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ORIGINAL RESEARCH

On families of graphs which are both adjacency equienergetic and distance equienergetic

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Abstract Let $\mathcal{A}(G)$ and $\mathfrak{D}(G)$ be the adjacency and distance matrices of a graph G respectively. The adjacency energy or $\mathcal{A}$-energy $\mathcal{E}_{\mathcal{A}}(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of $\mathcal{A}(G)$. Analogously, the $\mathfrak{D}$-energy $\mathcal{E}_{\mathfrak{D}}(G)$ is defined to be the sum of the absolute values of the eigenvalues of $\mathfrak{D}(G)$. One of the interesting problems on graph energy is to characterize those graphs which are equienergetic with respect to both the adjacency and distance matrices. A weaker problem is to construct the families of graphs which are equienergetic with respect to both the adjacency and distance matrices. In this paper, we find the explicit relations between $\mathcal{A}$-energy and $\mathfrak{D}$-energy of certain families of graphs. As a consequence, we provide an answer to the above open problem (Indulal in https://icgc2020.wordpress.com/invitedlectures, 2020; http://www.facweb.iitkgp.ac.in/rkannan/gma.html, 2020)

Keywords Graph energy · Distance energy · Equienergetic graphs · Distance equienergetic graphs

Mathematics Subject Classification 05C50 · 05C76

1 Introduction

In the spectral graph theory, the concept of graph energy has emerged as one of the interesting directions due to its applications in molecular chemistry, see for instance [11,21]. For a connected graph G of order n, Indulal et al. [20] defined the distance energy as $\sum_{i=1}^{n}|\xi_i|$, where ξ_i, $1 \leq i \leq n$, are the distance eigenvalues of G. In the literature, there are many energy variants defined on graphs from which the distance energy has also

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received much attention, as it has been seen to be useful in QSPR modeling [6]. The investigation of the distance eigenvalues and the distance energy of a graph can be seen in [1,2,14,18–20,24,26,32,33,36] and the references therein. An interesting problem is to find the non-isomorphic graphs with the same distance energy, such graphs are called as distance equienergetic graphs. So far the graphs which are equienergetic with respect to different matrices associated to the graphs are not fully investigated [15]. Also, the explicit relations for the distance energy for many large class of graphs are not fully known. Indulal [16,17] posed an open problem to characterize or construct the families of graphs which are equienergetic with respect to both the adjacency and distance matrices. This motivates us to investigate the distance energy of the families of the graphs which in turns provide an answer to the open problem posed by Indulal by presenting several families of the non-cospectral graphs which are equienergetic with respect to both the adjacency and distance matrices.

2 Preliminaries

All the graphs considered in this paper are simple, connected and undirected. The number of vertices is the order and the number of edges is the size of the graph G. The degree of a vertex x_i is denoted by d_i. A graph is regular of degree r if $d_i = r$ for each vertex x_i. The *adjacency matrix* of a graph G, denoted by $\mathcal{A}(G)$ or simply by $\mathcal{A}$, is defined as $\mathcal{A} = [\mathfrak{a}_{ij}]$, where $\mathfrak{a}_{ij} = 1$ if the vertex x_i is adjacent to the vertex x_j and $\mathfrak{a}_{ij} = 0$, otherwise. The *complement* $\overline{G}$ of a graph G is the graph with the vertex set same as that of G, and in which two vertices are adjacent if and only if they are not adjacent in G. The eigenvalues of $\mathcal{A}(G)$ are the *eigenvalues* or $\mathcal{A}$-*eigenvalues* of the graph G. The eigenvalues of a graph G of order n are denoted by $\mu_1, \mu_2, \ldots, \mu_n$. Two graphs are said to be $\mathcal{A}$-*cospectral* if they have same $\mathcal{A}$-eigenvalues. The *distance matrix* of a graph G, denoted by $\mathfrak{D}(G)$ or simply by $\mathfrak{D}$, is defined as $\mathfrak{D} = [\mathfrak{d}_{ij}]$, where $\mathfrak{d}_{ij} = d_G(x_i, x_j)$ is the distance between the vertices x_i and x_j. The *distance eigenvalues* or $\mathfrak{D}$-*eigenvalues* of a graph G are the eigenvalues of $\mathfrak{D}(G)$. The distance eigenvalues of a graph G of order n are denoted by $\xi_1, \xi_2, \ldots, \xi_n$. Two graphs are said to be $\mathfrak{D}$-*cospectral* if they have same $\mathfrak{D}$-eigenvalues. The *diameter* of a graph G is $diam(G) = \max\{d_G(x_i, x_j) : x_i, x_j \in V(G)\}$, where $V(G)$ is the vertex set of a graph G. The *adjacency energy* or $\mathcal{A}$-*energy* $\mathcal{E}_{\mathcal{A}}(G)$ of a graph G of order n was introduced by Ivan Gutman [10] in 1978, in order to approximate a molecule's total π-electron energy. It is defined as $\mathcal{E}_{\mathcal{A}}(G) = \sum_{i=1}^{n} |\mu_i|$. Two graphs G_1 and G_2 of the same order are said to be *adjacency equienergetic* or $\mathcal{A}$-*equienergetic* if $\mathcal{E}_{\mathcal{A}}(G_1) = \mathcal{E}_{\mathcal{A}}(G_2)$. Indulal et al. [20] defined the *distance energy* or $\mathfrak{D}$-*energy* $\mathcal{E}_{\mathfrak{D}}(G)$ of a graph G of order n as $\mathcal{E}_{\mathfrak{D}}(G) = \sum_{i=1}^{n} |\xi_i|$. Two graphs G_1 and G_2 of the same order are said to be *distance equienergetic* or $\mathfrak{D}$-*equienergetic* if $\mathcal{E}_{\mathfrak{D}}(G_1) = \mathcal{E}_{\mathfrak{D}}(G_2)$. The cospectral graphs are always equienergetic. It is worth to study equienergetic graphs which are non-cospectral. We denote the number of positive, negative and zero $\mathcal{A}$-eigenvalues and $\mathfrak{D}$-eigenvalues of a graph G, respectively, by $n^+_{\mathcal{A}}, n^-_{\mathcal{A}}$ and $n^0_{\mathcal{A}}$; $n^+_{\mathfrak{D}}, n^-_{\mathfrak{D}}$ and $n^0_{\mathfrak{D}}$. The complete graph of order p and the complete bipartite graph of order $s + r$ are, respectively, denoted by K_p and $K_{s,r}$. For other notations and terminology, we follow [7].

Definition 2.1 [12] The line graph $\mathcal{L}(G)$ of a graph G is the graph with the vertex set same as that of the edge set of G and in which two vertices are adjacent if the corresponding edges in G are adjacent to each other. The k-th iterated line graph of G for $k = 0, 1, 2, \ldots$ is defined as $\mathcal{L}^k(G) \equiv \mathcal{L}(\mathcal{L}^{k-1}(G))$, where $\mathcal{L}^0(G) \equiv G$ and $\mathcal{L}^1(G) \equiv \mathcal{L}(G)$.

Let n_k and m_k denote the order and size of $\mathcal{L}^k(G)$ respectively for $k = 0, 1, 2, \ldots$.

Definition 2.2 [5,35] Let G be a graph of order n with the vertices labeled as $x_1, x_2, \ldots, x_n$. The generalized composition or H-join of the graphs $H_1, H_2, \ldots, H_n$ is a graph $G[H_1, H_2, \ldots, H_n]$ which is obtained by joining every vertex of H_j to every vertex of H_k if x_j is adjacent to x_k in G.

Definition 2.3 [4] Let G be a graph of order n with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$. The extended bipartite double graph $Ebd(G)$ of G is a bipartite graph with its two vertex partite sets $V_1 = \{x_1^{'}, x_2^{'}, \ldots, x_n^{'}\}$ and $V_2 = \{x_1^{''}, x_2^{''}, \ldots, x_n^{''}\}$ in which two vertices $x_j^{'}$ and $x_k^{''}$ are adjacent if $j = k$ or x_j and x_k are adjacent in G.

The $Ebd(G)$ of a graph G is one of the nice graph operations available to create bipartite graphs. The bipartite graphs also arise from the alternante hydrocarbons in chemistry.

Definition 2.4 [23] Let G be a graph of order n with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$. The double graph $D_2(G)$ of G is a graph which is obtained by taking two copies of G with the set of vertices $V_1 = \{x_1^{'}, x_2^{'}, \ldots, x_n^{'}\}$ and $V_2 = \{x_1^{''}, x_2^{''}, \ldots, x_n^{''}\}$ respectively, in which the vertices $x_j^{'}$ and $x_k^{''}$ are adjacent if x_j and x_k are adjacent in G.

Definition 2.5 [7] Let G_1 and G_2 be two graphs with the vertex sets V_1 and V_2 respectively. The strong product of the graphs G_1 and G_2 is a graph $G_1 \boxtimes G_2$ with the vertex set $V_1 \times V_2$ in which two vertices (x_i, x_j) and (y_i, y_j) are adjacent if x_i is adjacent to y_i in G_1 and $x_j = y_j$ in G_2; or $x_i = y_i$ in G_1 and x_j is adjacent to y_j in G_2; or x_i is adjacent to y_i in G_1 and x_j is adjacent to y_j in G_2.

Now, we state some results which will be used in the sequel.

Theorem 2.6 [7] *Let G be a regular graph of degree r and order n with the $\mathcal{A}$-eigenvalues $r = \mu_1, \mu_2, \ldots, \mu_n$. Then*

1. *the complement graph $\overline{G}$ of G is a regular graph of degree $n - r - 1$ with the $\mathcal{A}$-eigenvalues $n - r - 1$ and $-1 - \mu_j$; $2 \leq j \leq n$.*
2. *the line graph $\mathcal{L}(G)$ of G is a regular graph of degree $2r - 2$ with the $\mathcal{A}$-eigenvalues $2r - 2$, $\mu_j + r - 2$; $2 \leq j \leq n$ and -2 $\left(\frac{n(r-2)}{2} \text{ times}\right)$.*

Theorem 2.7 [8] *Let G be a regular graph of degree r and order n with $diam(G) \leq 2$. If $r = \mu_1, \mu_2, \ldots, \mu_n$ are the $\mathcal{A}$-eigenvalues of G, then the $\mathfrak{D}$-eigenvalues of G are $2n - r - 2$ and $-(\mu_j + 2)$; $2 \leq j \leq n$.*

By Theorem 2.7 we can see that if the regular graphs of same order and of same degree having diameters less than or equal to 2 are $\mathcal{A}$-cospectral then they are $\mathfrak{D}$-cospectral also.

The following result involves the diameter of the graph $\overline{\mathcal{L}^k(G)}$.

Theorem 2.8 [18] *Let G be a regular graph of degree r and order n with $2r \leq n - 1$. Then $diam(\overline{\mathcal{L}^k(G)}) = 2$ for all $k \geq 0$.*

The next result is about the $\mathfrak{D}$-equienergetic graphs.

Theorem 2.9 [18] *Let G_1 and G_2 be two regular graphs both of same degree $r \geq 4$ and same order n. If $2r \leq n - 1$, then the graphs $\overline{\mathcal{L}^k(G_1)}$ and $\overline{\mathcal{L}^k(G_2)}$ are $\mathfrak{D}$-equienergetic for all $k \geq 2$.*

Theorem 2.10 [27] *Let G_1 and G_2 be two regular graphs both of same degree $r \geq 3$ and same order n. Then the graphs $\overline{\mathcal{L}^k(G_1)}$ and $\overline{\mathcal{L}^k(G_2)}$ are $\mathcal{A}$-equienergetic for all $k \geq 2$.*

Theorem 2.11 [28,30] *Let G be a graph of order n_0 and size m_0 with $d_i + d_j \geq 6$ to each edge $e = x_i x_j$ in G. Then the iterated line graphs $\mathcal{L}^k(G)$ have all the negative $\mathcal{A}$-eigenvalues equal to -2 with the multiplicity $m_{k-1} - n_{k-1}$ for all $k \geq 2$, where n_k and m_k denote the order and size of $\mathcal{L}^k(G)$ respectively.*

Theorem 2.12 [36] *Let G be a graph of order n with the vertices labeled as $x_1, x_2, \ldots, x_n$. Let H_i be regular graphs of degree r_i and order $\mathfrak{n}_i$ with the $\mathcal{A}$-eigenvalues $r_i = \mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,\mathfrak{n}_i}$; $1 \leq i \leq n$. Then the $\mathfrak{D}$-eigenvalues of $G[H_1, H_2, \ldots, H_n]$ are $-\mu_{i,j} - 2$; $1 \leq i \leq n$ and $2 \leq j \leq \mathfrak{n}_i$ together with the eigenvalues of an auxiliary matrix*

$$M = \begin{bmatrix} 2\mathfrak{n}_1 - r_1 - 2 & \mathfrak{n}_2 d_G(x_1, x_2) & \mathfrak{n}_3 d_G(x_1, x_3) & \ldots & \mathfrak{n}_n d_G(x_1, x_n) \\ \mathfrak{n}_1 d_G(x_2, x_1) & 2\mathfrak{n}_2 - r_2 - 2 & \mathfrak{n}_3 d_G(x_2, x_3) & \ldots & \mathfrak{n}_n d_G(x_2, x_n) \\ \mathfrak{n}_1 d_G(x_3, x_1) & \mathfrak{n}_2 d_G(x_3, x_2) & 2\mathfrak{n}_3 - r_3 - 2 & \ldots & \mathfrak{n}_n d_G(x_3, x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathfrak{n}_1 d_G(x_n, x_1) & \mathfrak{n}_2 d_G(x_n, x_2) & \mathfrak{n}_3 d_G(x_n, x_3) & \ldots & 2\mathfrak{n}_n - r_n - 2 \end{bmatrix}.$$

Proposition 2.13 [28] *Let G_1 and G_2 be two graphs of the same order n with the number of negative $\mathcal{A}$-eigenvalues $n^-_{\mathcal{A}_1}$ and $n^-_{\mathcal{A}_2}$ respectively. If G_1 and G_2 have no $\mathcal{A}$-eigenvalues in the interval $(-1, 0)$, then the bipartite graphs $Ebd(G_1)$ and $Ebd(G_2)$ are $\mathcal{A}$-equienergetic if and only if $n^-_{\mathcal{A}_1} = n^-_{\mathcal{A}_2}$.*

Theorem 2.14 [19] *Let G be a regular graph of degree r and order n with the $\mathcal{A}$-eigenvalues $r = \mu_1, \mu_2, \ldots, \mu_n$ and $diam(G) = 2$. Then the $\mathfrak{D}$-eigenvalues of $Ebd(G)$ are $5n - 2r - 4$, $2r - n$, $-2(\mu_i + 2)$; $2 \leq i \leq n$ and $2\mu_i$; $2 \leq i \leq n$.*

Theorem 2.15 [19] *Let G be a graph of order n with the $\mathfrak{D}$-eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$. Then the $\mathfrak{D}$-eigenvalues of double graph $D_2(G)$ are $2(\xi_i + 1)$; $1 \leq i \leq n$ and -2 (n times).*

Theorem 2.16 [7] *Let G_1 and G_2 be two graphs of order $\mathfrak{n}_1$ and $\mathfrak{n}_2$ with the $\mathcal{A}$-eigenvalues $\mu_{1,i}$ and $\mu_{2,j}$; $1 \leq i \leq \mathfrak{n}_1$ and $1 \leq j \leq \mathfrak{n}_2$ respectively. Then the $\mathcal{A}$-eigenvalues of $G_1 \boxtimes G_2$ are $\mu_{1,i} + \mu_{2,j} + \mu_{1,i}\mu_{2,j}$; $1 \leq i \leq \mathfrak{n}_1$ and $1 \leq j \leq \mathfrak{n}_2$.*

Theorem 2.17 [3] *Let G be a graph of order n with the $\mathfrak{D}$-eigenvalues ξ_i; $1 \leq i \leq n$. Then the $\mathfrak{D}$-eigenvalues of $G \boxtimes K_p$, $p \geq 2$ are $p\xi_i + p - 1$; $1 \leq i \leq n$ and -1 ($n(p-1)$ times).*

Proposition 2.18 [9] *Let G be a graph of order n. Then $\mathcal{E}_{\mathcal{A}}(D_2(G)) = 2\mathcal{E}_{\mathcal{A}}(G)$.*

3 Main results

In the following theorem, we present an explicit relation between the $\mathcal{A}$-energy and $\mathfrak{D}$-energy of a regular graph with diameter less than or equal to 2.

Theorem 3.1 *Let G be a regular graph of degree r and order n with $diam(G) \leq 2$. If $r = \mu_1, \mu_2, \ldots, \mu_n$ are the $\mathcal{A}$-eigenvalues of G, then*

$$\mathcal{E}_{\mathfrak{D}}(G) = 2\left(2n - r - 2 - 2n_{\mathcal{A}}^{-} + \sum_{\mu_i \in (-2,0)} (\mu_i + 2)\right) + \mathcal{E}_{\mathcal{A}}(G).$$

Proof Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\xi_1, \xi_2, \ldots, \xi_n$ be the $\mathcal{A}$-eigenvalues and $\mathfrak{D}$-eigenvalues of G respectively. Then, by Theorem 2.7, the $\mathfrak{D}$-energy of G

$$\begin{aligned}
\mathcal{E}_{\mathfrak{D}}(G) &= \sum_{i=1}^{n} |\xi_i| = 2n - r - 2 + \sum_{i=2}^{n} |-2 - \mu_i| \\
&= 2n - r - 2 - r - 2 + \sum_{i=1}^{n} |-2 - \mu_i| \\
&= 2n - 2r - 4 + \sum_{\mu_i \leq -2} (-2 - \mu_i) + \sum_{\mu_i > -2} (2 + \mu_i) \\
&= 2n - 2r - 4 - 2n_{\mu}([\mu_n, -2]) + \sum_{\mu_i \leq -2} |\mu_i| + 2n_{\mu}((-2, \mu_1]) \\
&\quad + \sum_{\mu_i \in (-2,0)} \mu_i + \sum_{\mu_i \geq 0} \mu_i \,,
\end{aligned}$$

where the number of $\mathcal{A}$-eigenvalues of G in the interval $\mathbf{I}$ is denoted by $n_{\mu}(\mathbf{I})$ and $n_{\mu}([\mu_n, -2]) = 0$ if $\mu_n > -2$. The $\mathcal{A}$-energy of a graph G can be written as

$$\mathcal{E}_{\mathcal{A}}(G) = \sum_{i=1}^{n} |\mu_i| = \sum_{\mu_i \leq -2} |\mu_i| + \sum_{\mu_i \in (-2,0)} |\mu_i| + \sum_{\mu_i \geq 0} \mu_i.$$

With this fact we have

$$\begin{aligned}
\mathcal{E}_{\mathfrak{D}}(G) &= 2n - 2r - 4 - 2n_{\mu}([\mu_n, -2]) + 2n_{\mu}((-2, \mu_1]) + \sum_{\mu_i \in (-2,0)} \mu_i \\
&\quad + \mathcal{E}_{\mathcal{A}}(G) - \sum_{\mu_i \in (-2,0)} |\mu_i| \\
&= 2n - 2r - 4 + \mathcal{E}_{\mathcal{A}}(G) - 2n_{\mu}([\mu_n, -2]) + 2n_{\mu}((-2, \mu_1]) + 2 \sum_{\mu_i \in (-2,0)} \mu_i \\
&= 2n - 2r - 4 + \mathcal{E}_{\mathcal{A}}(G) - 2n_{\mu}([\mu_n, -2]) + 2\Big(n - n_{\mu}([\mu_n, -2])\Big)
\end{aligned}$$

$$+2\sum_{\mu_i\in(-2,0)}\mu_i$$
$$=2\left(2n-r-2-2n_\mu([\mu_n,-2])+\sum_{\mu_i\in(-2,0)}\mu_i\right)+\mathcal{E}_{\mathcal{A}}(G). \quad (1)$$

The order n of G can be expressed as

$$n=n_\mu([\mu_n,-2])+n_\mu((-2,0))+n^0_{\mathcal{A}}+n^+_{\mathcal{A}}.$$

Therefore

$$n_\mu([\mu_n,-2])=n-n^+_{\mathcal{A}}-n^0_{\mathcal{A}}-n_\mu(-2,0)=n^-_{\mathcal{A}}-n_\mu((-2,0)). \quad (2)$$

Also, we have

$$\sum_{\mu_i\in(-2,0)}(\mu_i+2)=\sum_{\mu_i\in(-2,0)}\mu_i+2n_\mu((-2,0)). \quad (3)$$

Now using (2) and (3) in (1), we get

$$\mathcal{E}_{\mathfrak{D}}(G)=2\left(2n-r-2-2n^-_{\mathcal{A}}+\sum_{\mu_i\in(-2,0)}(\mu_i+2)\right)+\mathcal{E}_{\mathcal{A}}(G).$$

□

To each $\mathcal{A}$-eigenvalue $\mu_i\in(-2,0)$, we observe that $0<\mu_i+2<2$, which implies that $0<\sum_{\mu_i\in(-2,0)}(\mu_i+2)<2n^-_{\mathcal{A}}$ for any graph G. Also $\sum_{\mu_i\in(-2,0)}(\mu_i+2)=0$ if and only if a graph G has no $\mathcal{A}$-eigenvalue μ_i in the interval $(-2,0)$. With these facts we have the following.

Corollary 3.2 *Let G be a regular graph of degree r and order n with $diam(G)\le 2$. Then*

$$2\big(2n-r-2-2n^-_{\mathcal{A}}\big)+\mathcal{E}_{\mathcal{A}}(G)\le\mathcal{E}_{\mathfrak{D}}(G)<2\big(2n-r-2\big)+\mathcal{E}_{\mathcal{A}}(G).$$

Equality holds in the left side inequality if and only if G has no $\mathcal{A}$-eigenvalues in the interval $(-2,0)$.

If a regular graph has diameter less than or equal to 2, the following spectral characterization provides an answer to the open problem posed by Indulal [16,17].

Let the $\mathcal{A}$-eigenvalues of the regular graphs G_1 and G_2 be $\mu_1^{'},\mu_2^{'},\ldots,\mu_n^{'}$ and $\mu_1^{''},\mu_2^{''},\ldots,\mu_n^{''}$ respectively and let the number of negative $\mathcal{A}$-eigenvalues of G_1 and G_2 be $n^-_{\mathcal{A}_1}$ and $n^-_{\mathcal{A}_2}$ respectively.

Corollary 3.3 *Let G_1 and G_2 be two regular graphs of degree r_1 and r_2 respectively, both of same order n with diameter less than or equal to 2. If G_1 and G_2 are $\mathcal{A}$-equienergetic graphs, then G_1 and G_2 are $\mathfrak{D}$-equienergetic if and only if*

$$r_1+2n^-_{\mathcal{A}_1}-\sum_{\mu_j^{'}\in(-2,0)}(\mu_j^{'}+2)=r_2+2n^-_{\mathcal{A}_2}-\sum_{\mu_j^{''}\in(-2,0)}(\mu_j^{''}+2).$$

In particular, if G_1 and G_2 have no $\mathcal{A}$-eigenvalues in the interval $(-2,0)$, then G_1 and G_2 are $\mathfrak{D}$-equienergetic if and only if $r_1+2n^-_{\mathcal{A}_1}=r_2+2n^-_{\mathcal{A}_2}$.

Example 3.4 The graphs $K_{n,n}\square K_{n-1}$ and $K_{n-1,n-1}\square K_n$ are of same order $2n(n-1)$ and of same degree $2(n-1)$, where $\square$ denotes the Cartesian product. For all $n \geq 5$ and $k \geq 0$, the graphs $\mathcal{L}^k(K_{n,n}\square K_{n-1})$ and $\mathcal{L}^k(K_{n-1,n-1}\square K_n)$ are non-isomorphic regular $\mathcal{A}$-equienergetic graphs with the same order and same degree [31]. It can be observed that the graphs $K_{n,n}\square K_{n-1}$ and $K_{n-1,n-1}\square K_n$ satisfy the conditions of the Theorem 2.9. Therefore, the graphs $\overline{\mathcal{L}^k(K_{n,n}\square K_{n-1})}$ and $\overline{\mathcal{L}^k(K_{n-1,n-1}\square K_n)}$ are $\mathfrak{D}$-equienergetic for all $n \geq 5$ and $k \geq 2$ by Theorem 2.9. Also, for all $n \geq 4$ and $k \geq 0$, the graphs $\overline{\mathcal{L}^k(K_{n,n}\square K_{n-1})}$ and $\overline{\mathcal{L}^k(K_{n-1,n-1}\square K_n)}$ are non-isomorphic regular $\mathcal{A}$-equienergetic graphs with the same order and the same degree [31]. If $n \geq 6$, it can be observed that the graphs $\overline{\mathcal{L}(K_{n,n}\square K_{n-1})}$ and $\overline{\mathcal{L}(K_{n-1,n-1}\square K_n)}$ have equal number of negative $\mathcal{A}$-eigenvalues which are less than or equal to -2 [31]. Therefore these graphs are $\mathfrak{D}$-equienergetic by Corollary 3.3. Hence the graphs $\overline{\mathcal{L}^k(K_{n,n}\square K_{n-1})}$ and $\overline{\mathcal{L}^k(K_{n-1,n-1}\square K_n)}$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $n \geq 6$ and $k \geq 1$.

Remark 3.5 Consider a regular graph G of degree $r \geq 4$, order n_0 and size m_0. Then, by Theorem 2.11, the graphs $\mathcal{L}^k(G)$ have all negative $\mathcal{A}$-eigenvalues equal to -2 with the multiplicity $m_{k-1} - n_{k-1}$ for all $k \geq 2$. Also, by Theorem 2.6, the $\mathcal{A}$-eigenvalues of $\mathcal{L}^2(G)$ are $4r-6$, $3r+\mu_j-6$; $2 \leq j \leq n$, $2r-6$ $\left(\frac{n(r-2)}{2} \text{ times}\right)$ and -2 $\left(\frac{nr(r-2)}{2} \text{ times}\right)$ in which it is easy to observe that all the $\mathcal{A}$-eigenvalues except -2 are at least 2. In fact, if a regular graph G of degree $r \geq 4$ has an $\mathcal{A}$-eigenvalue $\mu_j \geq 0$, then by Theorem 2.6, its line graph $\mathcal{L}(G)$ has an $\mathcal{A}$-eigenvalue $\mu_j + r - 2 \geq 2$. Therefore, all the $\mathcal{A}$-eigenvalues of $\mathcal{L}^k(G)$ except -2 are at least 2 for all $k \geq 2$. Hence, for all $k \geq 2$, there are $n_k - m_{k-1} + n_{k-1} - 1$ negative $\mathcal{A}$-eigenvalues which are less than or equal to -3, positive $\mathcal{A}$-eigenvalues 1 with multiplicity $m_{k-1} - n_{k-1}$ and the spectral radius in $\overline{\mathcal{L}^k(G)}$, by Theorem 2.6. With these facts we can prove Theorem 2.9 easily by taking two regular graphs of the same degree and the same order.

The following theorem provides a spectral characterization to the graphs for which $\mathcal{A}$-energy is equal to $\mathfrak{D}$-energy.

Theorem 3.6 *Let G be a regular graph of degree r and order n with the $\mathcal{A}$-eigenvalues $r = \mu_1, \mu_2, \ldots, \mu_n$ and $diam(G) \leq 2$. Then $\mathcal{E}_{\mathfrak{D}}(G) = \mathcal{E}_{\mathcal{A}}(G)$ if and only if*

$$2n + \sum_{\mu_i \in (-2,0)} (\mu_i + 2) = r + 2 + 2n^-_{\mathcal{A}}.$$

In particular, if G has no $\mathcal{A}$-eigenvalue in the interval $(-2, 0)$, then $\mathcal{E}_{\mathfrak{D}}(G) = \mathcal{E}_{\mathcal{A}}(G)$ if and only if $2n = r + 2 + 2n^-_{\mathcal{A}}$.

In the following result, we present $\mathfrak{D}$-energy of graphs without any restriction on the diameter. Let $n^-_{\mathcal{A}_i}$ denotes the number of negative $\mathcal{A}$-eigenvalues of the graph H_i. Let $\mathcal{E}(M) = \sum_{i=1}^{n} |\sigma_i|$, where σ_i; $1 \leq i \leq n$, are the eigenvalues of the matrix M.

Theorem 3.7 *Let G be a graph of order n with vertices labeled as $x_1, x_2, \ldots, x_n$. Let H_i be regular graphs of degree r_i and order $\mathfrak{n}_i$ with the $\mathcal{A}$-eigenvalues $r_i = \mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,\mathfrak{n}_i}$; $1 \leq i \leq n$. Then the $\mathfrak{D}$-energy of $G[H_1, H_2, \ldots, H_n]$ is $\mathcal{E}_{\mathfrak{D}}(G[H_1, H_2, \ldots, H_n])$*

$$= \sum_{i=1}^{n} \left(2\mathfrak{n}_i - r_i - 2 - 4n^-_{\mathcal{A}_i} + 2 \sum_{\mu_{i,j} \in (-2,0)} (\mu_{i,j} + 2) + \mathcal{E}_{\mathcal{A}}(H_i) \right) + \mathcal{E}(M),$$

where M is a matrix given in Theorem 2.12.

Proof Let $r_i = \mu_{i,1} \geq \mu_{i,2} \geq \cdots \geq \mu_{i,\mathfrak{n}_i}$ be the $\mathcal{A}$-eigenvalues of H_i; $1 \leq i \leq n$. Then by definition of $\mathfrak{D}$-energy and Theorem 2.12, we have

$$\mathcal{E}_{\mathfrak{D}}\left(G[H_1, H_2, \ldots, H_n]\right) = \sum_{i=1}^{n} \sum_{j=2}^{\mathfrak{n}_i} |-2 - \mu_{i,j}| + \mathcal{E}(M)$$

$$= \sum_{i=1}^{n} \left(-2 - r_i + \sum_{j=1}^{\mathfrak{n}_i} |-2 - \mu_{i,j}| \right) + \mathcal{E}(M).$$

Now, proceeding similar to that of the proof of Theorem 3.1, we get the result. □

Again, it can be observed that to each $\mathcal{A}$-eigenvalue $\mu_{i,j} \in (-2, 0)$; $1 \leq i \leq n$ and $1 \leq j \leq \mathfrak{n}_i$, we have $0 < \mu_{i,j} + 2 < 2$ which implies that $0 < \sum_{\mu_{i,j} \in (-2,0)} (\mu_{i,j} + 2) < 2n^-_{\mathcal{A}_i}$ for any graph H_i. With this fact we have the following.

Corollary 3.8 *Let G be a graph of order n with vertices labeled as $x_1, x_2, \ldots, x_n$. Let H_i be regular graphs of degree r_i and order $\mathfrak{n}_i$ with the $\mathcal{A}$-eigenvalues $r_i = \mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,\mathfrak{n}_i}$; $1 \leq i \leq n$. Then*

$$\sum_{i=1}^{n} \left(2\mathfrak{n}_i - r_i - 2 - 4n^-_{\mathcal{A}_i} + \mathcal{E}_{\mathcal{A}}(H_i) \right) + \mathcal{E}(M) \leq \mathcal{E}_{\mathfrak{D}}(G[H_1, H_2, \ldots, H_n])$$
$$< \sum_{i=1}^{n} (2\mathfrak{n}_i - r_i - 2 + \mathcal{E}_{\mathcal{A}}(H_i)) + \mathcal{E}(M).$$

Equality holds in the left side of the inequality if and only if each H_i; $1 \leq i \leq n$ has no $\mathcal{A}$-eigenvalues in the interval $(-2, 0)$.

There are many non-isomorphic regular $\mathcal{A}$-equienergetic graphs with the same degree and same order, see [13,25,27,31,34]. Also, Theorem 2.11 and Remark 3.5 ensure that there are many graphs with exact number of negative $\mathcal{A}$-eigenvalues which do not lie in the interval $(-2, 0)$. It is now easy to construct $\mathfrak{D}$-equienergetic graphs by using such graphs. Let the graph $G[H_1, H_2, \ldots, H_{e_k}, \ldots, H_n]$ be obtained from the graph $G[H_1, H_2, \ldots, H_k, \ldots, H_n]$ by replacing the graph H_k; $1 \leq k \leq n$, with the graph H_{e_k}. Let the $\mathcal{A}$-eigenvalues of the regular graphs H_{e_1} and H_{e_2} be $\mu'_{e1}, \mu'_{e2}, \ldots, \mu'_{en}$ and $\mu''_{e1}, \mu''_{e2}, \ldots, \mu''_{en}$ and let the number of negative $\mathcal{A}$-eigenvalues of H_{e_1} and H_{e_2} be $n^-_{\mathcal{A}_{e1}}$ and $n^-_{\mathcal{A}_{e2}}$ respectively.

Corollary 3.9 *Let G be a graph of order n with labeled vertices $x_1, x_2, \ldots, x_n$. Let H_i be regular graphs of degree r_i and order $\mathfrak{n}_i$ with the $\mathcal{A}$-eigenvalues $r_i = \mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,\mathfrak{n}_i}$; $1 \leq i \leq n$. Let H_{e_k}; $k = 1, 2$ be two regular $\mathcal{A}$-equienergetic graphs of same order and same degree. Then the graphs $G[H_1, H_2, \ldots, H_{e_k}, \ldots, H_n]$, $k = 1, 2$ are $\mathfrak{D}$-equienergetic if and only if*

$$2n^-_{\mathcal{A}_{e1}} - \sum_{\mu'_{ej} \in (-2,0)} (\mu'_{ej} + 2) = 2n^-_{\mathcal{A}_{e2}} - \sum_{\mu''_{ej} \in (-2,0)} (\mu''_{ej} + 2).$$

In particular, if H_{e_k}; $k = 1, 2$ have no $\mathcal{A}$-eigenvalues in the interval $(-2, 0)$, then the graphs $G[H_1, H_2, \ldots, H_{e_k}, \ldots, H_n]$; $k = 1, 2$ are $\mathfrak{D}$-equienergetic if and only if $n^-_{\mathcal{A}_{e1}} = n^-_{\mathcal{A}_{e2}}$.

Proof The proof follows directly from Theorem 3.7 by taking two regular $\mathcal{A}$-equienergetic graphs $H_{e_k}, k = 1, 2$ of the same order and the same degree and the fact that $G[H_1, H_2, \ldots, H_{e_k}, \ldots, H_n]$, $k = 1, 2$ contributes the same auxiliary matrix M. □

Remark 3.10 Ramane et al. [31] have shown that the graphs $G[H_1, H_2, \ldots, H_{e_k}, \ldots, H_n]$; $k = 1, 2$ are $\mathcal{A}$-equienergetic if the graphs H_{e_k}; $k = 1, 2$ are $\mathcal{A}$-equienergetic by using the concept of equitable partitions. Therefore the graphs $G[H_1, H_2, \ldots, H_{e_k}, \ldots, H_n]$; $k = 1, 2$, are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic if the conditions in the Corollary 3.9 are satisfied.

In the following theorem, we obtain $\mathfrak{D}$-energy of $Ebd(G)$, where G is a regular graph with $diam(G) = 2$.

Theorem 3.11 *Let G be a regular graph of degree r and order n with the $\mathcal{A}$-eigenvalues $r = \mu_1, \mu_2, \ldots, \mu_n$ and $diam(G) = 2$. Then*

$$\mathcal{E}_{\mathfrak{D}}(Ebd(G)) = 9n - 6r - 8 + |2r - n| + 4\mathcal{E}_{\mathcal{A}}(G) - 8n^-_{\mathcal{A}} + 4 \sum_{\mu_i \in (-2,0)} (\mu_i + 2).$$

Proof Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the $\mathcal{A}$-eigenvalues of G. Then by definition of $\mathfrak{D}$-energy and Theorem 2.14, we have

$$\begin{aligned}\mathcal{E}_{\mathfrak{D}}(Ebd(G)) &= 5n - 2r - 4 + |2r - n| + 2\sum_{i=2}^{n}|-2 - \mu_i| + 2\sum_{i=2}^{n}|\mu_i| \\ &= 5n - 2r - 4 + |2r - n| - 2(r + 2) + 2\sum_{i=1}^{n}|-2 - \mu_i| - 2r + 2\sum_{i=1}^{n}|\mu_i| \\ &= 5n - 6r - 8 + |2r - n| + 2\mathcal{E}_{\mathcal{A}}(G) + 2\sum_{i=1}^{n}|-2 - \mu_i|.\end{aligned}$$

Now, proceeding similar to that of the proof of Theorem 3.1, we get the result. □

Similar to the Corollaries 3.2 and 3.3, we have following two corollaries.

Corollary 3.12 *Let G be a regular graph of degree r and order n with $diam(G) = 2$. Then*

$$9n - 6r - 8 + |2r - n| + 4\mathcal{E}_{\mathcal{A}}(G) - 8n^{-}_{\mathcal{A}} \leq \mathcal{E}_{\mathfrak{D}}(Ebd(G)) < 9n - 6r - 8 + |2r - n| + 4\mathcal{E}_{\mathcal{A}}(G).$$

Equality holds in the left side of the inequality if and only if G has no $\mathcal{A}$-eigenvalues in the interval $(-2, 0)$.

Indulal and Gutman [19] reported some particular class of bipartite $\mathfrak{D}$-equienergetic graphs. The following provides a general way to construct bipartite $\mathfrak{D}$-equienergetic graphs.

Corollary 3.13 *Let G_1 and G_2 be two regular graphs, both of degree r and order n with diameter 2. If G_1 and G_2 are $\mathcal{A}$-equienergetic graphs, then $Ebd(G_1)$ and $Ebd(G_2)$ are bipartite $\mathfrak{D}$-equienergetic graphs if and only if*

$$2n^{-}_{\mathcal{A}_1} - \sum_{\mu_j^{'} \in (-2,0)} (\mu_j^{'} + 2) = 2n^{-}_{\mathcal{A}_2} - \sum_{\mu_j^{''} \in (-2,0)} (\mu_j^{''} + 2).$$

In particular, if G_1 and G_2 have no $\mathcal{A}$-eigenvalues in the interval $(-2, 0)$, then $Ebd(G_1)$ and $Ebd(G_2)$ are bipartite $\mathfrak{D}$-equienergetic graphs if and only if $n^{-}_{\mathcal{A}_1} = n^{-}_{\mathcal{A}_2}$.

Remark 3.14 By Proposition 2.13 the graphs $Ebd(G_1)$ and $Ebd(G_2)$ are bipartite $\mathcal{A}$-equienergetic if and only if $n^{-}_{\mathcal{A}_1} = n^{-}_{\mathcal{A}_2}$, whenever the graphs G_1 and G_2 have no $\mathcal{A}$-eigenvalues in the interval $(-1, 0)$. Hence the graphs $Ebd(G_1)$ and $Ebd(G_2)$ are bipartite $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic if the graphs G_1 and G_2 in the Corollary 3.13 have no $\mathcal{A}$-eigenvalues in the interval $(-2, 0)$.

Proposition 3.15 *Let G_1 and G_2 be two regular graphs, both of degree r and order n with $2r \leq n - 1$; $r \geq 4$. Then the graphs $Ebd\left(\overline{\mathcal{L}^k(G_1)}\right)$ and $Ebd\left(\overline{\mathcal{L}^k(G_2)}\right)$ are bipartite $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $k \geq 2$.*

Proof If G_1 and G_2 are two regular graphs, both of degree r and order n with $2r \leq n - 1$, then by Theorem 2.8, $diam(\overline{\mathcal{L}^k(G_1)}) = diam(\overline{\mathcal{L}^k(G_2)}) = 2$. Also if $r \geq 4$, then by Theorem 2.10 and Remark 3.5, the graphs $\overline{\mathcal{L}^k(G_1)}$ and $\overline{\mathcal{L}^k(G_2)}$ are $\mathcal{A}$-equienergetic and have equal number of negative $\mathcal{A}$-eigenvalues with no $\mathcal{A}$-eigenvalue in the interval $(-2, 0)$ for all $k \geq 2$. Therefore, by Remark 3.14, the graphs $Ebd\left(\overline{\mathcal{L}^k(G_1)}\right)$ and $Ebd\left(\overline{\mathcal{L}^k(G_2)}\right)$ are bipartite $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $k \geq 2$. □

Example 3.16 Consider the graphs $\mathcal{L}^k(K_{n,n}\square K_{n-1})$ and $\mathcal{L}^k(K_{n-1,n-1}\square K_n)$ as given in the Example 3.4. Now following similar arguments as in the proof of Proposition 3.15, we get that the graphs $Ebd\left(\overline{\mathcal{L}^k(K_{n,n}\square K_{n-1})}\right)$ and $Ebd\left(\overline{\mathcal{L}^k(K_{n-1,n-1}\square K_n)}\right)$ are bipartite $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $n \geq 6$ and $k \geq 1$.

Now, we obtain the $\mathcal{A}$-energy of $G \boxtimes K_p$. Let $\beta_p = 1 - \frac{1}{p}$; $p \geq 2$.

Theorem 3.17 *Let G be a graph of order n with the $\mathcal{A}$-eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. Then for $p \geq 2$,*

$$\mathcal{E}_{\mathcal{A}}(G \boxtimes K_p) = p\left(2n\beta_p + \mathcal{E}_{\mathcal{A}}(G) - 2\beta_p n_{\mathcal{A}}^- + 2\sum_{\mu_i \in (-\beta_p, 0)} (\mu_i + \beta_p)\right).$$

Proof Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the $\mathcal{A}$-eigenvalues of G. The $\mathcal{A}$-eigenvalues of the complete graph K_p are $p-1$ and -1 ($p-1$ times). Therefore, by Theorem 2.16, the $\mathcal{A}$-eigenvalues of $G \boxtimes K_p$, $p \geq 2$ are $p\mu_i + p - 1$, $1 \leq i \leq n$ and -1 ($n(p-1)$ times). By definition of $\mathcal{A}$-energy of a graph, we have

$$\begin{aligned}\mathcal{E}_{\mathcal{A}}(G \boxtimes K_p) &= n(p-1) + \sum_{i=1}^{n} |p\mu_i + p - 1| \\ &= pn\beta_p + p\sum_{i=1}^{n} |\mu_i + \beta_p|.\end{aligned}$$

Now, proceeding similar to that of the proof of Theorem 3.1, we get the result. □

Note that $(-\beta_p, 0) \subseteq (-1, 0)$ for $p \geq 2$. To each $\mathcal{A}$-eigenvalue $\mu_i \in (-\beta_p, 0)$, we have $0 < \mu_i + \beta_p < \beta_p$, which leads to $0 < \sum_{\mu_i \in (-\beta_p, 0)} (\mu_i + \beta_p) < \beta_p n_{\mathcal{A}}^-$ for any graph G. With this fact we have the following.

Corollary 3.18 *Let G be a graph of order n. Then for $p \geq 2$,*

$$2(p-1)(n - n_{\mathcal{A}}^-) + p\mathcal{E}_{\mathcal{A}}(G) \leq \mathcal{E}_{\mathcal{A}}(G \boxtimes K_p) < 2n(p-1) + p\mathcal{E}_{\mathcal{A}}(G).$$

Equality holds in the left side of the inequality if and only if G has no $\mathcal{A}$-eigenvalues in the interval $(-\beta_p, 0)$.

There are many graphs with no $\mathcal{A}$-eigenvalues in the interval $(-1, 0)$ such as iterated line graphs $\mathcal{L}^k(G)$, for each $k \geq 2$ as in Theorem 2.11, cographs [22], integral graphs, strong double graphs [29] etc. Now, we construct the $\mathcal{A}$-equienergetic graphs by using Theorem 3.17.

Corollary 3.19 *Let G_1 and G_2 be $\mathcal{A}$-equienergetic graphs of the same order n. Then, for $p \geq 2$, the graphs $G_1 \boxtimes K_p$ and $G_2 \boxtimes K_p$ are $\mathcal{A}$-equienergetic if and only if*

$$\beta_p n_{\mathcal{A}_1}^- - \sum_{\mu_j' \in (-\beta_p, 0)} (\mu_j' + \beta_p) = \beta_p n_{\mathcal{A}_2}^- - \sum_{\mu_j'' \in (-\beta_p, 0)} (\mu_j'' + \beta_p).$$

In particular, if G_1 and G_2 have no $\mathcal{A}$-eigenvalues in the interval $(-\beta_p, 0)$, then for $p \geq 2$, the graphs $G_1 \boxtimes K_p$ and $G_2 \boxtimes K_p$ are $\mathcal{A}$-equienergetic if and only if $n_{\mathcal{A}_1}^- = n_{\mathcal{A}_2}^-$.

Proof The proof follows directly from Theorem 3.17 by taking two $\mathcal{A}$-equienergetic graphs of same order and the fact that $\sum_{\mu \in (-\beta_p, 0)} (\mu + \beta_p) = 0$ if and only if G has no $\mathcal{A}$-eigenvalue μ in the interval $(-\beta_p, 0)$. □

Example 3.20 The graphs $\mathcal{L}^k(K_{n,n} \square K_{n-1})$ and $\mathcal{L}^k(K_{n-1,n-1} \square K_n)$ are integral $\mathcal{A}$-equienergetic graphs with the same number of negative $\mathcal{A}$-eigenvalues for all $n \geq 5, k \geq 0$ [31]. Therefore by Corollary 3.19, the graphs $\mathcal{L}^k(K_{n,n} \square K_{n-1}) \boxtimes K_p$ and $\mathcal{L}^k(K_{n-1,n-1} \square K_n) \boxtimes K_p$ are $\mathcal{A}$-equienergetic for all $n \geq 5, k \geq 0$ and $p \geq 2$.

In the next theorem, we find the $\mathfrak{D}$-energy of $G \boxtimes K_p$, where G is any graph.

Theorem 3.21 *Let G be a graph of order n with the $\mathfrak{D}$-eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$. Then for $p \geq 2$,*

$$\mathcal{E}_{\mathfrak{D}}(G \boxtimes K_p) = p\left(2n\beta_p + \mathcal{E}_{\mathfrak{D}}(G) - 2\beta_p n_{\mathfrak{D}}^- + 2\sum_{\xi_i \in (-\beta_p, 0)} (\xi_i + \beta_p)\right).$$

Proof Let $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ be the $\mathfrak{D}$-eigenvalues of G. Then, by the definition of $\mathfrak{D}$-energy of a graph and Theorem 2.17, we have

$$\begin{aligned}\mathcal{E}_{\mathfrak{D}}(G \boxtimes K_p) &= n(p-1) + \sum_{i=1}^{n} |p\xi_i + p - 1| \\ &= pn\beta_p + p\sum_{i=1}^{n} |\xi_i + \beta_p|.\end{aligned}$$

Now, proceeding similar to that of proof of Theorem 3.1, we get the result. □

To each $\mathfrak{D}$-eigenvalue $\xi_i \in (-\beta_p, 0)$, we have $0 < \xi_i + \beta_p < \beta_p$, which leads to $0 < \sum_{\xi_i \in (-\beta_p, 0)} (\xi_i + \beta_p) < \beta_p n^-_{\mathfrak{D}}$ for any graph G. With this fact, we have the following.

Corollary 3.22 *Let G be a graph of order n. Then for $p \geq 2$,*

$$2(p-1)(n - n^-_{\mathfrak{D}}) + p\mathcal{E}_{\mathfrak{D}}(G) \leq \mathcal{E}_{\mathfrak{D}}(G \boxtimes K_p) < 2n(p-1) + p\mathcal{E}_{\mathfrak{D}}(G).$$

Equality holds in the left side of the inequality if and only if G has no $\mathfrak{D}$-eigenvalues in the interval $(-\beta_p, 0)$.

Next, we construct the $\mathfrak{D}$-equienergetic graphs by using Theorem 3.21. Let the $\mathfrak{D}$-eigenvalues of the graphs G_1 and G_2 be $\xi_1^{'}, \xi_2^{'}, \ldots, \xi_n^{'}$ and $\xi_1^{''}, \xi_2^{''}, \ldots, \xi_n^{''}$ respectively and let the number of negative $\mathfrak{D}$-eigenvalues of G_1 and G_2 be $n^-_{\mathfrak{D}_1}$ and $n^-_{\mathfrak{D}_2}$ respectively.

Corollary 3.23 *Let G_1 and G_2 be the $\mathfrak{D}$-equienergetic graphs of the same order n. Then, for $p \geq 2$, the graphs $G_1 \boxtimes K_p$ and $G_2 \boxtimes K_p$ are $\mathfrak{D}$-equienergetic if and only if*

$$\beta_p n^-_{\mathfrak{D}_1} - \sum_{\xi_j^{'} \in (-\beta_p, 0)} (\xi_j^{'} + \beta_p) = \beta_p n^-_{\mathfrak{D}_2} - \sum_{\xi_j^{''} \in (-\beta_p, 0)} (\xi_j^{''} + \beta_p).$$

In particular, if G_1 and G_2 have no $\mathfrak{D}$-eigenvalues in the interval $(-\beta_p, 0)$, then for $p \geq 2$, the graphs $G_1 \boxtimes K_p$ and $G_2 \boxtimes K_p$ are $\mathfrak{D}$-equienergetic if and only if $n^-_{\mathfrak{D}_1} = n^-_{\mathfrak{D}_2}$.

Proof The proof follows directly from Theorem 3.21 by taking two $\mathfrak{D}$-equienergetic graphs of the same order and the fact that $\sum_{\xi \in (-\beta_p, 0)} (\xi + \beta_p) = 0$ if and only if G has no $\mathfrak{D}$-eigenvalue ξ in the interval $(-\beta_p, 0)$. □

Remark 3.24 If G_1 and G_2 are two $\mathcal{A}$-equienergetic and $\mathfrak{D}$-equienergetic graphs with the same number of the negative $\mathcal{A}$-eigenvalues and $\mathfrak{D}$-eigenvalues and no $\mathcal{A}$-eigenvalues and $\mathfrak{D}$-eigenvalues in the interval $(-\beta_p, 0)$, then for $p \geq 2$, the graphs $G_1 \boxtimes K_p$ and $G_2 \boxtimes K_p$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic by Corollaries 3.19 and 3.23.

Example 3.25 The graphs $\overline{\mathcal{L}^k(K_{n,n} \square K_{n-1})}$ and $\overline{\mathcal{L}^k(K_{n-1,n-1} \square K_n)}$ in the Example 3.4 are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $n \geq 6$ and $k \geq 1$. For all $n \geq 6$ and $k \geq 1$, by Remark 3.5, the graphs $\overline{\mathcal{L}^k(K_{n,n} \square K_{n-1})}$ and $\overline{\mathcal{L}^k(K_{n-1,n-1} \square K_n)}$ have equal number of the negative $\mathcal{A}$-eigenvalues which are less than are equal to -2 and also have equal number of the negative $\mathfrak{D}$-eigenvalues which are equal to -3. Therefore, the graphs $\overline{\mathcal{L}^k(K_{n,n} \square K_{n-1})} \boxtimes K_p$ and $\overline{\mathcal{L}^k(K_{n-1,n-1} \square K_n)} \boxtimes K_p$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $n \geq 6$, $k \geq 1$ and $p \geq 2$.

Proposition 3.26 *Let G_1 and G_2 be two regular graphs both of the same degree $r \geq 4$ and the same order n. If $2r \leq n - 1$, then the graphs $\overline{\mathcal{L}^k(G_1)} \boxtimes K_p$ and $\overline{\mathcal{L}^k(G_2)} \boxtimes K_p$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $k \geq 2$ and $p \geq 2$.*

Proof If G a regular graph of degree $r \geq 4$ and order n with $2r \leq n - 1$, then by Theorem 2.8, the diameter of the graphs $\overline{\mathcal{L}^k(G)}$ is equal to 2. Also, by Remark 3.5, the graphs $\overline{\mathcal{L}^k(G)}$ have the same number of negative $\mathcal{A}$-eigenvalues which are less than or equal to -3 and negative $\mathfrak{D}$-eigenvalues which are equal to -3. With these facts the proof follows from Theorems 2.10 and 2.9 and Corollaries 3.19 and 3.23 by taking two regular graphs G_1 and G_2 both of same degree r and same order n. □

In the following theorem, the $\mathfrak{D}$-energy of the double graph $D_2(G)$ is given, where G is any graph.

Theorem 3.27 *Let G be a graph of order n with $\mathfrak{D}$-eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$. Then*

$$\mathcal{E}_{\mathfrak{D}}(D_2(G)) = 2\Big(2n + \mathcal{E}_{\mathfrak{D}}(G) - 2n^-_{\mathfrak{D}} + 2\sum_{\xi_j \in (-1,0)} (\xi_j + 1)\Big).$$

Proof Let $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ be the $\mathfrak{D}$-eigenvalues of G. By the definition of $\mathfrak{D}$-energy of a graph and Theorem 2.15, we have

$$\mathcal{E}_{\mathfrak{D}}(D_2(G)) = 2n + 2\sum_{j=1}^{n} |\xi_j + 1|.$$

Now, proceeding similar to that of the proof of Theorem 3.1, we get the result. □

The following results are similar to that of the Corollaries 3.22 and 3.23, Remark 3.24, Example 3.25 and Proposition 3.26.

Corollary 3.28 *Let G be a graph of order n. Then*

$$2\big(2n + \mathcal{E}_{\mathfrak{D}}(G) - 2n^-_{\mathfrak{D}}\big) \leq \mathcal{E}_{\mathfrak{D}}(D_2(G)) < 2\big(2n + \mathcal{E}_{\mathfrak{D}}(G)\big).$$

Equality holds in the left side of the inequality if and only if G has no $\mathfrak{D}$-eigenvalues in the interval $(-1, 0)$.

Corollary 3.29 *Let G_1 and G_2 be $\mathfrak{D}$-equienergetic graphs of the same order n. Then the graphs $D_2(G_1)$ and $D_2(G_2)$ are $\mathfrak{D}$-equienergetic if and only if*

$$n^-_{\mathfrak{D}_1} - \sum_{\xi_j^{'} \in (-1,0)} (\xi_j^{'} + 1) = n^-_{\mathfrak{D}_2} - \sum_{\xi_j^{''} \in (-1,0)} (\xi_j^{''} + 1).$$

In particular, if G_1 and G_2 have no $\mathfrak{D}$-eigenvalues in the interval $(-1, 0)$, then the graphs $D_2(G_1)$ and $D_2(G_2)$ are $\mathfrak{D}$-equienergetic if and only if $n^-_{\mathfrak{D}_1} = n^-_{\mathfrak{D}_2}$.

Remark 3.30 If G_1 and G_2 are two $\mathcal{A}$-equienergetic graphs of the same order, then the graphs $D_2(G_1)$ and $D_2(G_2)$ are $\mathcal{A}$-equienergetic by Proposition 2.18. Therefore, if G_1 and G_2 are two $\mathcal{A}$-equienergetic and $\mathfrak{D}$-equienergetic graphs with the same number of negative $\mathfrak{D}$-eigenvalues and no $\mathfrak{D}$-eigenvalues in the interval $(-1, 0)$, then the graphs $D_2(G_1)$ and $D_2(G_2)$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic.

Example 3.31 For all $n \geq 6$ and $k \geq 1$, the graphs $D_2\left(\overline{\mathcal{L}^k(K_{n,n} \square K_{n-1})}\right)$ and $D_2\left(\overline{\mathcal{L}^k(K_{n-1,n-1} \square K_n)}\right)$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic.

Proposition 3.32 *Let G_1 and G_2 be two regular graphs both of the same degree $r \geq 4$ and the same order n. If $2r \leq n - 1$, then the graphs $D_2(\overline{\mathcal{L}^k(G_1)})$ and $D_2(\overline{\mathcal{L}^k(G_2)})$ are $\mathcal{A}$-equienergetic as well as $\mathfrak{D}$-equienergetic for all $k \geq 2$.*

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