alpha \in [\beta'_j, \beta''_j]$ we let

$$\begin{aligned} q(e^{i\alpha}) &= (1 - m_j(\alpha))(q_j(e^{i\beta'_j}) - q_j(e^{i\alpha_j})) + \sigma_j(m_j(\alpha)) \\ &\quad + m_j(\alpha)(q_{j+1}(e^{i\beta''_j}) - q_{j+1}(e^{i\alpha_j})) + g_1(e^{i\alpha}) \\ &\quad - (1 - m_j(\alpha))g_1(e^{\beta'_j}) - m_j(\alpha)g_1(e^{\beta''_j}). \end{aligned}$$

It easy to verify that q is continuous and $PL_{E'}q = h_1$. We may also choose points β'_j and β''_j so close that q maps S into V.

Let $f = L_{E'}q$. We denote by G the complement of G' in S. Recalling the choice of the sets G' and of the mapping q and also (4.5), we see that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))d\theta &= \frac{1}{2\pi}\int_G \phi(p'(e^{i\theta}))d\theta + \frac{1}{2\pi}\int_{G'} \phi(f(e^{i\theta}))d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \phi(p'(e^{i\theta}))d\theta + 2\epsilon \\ &\le \frac{1}{2\pi}\int_0^{2\pi} u(p'(e^{i\theta}))d\theta + 5\epsilon. \end{aligned}$$

□

Let us return to prove Theorem 4.1 and prove (4.2). We apply Lemma 4.4 to ϕ_i, take any s satisfying the conclusion of the lemma, and denote $h(\zeta) + \zeta z_s$ by $h_1(\zeta)$. Let

$$\psi_t(z) = \int_{P_{tE'}} \phi_i(z + y)d\tau_{tE'}(y)$$

and let $\psi'_t = \psi_t \circ L_{E'}$. Since ϕ_i is $\mathcal{E}$-ausc, we can find $t > 0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_t(f(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \phi_i(f(e^{i\theta}))d\theta + \epsilon. \tag{4.6}$$

We can choose t so small that $f(S)$ lies in the set $W^t = \{x \in W : x + tP_{E'} \subset W\}$. The function ψ'_t is usc on $W_t = L_{E'}^{-1}(W^t)$. By Lemma 2.4 in [18], there exist a mapping

$$q_1(\zeta) = g(\zeta) + \zeta w_s + \sum_{k=-N}^{N} c_k \zeta^k$$

of $\mathbb{C} \setminus \{0\}$ into ℓ^1 such that $L_{E'}(g(\zeta) + \zeta w_s) = h_1(\zeta)$, $P_i L_{E'} q_1 = h_1$ on S, $q_1(S) \subset W_t$, and

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_t'(q_1(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \psi_t(f(e^{i\theta}))d\theta + \epsilon. \tag{4.7}$$

Let

$$f_1(\zeta) = L_{E'}q_1(\zeta) = h_1(\zeta) + \sum_{k=-N}^{N} d_k \zeta^k.$$

Fubini's theorem and the mean value theorem applied to (4.6) and (4.7) show that there is $y_0 \in P_{tE'}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_i(f_1(e^{i\theta}) + e^{i\theta} y_0)d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \phi_i(f(e^{i\theta}))d\theta + 3\epsilon.$$

The mapping $f_2(\zeta) = f_1(\zeta) + \zeta y_0$ maps S into W_i, and has a pole of order at most N at the origin. Hence the mapping $f_3 = F_i \circ f_2$ has no poles on S and only finitely many in U. By Lemma 2.3 there is a Blaschke product B such that the mapping $f_4(\zeta) = A(B(\zeta), f_3(\zeta))$ is in $H(Z)$. Since the function ϕ is S-invariant, $\phi(f_4(\zeta)) = \phi(f_3(\zeta)) \le \phi_i(f_2(\zeta))$ when $|\zeta| = 1$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(f_4(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \phi_i(f_2(e^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u_i(h(e^{i\theta}))d\theta + 4\epsilon.$$

Note that on S we have:

$$Pf_4(\zeta) = Pf_3(\zeta) = P_i f_2(\zeta) = h_1(\zeta) + \zeta P_i y_0.$$

Thus $Pf_4(0) = z_0$. The function ϕ is P-psh, and therefore

$$u(h(0)) \le \frac{1}{2\pi}\int_0^{2\pi} u_i(h(e^{i\theta}))d\theta + 4\epsilon.$$

This proves (4.2) and the theorem. □

According to this theorem, we must verify that a function ϕ is P-psh to establish the minimum principle. Clearly it could be done easier if it will suffice to verify the weak P-plurisubharmonicity. The following, we are going to prove in the case of $\overline{U}$-invariant domains, i. e., $A(\zeta, z) \in W$ whenever $z \in W$ and $|\zeta| \le 1$, the weak P-plurisubharmonicity implies P-plurisubharmonicity.

Theorem 4.5 *Let Z be a Hausdorff sequential complete local convex space with a holomorphic $\overline{U}$-action A and let $Pz = A(0, z)$. Let W be an $\overline{U}$-invariant open set in Z and let ϕ be a weakly P-psh S-invariant function on W. Then ϕ is P-psh.*

Proof Note that now $W_0 = P(W) \subset W$. For $f \in H(Z)$ such that $f(0) = z$, $P \circ f = h \in H(W_0)$, and $f(S) \subset W$ we consider $F(\zeta, \xi) = A(\zeta, f(\xi))$. The preimage D of W under the mapping F is an open set containing neighborhoods of sets $\{(\zeta, \xi) : |\xi| \le 1, \zeta = 0\}$. Therefore there is an integer $N > 0$ such that the mapping $g(\xi) = (\xi^N, \xi)$ is in $H(D)$. Let $G = F \circ g$. Then $G \in H(W)$, $P \circ G = h$, and $\phi(G(\xi)) = \phi(f(\xi))$ when $|\xi| = 1$. Since ϕ is weakly P-psh,

$$I_P\phi(z) \le \frac{1}{2\pi}\int_0^{2\pi} \phi(G(e^{i\theta}))d\theta = \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))d\theta,$$

and therefore ϕ is P-psh. □

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