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ORIGINAL RESEARCH

L^2_α-solutions for nonlinear Schrödinger equations associated with the Weinstein operator

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Abstract In this article, we study the Schrödinger equation associated with the Weinstein operator (SW). We prove that the problem (SW) is locally well-posed in $L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ of the subcritical case $p \in \left(0, \frac{4}{d+2\alpha+2}\right]$ and we show that the equation (SW) is globally well-posed in the critical case $p = \frac{4}{d+2\alpha+2}$.

Keywords Schrödinger-Weinstein equations · Weinstein transform · Strichartz-type Schrödinger-Weinstein estimates · Dispersion phenomena

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JEL classification D8 · H51

1 Introduction

This work is devoted to define and study the Schrödinger equations associated with the Weinstein operator (SW) in $\mathbb{R}^{d+1}_+$:

$$\begin{cases} \partial_t u(t,x) - i\,\Delta_W^{\alpha,d} u(t,x) = F(u(t,x)), \quad (t,x) \in I \times \mathbb{R}^{d+1}_+ \\ u_{|t=0} \qquad\qquad\qquad\qquad = g \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right), \end{cases} \tag{SW}$$

where $d \geq 2$, u a is complex-valued function defined on $I \times \mathbb{R}^{d+1}_+$, $F \in C(\mathbb{C}, \mathbb{C})$ satisfying

$$F(0) = 0, \quad |F(u) - F(v)| \leq C\left(|u|^p + |v|^p\right)|u - v|, \quad p > 0 \tag{1.1}$$

and $\Delta_W^{\alpha,d}$ is the Weinstein differential operator on $\mathbb{R}^{d+1}_+$ introduced by Weinstein in [4]. Over the last few years, much attention has been paid to this operator in various mathematical and physical directions. The harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem ([9,11]). In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. This transform is called the Weinstein transform. Several authors have been interested in functional

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spaces related to this operator, for example in [3] the authors introduced the Sobolev space associated with the Weinstein operator $H^{s,\alpha}_{\mathcal{S}_*}\left(\mathbb{R}^{d+1}_+\right)$ and investigated their properties, and in [8] the the authors established refined Sobolev inequality between the homogeneous Weinstein-Besov spaces $\dot{\mathcal{B}}^{s,j\,\beta}_{p,q}\left(\mathbb{R}^{d+1}_+\right)$ and many spaces, as the homogenous Weinstein-Riesz spaces $\mathcal{R}^{-s}_\beta\left(L^p_\beta\left(\mathbb{R}^{d+1}_+\right)\right)$ and the generalized Lorentz spaces. B. Youssef [14] studied the Navier-Stokes equation associated with the Weinstein operator.

In this paper we study a general theory for the (SW) equation focusing on the following problems:

(I) The dispersive phenomena and the Strichartz estimates for this equation.
(II) Local well-posedness of the nonlinear Cauchy problem (SW) in $L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ in the subcritical case $p \in \left(0, \frac{4}{d+2\alpha+2}\right]$.
(III) Global well-posedness of the nonlinear Cauchy problem (SW) in $L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ in the critical case $p = \frac{4}{d+2\alpha+2}$.

Therefore, the aim of this work is to demonstrate the following results.

Theorem 1.1 (Strichartz-type Schrödinger-Weinstein estimates). *Suppose that $d \geq 1$, and that (q, r) and (q_1, r_1) are $\frac{d+2\alpha+2}{2}$-admissible pairs. If u is a solution of the problem*

$$\begin{cases} \partial_t u - i\Delta^{\alpha,d}_W u = F(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^{d+1}_+ \\ u_{|t=0} = g \end{cases} \tag{1.2}$$

for some data, g, F and time $0 < T < \infty$, then

$$\begin{aligned} &\|u\|_{L^q\left([0,T];L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + \|u\|_{C\left([0,T];L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \\ &\leq C\left(\|g\|_{\alpha,2} + \|F\|_{L^{q_1'}\left([0,T];L^{r_1'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\right) \end{aligned}$$

Conversely, if the above estimate holds for all g, F, T, then (q, r) and (q_1, r_1) must be $\frac{d+2\alpha+2}{2}$-admissible.

Theorem 1.2 *If $p \in \left(0, \frac{4}{d+2\alpha+2}\right]$, then for every $g \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$, there exist $T^*, T_* \in (0, \infty]$ and a unique maximal solution u of* (SW) *belonging to*

$$C\left(\left(-T_*, T^*\right); L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) \cap L^q_{loc}\left(\left(-T_*, T^*\right); L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$$

for every $\frac{d+2\alpha+2}{2}$-admissible pair (q, r).

Theorem 1.3 *If $p = \frac{4}{d+2\alpha+2}$, there exists $\delta > 0$ such that if $g \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ satisfying $\|g\|_{\alpha,2} \leq \delta$, (i.e. $\|g\|_{\alpha,2}$ small enough), then the corresponding maximal L^2_α-solution is global, i.e., $T^* = T_* = \infty$. Moreover, u belongs to $L^{p+2}_{loc}\left(\mathbb{R}; L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$.*

This paper is organized as follows : In section 2, we recall some elements of harmonic analysis associated with the Weinstein operator, which will be needed in the sequel. Finally, Section 3 is deroted to establish the Strichartz estimates for the Weinsein-Schrödinger equation, and to introduce a class of nonlinear Schrödinger equations associated with the Weinstein operator. In this regard, we study local and global well-posedness and scattering theory associated with these equations.

2 Harmonic analysis associated with the Weinstein operator

Notations. In what follows, we need the following notations:

- $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times]0, \infty[$.
- $x = (x_1, ..., x_d, x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}_+^{d+1}$
- $-x = (-x', x_{d+1})$
- $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_{d+1}^2}$.
- $\mathscr{C}_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{C}_{*,c}(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$ with compact support, even with respect to the last variable.
- $\mathscr{C}_*^p(\mathbb{R}^{d+1})$, the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact supports, even with respect to the last variable.
- $\mathcal{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.
- $L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\|f\|_{\alpha,p} = \left[\int_{\mathbb{R}_+^{d+1}} |f(x)|^p d\mu_{\alpha,d}(x)\right]^{\frac{1}{p}} < \infty, \text{ if } 1 \le p < \infty,$$

$$\|f\|_{\alpha,\infty} = \operatorname{ess}\sup_{x \in \mathbb{R}_+^{d+1}} |f(x)| < \infty,$$

  where $\mu_{\alpha,d}$ is the measure defined on $\mathbb{R}_+^{d+1}$ by

$$d\mu_{\alpha,d}(x) = \frac{x_{d+1}^{2\alpha+1}}{(2\pi)^{\frac{d}{2}}\, 2^\alpha \Gamma(\alpha+1)} dx, \tag{2.1}$$

  and dx the Lebesgue measure on $\mathbb{R}^{d+1}$.
- $\langle .\,, . \rangle_\alpha$, the scalar product on the Hilbert space $L_\alpha^2(\mathbb{R}_+^{d+1})$, given by

$$\langle f, g \rangle_\alpha = \int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)} d\mu_{\alpha,d}(x).$$

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Weinstein operator developed in [10].

The Weinstein operator $\Delta_W^{\alpha,d}$ is defined on $\mathbb{R}_+^{d+1}$, by:

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \quad \alpha > -\frac{1}{2},$$

where Δ_d is the Laplacian for the d first variables and L_α is the Bessel operator for the last variable defined on $]0, +\infty[$ by :

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}} = \frac{1}{x_{d+1}^{2\alpha+1}} \frac{\partial}{\partial x_{d+1}} \left[x_{d+1}^{2\alpha+1} \frac{\partial u}{\partial x_{d+1}}\right].$$

The Weinstein operator $\Delta_W^{\alpha,d}$, mostly referred to as the Weinstein differential operator is now known as an important operator in analysis. The relevant harmonic analysis associated with the Bessel differential operatorL_α goes back to S. Bochner, J. Delsarte, B.M. Levitan and has been studied by many other authors such as J. Löfström and J. peetre [7], I. Kipriyanov [6], K. Trimèche [12], I.A. Aliev and B. Youssef [13].

Let us begin by the following result, which gives the eigenfunction $\Psi_d^\alpha(\lambda, .)$ of the Weinstein operator $\Delta_W^{\alpha,d}$.

Proposition 2.1 [6] *For all* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{d+1}) \in \mathbb{C}^{d+1}$, *the system*

$$\begin{cases} \frac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), & if \quad 1 \le j \le d, \\ L_\alpha u(x) = -\lambda_{d+1}^2 u(x), \\ u(0) = 1, \ \frac{\partial u}{\partial x_{d+1}}(0) = 0, \\ \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, & if \quad 1 \le j \le d \end{cases}$$

has a unique solution $\Psi_d^\alpha(\lambda, .)$ *given by :*

$$\forall z \in \mathbb{C}^{d+1}, \ \Psi_d^\alpha(\lambda, z) = e^{-i\langle z', \lambda' \rangle} j_\alpha(\lambda_{d+1} z_{d+1}),$$

where $z = (z', x_{d+1})$, $z' = (z_1, z_2, ..., z_d)$ *and* j_α *is the normalized Bessel function of index* α, *defined by*

$$\forall \xi \in \mathbb{C}, \ j_\alpha(\xi) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\alpha+1)} (\frac{\xi}{2})^{2n}.$$

Proposition 2.2 [12]

(i) For all λ, $z \in \mathbb{C}^{d+1}$ *and* $t \in \mathbb{R}$, *we have*

$$\Psi_d^\alpha(\lambda, 0) = 1, \ \Psi_d^\alpha(\lambda, z) = \Psi_d^\alpha(z, \lambda) \ \text{and} \ \Psi_d^\alpha(\lambda, tz) = \Psi_d^\alpha(t\lambda, z).$$

(ii) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}_+^{d+1}$ *and* $z \in \mathbb{C}^{d+1}$, *we have*

$$|D_z^\nu \Psi_d^\alpha(x, z)| \le |x|^{|\nu|} \exp(|x||\operatorname{Im} z|),$$

where $D_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1} ... \partial z_{d+1}^{\nu_{d+1}}}$ *and* $|\nu| = \nu_1 + ... + \nu_{d+1}$. *In particular*

$$\forall x, y \in \mathbb{R}_+^{d+1}, \ |\Psi^\alpha_{d(x,y)}| \le 1.$$

Definition 2.1 The Weinstein transform is given for $f \in L^1_\alpha(\mathbb{R}_+^{d+1})$ by

$$\forall \lambda \in \mathbb{R}_+^{d+1}, \ \mathscr{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x) \Psi_d^\alpha(x, \lambda) d\mu_{\alpha,d}(x).$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by the relation (2.1).

The Weinstein tansform, referred to as the Fourier-Bessel transform, has been investigated by I. Kipriyanov [6], I.A. Aliev [1] and others ([2,9,10] and [3]).

Using the properties of the classical Fourier transform on $\mathbb{R}^d$ and of the Bessel transform, one can easily see the following relation, which will play an important role in the sequel.

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d}$ are summarized in the following results. For the proofs, we refer to [9,10].

Proposition 2.3 [9,10] *i) For all* $f \in L^1_\alpha(\mathbb{R}_+^{d+1})$, *we have*

$$\|\mathscr{F}_W^{\alpha,d}(f)\|_{\alpha,\infty} \le \|f\|_{\alpha,1}.$$

ii) For all $f \in L^1_\alpha(\mathbb{R}_+^{d+1})$ *such that* $\Delta_W^{\alpha,d} f \in L^1_\alpha(\mathbb{R}_+^{d+1})$, *we have*

$$\mathscr{F}_W^{\alpha,d}(\Delta_W^{\alpha,d} f)(x) = -|x|^2 \mathscr{F}_W^{\alpha,d}(f)(x).$$

Theorem 2.1 [9,10]

(i) The Weinstein transform $\mathscr{F}_W^{\alpha,d}$ *is a topological isomorphism from* $\mathscr{S}_*(\mathbb{R}^{d+1})$ *onto itself and from* $\mathscr{D}_*(\mathbb{R}^{d+1})$ *onto* $\mathcal{H}_*(\mathbb{C}^{d+1})$.

(ii) Let $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$. The inverse transform $\left(\mathcal{F}_W^{\alpha,d}\right)^{-1}$ is given by

$$\forall x \in \mathbb{R}_+^{d+1},\ \left(\mathcal{F}_W^{\alpha,d}\right)^{-1}(f)(x) = \mathcal{F}_W^{\alpha,d}(f)(-x).$$

(iii) Let $f \in L^1_\alpha(\mathbb{R}_+^{d+1})$. If $\mathcal{F}_W^{\alpha,d}(f) \in L^1_\alpha(\mathbb{R}_+^{d+1})$, then we have

$$f(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(y)\,\Psi_d^\alpha(-x, y)d\mu_{\alpha,d}(y),\ a.e\ x \in \mathbb{R}_+^{d+1}.$$

Theorem 2.2 [9,10]

(i) For all $f, g \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have the following Parseval formula

$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(\lambda)\overline{\mathcal{F}_W^{\alpha,d}(g)(\lambda)}d\mu_{\alpha,d}(\lambda).$$

(ii) (Plancherel formula).
For all $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have :

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2\, d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} \left|\mathcal{F}_W^{\alpha,d}(f)(\lambda)\right|^2 d\mu_{\alpha,d}(\lambda).$$

(iii) (Plancherel Theorem) :
The transform $\mathcal{F}_W^{\alpha,d}$ extends uniquely to an isometric isomorphism on $L^2_\alpha(\mathbb{R}_+^{d+1})$.

Definition 2.2 The translation operator $T_x,\ x \in \mathbb{R}_+^{d+1}$, associated with the Weinstein operator, is defined on $\mathcal{C}_*(\mathbb{R}^{d+1})$, for all $y \in \mathbb{R}_+^{d+1}$, by :

$$T_x f(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_0^\pi f\left(x' + y',\ \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha}\, d\theta,$$

where $x' + y' = (x_1 + y_1, ..., x_d + y_d)$.

The following proposition summarizes some properties of the Weinstein translation operator.

Proposition 2.4 [9,10]

(i) For $f \in \mathcal{C}_*(\mathbb{R}^{d+1})$, we have

$$\forall x,\ y \in \mathbb{R}_+^{d+1},\ T_x f(y) = T_y f(x)\ \text{and}\ T_0 f = f.$$

(ii) For all $f \in \mathcal{E}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$, the function $x \mapsto T_x f(y)$ belongs to $\mathcal{E}_*(\mathbb{R}^{d+1})$.
(iii) We have

$$\forall x \in \mathbb{R}_+^{d+1},\ \Delta_W^{\alpha,d} \circ T_x = T_x \circ \Delta_W^{\alpha,d}. \tag{2.2}$$

(iv) Let $f \in L^p_\alpha(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$ and $x \in \mathbb{R}_+^{d+1}$. Then $T_x f$ belongs to $L^p_\alpha(\mathbb{R}_+^{d+1})$ and we have

$$\|T_x f\|_{\alpha,p} \le \|f\|_{\alpha,p}.$$

(v) The function $\Psi_d^\alpha(., \lambda)$, $\lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}_+^{d+1}$ the following product formula:

$$\forall y \in \mathbb{R}_+^{d+1},\ \Psi_d^\alpha(x, \lambda)\,\Psi_d^\alpha(y, \lambda) = T_x\left[\Psi_d^\alpha(., \lambda)\right](y).$$

(vi) Let $f \in L^p_\alpha(\mathbb{R}_+^{d+1})$, $p = 1$ or 2 and $x \in \mathbb{R}_+^{d+1}$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ \mathcal{F}_W^{\alpha,d}(T_x f)(y) = \Psi_d^\alpha(x, y)\,\mathcal{F}_W^{\alpha,d}(f)(y).$$

(vii) The space $\mathcal{S}_*(\mathbb{R}^{d+1})$ is invariant under the operator $T_x,\ x \in \mathbb{R}_+^{d+1}$.

Definition 2.3 [10] The Weinstein convolution product of $f, g \in \mathscr{C}_*(\mathbb{R}^{d+1})$ is given by:

$$\forall x \in \mathbb{R}^{d+1}_+,\ f *_W g\,(x) = \int_{\mathbb{R}^{d+1}_+} T_x f\,(-y)\, g\,(y)\, d\mu_{\alpha,d}(y).$$

Proposition 2.5 [9,10]

(i) Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $f \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ and $g \in L^q_\alpha(\mathbb{R}^{d+1}_+)$, the function $f *_W g \in L^r_\alpha(\mathbb{R}^{d+1}_+)$ and we have

$$\|f *_W g\|_{\alpha,r} \leq \|f\|_{\alpha,p}\|g\|_{\alpha,q}.$$

(ii) For all $f, g \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, $\left(resp.\ \mathscr{S}_*(\mathbb{R}^{d+1})\right)$, $f *_W g \in L^1_\alpha(\mathbb{R}^{d+1}_+)$ $\left(resp.\ \mathscr{S}_*(\mathbb{R}^{d+1})\right)$ and we have

$$\mathscr{F}^{\alpha,d}_W(f *_W g) = \mathscr{F}^{\alpha,d}_W(f)\mathscr{F}^{\alpha,d}_W(g).$$

Examples 2.1

1) Let $G_s,\ s > 0$, be the function defined by

$$\forall x \in \mathbb{R}^{d+1},\ G_s\,(x) = e^{-s|x|^2}.$$

Then the Weinstein transform $\mathscr{F}^{\alpha,d}_W$ of G_s is given by :

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d}_W(G_s)(\lambda) = \frac{1}{(2s)^{\alpha+\frac{d}{2}+1}} e^{-\frac{|\lambda|^2}{4s}}.$$

From (2.1), we have for any positive s

$$, \forall \lambda \in \mathbb{R}^{d+1}_+\ \mathscr{F}^{\alpha,d}_W(G_s)(\lambda) = \frac{1}{(2s)^{\alpha+\frac{d}{2}+1}} e^{-\frac{|\lambda|^2}{4s}}.$$

Let us observe that for $z \in \mathbb{C}, z = |z|e^{i\theta}$ with positive real part, by taking the branch $z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, the two functions

$$z \mapsto \mathscr{F}^{\alpha,d}_W(G_z) \quad \text{and} \quad z \mapsto \frac{1}{(2z)^{\alpha+\frac{d}{2}+1}} e^{-\frac{|.|^2}{4z}}$$

are holomorphic on the domain $\mathrm{Re}\, z > 0$. As they coincide on the real axis, they coincide in the whole domain. Now, if t positive, considering a sequence of z_n with positive real part which tends to it, we get, as the Weinstein transform is continuous on tempered distributions that

$$\begin{aligned}\forall \lambda \in \mathbb{R}^{d+1}_+, \quad \mathscr{F}^{\alpha,d}_W(G_{is})(\lambda) &= \frac{1}{(2is)^{\alpha+\frac{d}{2}+1}} e^{i\frac{|\lambda|^2}{4s}} \\ &= \frac{1}{(2s)^{\alpha+\frac{d}{2}+1}} e^{-i\frac{\pi}{2}(\alpha+\frac{d}{2}+1)} e^{i\frac{|\lambda|^2}{4s}}\end{aligned}$$

If $s \in \mathbb{R}$ with $s \neq 0$, we have

$$\forall \lambda \in \mathbb{R}^{d+1}_+, \quad \mathscr{F}^{\alpha,d}_W(G_{is})(\lambda) = \frac{1}{|2s|^{\alpha+\frac{d}{2}+1}} e^{-i\frac{\pi}{2}(\alpha+\frac{d}{2}+1)\,\mathrm{sgn}(s)} e^{i\frac{|\lambda|^2}{4s}}$$

2) Let $E_t,\ t \neq 0$, be the function defined by

$$\forall x \in \mathbb{R}^{d+1},\ E_t\,(x) = \frac{1}{|2t|^{\alpha+\frac{d}{2}+1}} e^{-i\frac{\pi}{2}(\alpha+\frac{d}{2}+1)sgn(t)} e^{i\frac{|x|^2}{4t}}.$$

Then, by application of the Weinstein translation, we obtain

$$\forall x, y \in \mathbb{R}^{d+1}_+,\ T_x\,(E_t)\,(y) = \frac{1}{|2t|^{\alpha+\frac{d}{2}+1}} e^{-i\frac{\pi}{2}(\alpha+\frac{d}{2}+1)sgn(t)} e^{i\frac{(|x|^2+|y|^2)}{4t}} \Psi^\alpha_d\left(x, -\frac{y}{2t}\right)$$

So, for all $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$, we have

$$(E_t *_W f)\,(x) = E_t(x)\mathscr{F}^{\alpha,d}_W(e^{i\frac{|.|^2}{4t}} f)(\frac{x}{2t}).$$

Remark 2.1 It is no hand to prove ther the kernel E_t is a solution of the following equation:

$$\partial_t E_t(x) - i\Delta_W^{\alpha,d} E_t(x) = 0.$$

3 Schrödinger equation associated with the Weinstein operator

Notations. For an interval I of $\mathbb{R}$ and a Banach space X, we adopt the following notations

- $L^q(I;X)$ Banach space of (classes of) measurable functions $u : I \to X$ such that $\|u\|_{L^q(I;X)} < \infty$, with

$$\|u\|_{L^q(I;X)} = \left(\int_I \|u(t,\cdot)\|_X^q dt\right)^{\frac{1}{q}}, \quad \text{if } 1 \le q < \infty$$
$$\|u\|_{L^\infty(I;X)} = ess\sup_{t\in I} \|u(t,\cdot)\|_X$$

- $C(\bar{I};X)$ the space of continuous functions from $\bar{I}$ to X. Note that when I is bounded, then equipped with the norm $\|u\|_{L^\infty(I;X)}$, $C(\bar{I};X)$ is a Banach space.
- $C_c\left(I, \mathcal{S}\left(\mathbb{R}^d\right)\right)$ is the space of continuous functions from I into $\mathcal{S}\left(\mathbb{R}^d\right)$ compactly supported in I, equipped with the topology of uniform convergence on the compact subintervals of I.

3.1 Dispersion phenomena

Definition 3.1 We say that the exponent pair (q,r) is $\frac{d+2\alpha+2}{2}$-admissible if $q, r \ge 2$, $\left(q, r, \frac{d+2\alpha+2}{2}\right) \ne (2,\infty,1)$ and

$$\frac{1}{q} + \frac{d+2\alpha+2}{2r} = \frac{d+2\alpha+2}{4}. \tag{3.1}$$

Theorem 3.1 *Let $(U(t))_{t\in\mathbb{R}}$ be a bounded family of continuous operators of $L^2_\alpha(\mathbb{R}^{d+1}_+)$ on $L^2_\alpha(\mathbb{R}^{d+1}_+)$ such that*

$$\left\|U(t)U^*\left(t'\right) f\right\|_{\alpha,\infty} \le \frac{C}{|t-t'|^{\frac{d+2\alpha+2}{2}}} \|f\|_{\alpha,1}.$$

Then, the estimates

$$\|U(t)g\|_{L^q\left(\mathbb{R};L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \le C\,\|g\|_{\alpha,2} \tag{3.2}$$

and

$$\left\|\int_{\mathbb{R}} U^*(t) f(t,\cdot)dt\right\|_{\alpha,2} \le C\|f\|_{L^{q'}\left(\mathbb{R};L^{r'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \tag{3.3}$$

hold for any $\frac{d+2\alpha+2}{2} - admissible$ exponent (q,r), where q', r' are the conjugate exponents of q and r and U^* is the adjoint operator of U.

Moreover, for any $\frac{d+2\alpha+2}{2}$-admissible exponent pairs (q,r) and (q_1,r_1), we have

$$\left\|\int_{\mathbb{R}} U(t)U^*\left(t'\right) f\left(t',\cdot\right) dt'\right\|_{L^q\left(\mathbb{R};L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \le C\|f\|_{L^{q_1'}\left(\mathbb{R};L^{r_1'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}. \tag{3.4}$$

Furthermore, if

$$\left\|U(s)U^*(t) f\right\|_{\alpha,\infty} \le \frac{C}{(1+|t-s|)^{\frac{d+2\alpha+2}{2}}} \|f\|_{\alpha,1}$$

then (3.2),(3.3) *and* (3.4) *hold for all $\frac{d+2\alpha+2}{2} - admissible (q,r)$ and (q_1,r_1).*

Proof Using some the ideas and steps are in [5] one can prove that following result. □

3.2 Strichartz-type Schrödinger-Weinstein estimates

We consider in the rest of this article the incompressible Weinstein-Schrödinger system given by:

$$\begin{cases} \partial_t u(t,x) - i\Delta_W^{\alpha,d} u(t,x) = F(t,x), & \text{in} \quad \mathbb{R}\times\mathbb{R}^{d+1}_+ \\ u(0,x) = g(x) & \text{in} \quad \mathbb{R}^{d+1}_+. \end{cases} \tag{3.5}$$

Moreover, under the same conditions, Duhamel's formula implies

$$u(t,x) = \mathcal{I}_\alpha(t)g(x) + \int_0^t \mathcal{I}_\alpha(t-s)F(s,x)ds, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^{d+1}_+,$$

where $\mathcal{I}_\alpha(t)$ is the unitary operator defined by

$$\mathcal{I}_\alpha(t)u = \left(\mathcal{F}_W^{\alpha,d}\right)^{-1}\left(e^{-it|\xi|^2}\mathcal{F}_W^{\alpha,d}u\right). \tag{3.6}$$

We notice that:

$$\mathcal{I}_\alpha(t+s) = \mathcal{I}_\alpha(t)\mathcal{I}_\alpha(s)\,, \mathcal{I}^*_\alpha(t) = \mathcal{I}_\alpha(-t),$$

and $\mathcal{I}_\alpha(t)g$ is a solution of the linear Schrödinger-Weinstein equation:

$$\begin{cases} \partial_t v(t,x) - i\Delta_W^{\alpha,d} v(t,x) = 0, & \text{in} \quad \mathbb{R}\times\mathbb{R}^{d+1}_+ \\ v(0,x) = g\in\mathscr{S}_*(\mathbb{R}^{d+1}) & \text{in} \quad \mathbb{R}^{d+1}_+. \end{cases}$$

From Examples 2.1, Remark 2.1 and equation (2.2), we have

$$\begin{aligned} \mathcal{I}_\alpha(t)g =& E_t *_W g \\ =& \frac{1}{|2t|^{\frac{d+2\alpha+2}{2}}} e^{-i(d+2\alpha+2)\frac{\pi}{4}sgn(t)} e^{i\frac{|\cdot|^2}{4t}}\left[\mathcal{F}_W^{\alpha,d}\left(e^{i\frac{|\cdot|^2}{4t}}g\right)\right]\left(\frac{\cdot}{2t}\right). \end{aligned} \tag{3.7}$$

Proof of Theorem 1.1 Let ar begin by proving the sufficient condition. Assume that (q,r) satisfies the condition of the theorem, and that u is a solution of (1.2).

According to Plancheral formula and the equation (3.6), we have

$$\|\mathcal{I}_\alpha(t)g\|_{\alpha,2} = \|g\|_{\alpha,2}. \tag{3.8}$$

The estimate

$$\|\mathcal{I}_\alpha(t-s)g\|_{\alpha,\infty} \le \frac{C}{|t-s|^{\frac{d+2\alpha+2}{2}}}\|g\|_{\alpha,1} \tag{3.9}$$

follows from the equation (3.7) and the density of $\mathscr{S}_*(\mathbb{R}^{d+1})$ in $L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1\le p<\infty$. So $\mathcal{I}_\alpha(t)$ satisfies the conditions of Theorem 3.1. Using the equations (3.2) and (3.2), we have

$$\|u\|_{L^q\left([0,T];L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \le C\left(\|g\|_{\alpha,2} + \|F\|_{L^{q'_1}\left([0,T];L^{r'_1}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\right).$$

Let Φ_α be the operator defined by $\in$

$$\Phi_\alpha(F)(t,x) := \int_0^t \mathcal{I}_\alpha(t-s)F(s,x)ds$$

We now address the question of continuity in L^2_α. The continuity of $\mathcal{I}_\alpha(\cdot)g$ follows from Plancherel formula, and we have

$$\|\mathcal{I}_\alpha(t)g\|_{C\left([0,T];L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \le \|g\|_{\alpha,2}.$$

On the other hand, for every $\frac{d+2\alpha+2}{2}$-admissible pair (q_1, r_1), we have

$$\begin{aligned}\|\Phi_\alpha F(t)\|^2_{\alpha,2} &= \left\langle \int_0^t \mathcal{I}_\alpha(t-s)F(s)ds, \int_0^t \mathcal{I}_\alpha(t-s)F(\sigma)d\sigma \right\rangle_\alpha \\ &= \int_0^t \int_0^t \langle \mathcal{I}_\alpha(t-s)F(s), \mathcal{I}_\alpha(t-\sigma)F(\sigma)\rangle_\alpha \, d\sigma ds \\ &\leq \int_0^t \int_{\mathbb{R}} \langle F(s), \mathcal{I}_\alpha(s-\sigma)F(\sigma)\rangle_\alpha \, d\sigma ds \\ &= \int_0^t \left\langle f(s), \int_{\mathbb{R}} \mathcal{I}_\alpha(s-\sigma)F(\sigma)d\sigma \right\rangle_\alpha ds\end{aligned}$$

where we used the property $\mathcal{I}_\alpha(t)^\star = \mathcal{I}_\alpha(-t)$. Applying Hölder's inequality in space, then in time, and applying equation (3.3), we deduce that

$$\begin{aligned}\|\Phi_\alpha F(t)\|^2_{\alpha,2} &\leq \|F\|_{L^{q_1'}\left([0,T];L_\alpha^{r_1'}\left(\mathbb{R}_+^{d+1}\right)\right)} \left\| \int_{\mathbb{R}} \mathcal{I}_\alpha(\cdot-\sigma)F(\sigma)d\sigma \right\|_{L^{q_1}\left([0,T];L_\alpha^{r_1}\left(\mathbb{R}_+^{d+1}\right)\right)} \\ &\leq C\|F\|^2_{L^{q_1'}\left([0,T];L_\alpha^{r_1'}\left(\mathbb{R}_+^{d+1}\right)\right)}.\end{aligned}$$

To show that the quantity $\Phi_\alpha(F)$ is continuous in $L^2_\alpha\left(\mathbb{R}_+^{d+1}\right)$, one can use the identity

$$\Phi_\alpha(F)(t+\varepsilon) = \mathcal{I}_\alpha(\varepsilon)\left[\Phi_\alpha(F)(t) + \Phi_\alpha\left(1_{[t,t+\varepsilon]}F\right)(t)\right]$$

the continuity of $\mathcal{I}_\alpha(\varepsilon)$ as an operator on $L^2_\alpha\left(\mathbb{R}_+^{d+1}\right)$, and the fact that

$$\left\|1_{[t,t+\varepsilon]}F\right\|_{L^{q_1'}\left([0,T];L_\alpha^{r_1'}\left(\mathbb{R}_+^{d+1}\right)\right)} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

We finish the proof of necessity as in [5]. □

Corollary 3.1 *Let I be an interval of $\mathbb{R}$. If (q, r) and (q_1, r_1) are $\frac{d+2\alpha+2}{2}$-admissible pairs, then there exits a constant C such that*

$$\|\mathcal{I}_\alpha(\cdot)g\|_{L^q\left(\mathbb{R};L_\alpha^r\left(\mathbb{R}_+^{d+1}\right)\right)} \leq C\|g\|_{\alpha,2}$$

and

$$\|\Phi_\alpha(F)\|_{L^q\left(I;L_\alpha^r\left(\mathbb{R}_+^{d+1}\right)\right)} \leq C\|F\|_{L^{q_1'}\left(I;L_\alpha^{r_1'}\left(\mathbb{R}_+^{d+1}\right)\right)}.$$

Proposition 3.1 *If $p \in [2, \infty]$ and $t \neq 0$, then $\mathcal{I}_\alpha(t)$ maps $L^{p'}_\alpha\left(\mathbb{R}_+^{d+1}\right)$ continuously to $L^p_\alpha\left(\mathbb{R}_+^{d+1}\right)$ and*

$$\|\mathcal{I}_\alpha(t)g\|_{\alpha,p} \leq \frac{1}{(2|t|)^{(d+2\alpha+2)\left(\frac{1}{2}-\frac{1}{p}\right)}}\|g\|_{\alpha,p'}. \tag{3.10}$$

Proof It follows from (3.8) and (3.9) that

$$\|\mathcal{I}_\alpha(t)g\|_{\alpha,\infty} \leq \frac{1}{|2t|^{\frac{d+2\alpha+2}{2}}}\|g\|_{\alpha,1} \quad \text{and} \quad \|\mathcal{I}_\alpha(t)g\|_{\alpha,2} = \|g\|_{\alpha,2}$$

The general case is obtained by interpolation between the cases $p = 2$ and $p = \infty$. □

Proposition 3.2 *Let I be an interval of $\mathbb{R}$ (bounded or not).*
Assume that $2 < r < \frac{2d+8\alpha+6}{d+4\alpha-1}$ and let $(q_1, r_1) \in (1,\infty)^2$ satisfies

$$\frac{1}{q_1} + \frac{1}{r_1} = (d+2\alpha+2)\left(\frac{1}{2} - \frac{1}{r}\right).$$

Then $\Phi_\alpha(F) \in L^{q_1}\left(I; L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$ for every $F \in L^{r_1'}\left(I; L^{r'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$. Moreover, there exits a constant C independent of I such that for $F \in L^{r_1'}\left(I; L^{r'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$, we have

$$\|\Phi_\alpha(F)\|_{L^{q_1}\left(I;L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \le C\|F\|_{L^{r_1'}\left(I;L^{r'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}. \tag{3.11}$$

Proof By density, it prove to prove (3.11) for $F \in C_c\left(I; \mathcal{S}_*\left(\mathbb{R}^{d+1}\right)\right)$. It follows from (3.10) that

$$\|\Phi_\alpha(F)(t,\cdot)\|_{\alpha,r} \le \int_0^t \frac{1}{\left((2|t-s|)^{2\alpha+d+2}\right)^{\left(\frac{1}{2}-\frac{1}{r}\right)}}\|F(s,\cdot)\|_{\alpha,r'}ds.$$

So (3.11) is an immediate consequence of Hardy-Littlewood-Sobolev inequality. □

3.3 L^2_α−Solutions for nonlinear Weinstein-Schrödinger equations

In this subsection, Strichartz estimates are a powerful tool to prove local and global well-posedness results for the nonlinear Weinstein-Schrödinger equations (SW), for this we recall the definition of well-posedness.

Definition 3.2 We say that the problem (SW) is locally well-posed in $L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ if, for every g in $L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$, one can find time $T > 0$ and a unique solution $u \in C\left([-T,T], L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) \cap X$ to (SW) which depends continuously on the data, with X some additional Banach space. The equation is globally well-posed if these properties hold for $T = \infty$.

Proof of Theorem 1.2 We proceed in two steps.

Step 1: Local existence. For the existence, we use a fixed point argument.
• If $p \in \left(0, \frac{4}{d+2\alpha+2}\right)$, fix $T > 0$, $M > \|g\|_{\alpha,2}$ and set

$$X_{M,T} := \Big\{u \in L^q\left((-T,T); L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) \cap C\left((-T,T); L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) : \|u\|_{L^\infty\left((-T,T);L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + \|u\|_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \le M\Big\}$$

where $(q, p+2)$ is $\frac{d+2\alpha+2}{2}$-admissible pair. Note that by Theorem 1.1 and Corollary 3.1 , this space is empty. Moreover, we can verify that $X_{M,T}$ is a complete metric space when it is equipped with the distance:

$$d(u,v) = \|u-v\|_{L^\infty\left((-T,T);L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + \|u-v\|_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}.$$

To simplify, put

$$\|v\|_{X_{M,T}} = \|v\|_{L^\infty\left((-T,T);L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + \|v\|_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}$$

if $v \in X_{M,T}$. Take $g \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$. We wish find some conditions on T and M such that the operator $\mathcal{H}_\alpha$, given by

$$\mathcal{H}_\alpha(u)(t,\cdot) := \mathcal{I}_\alpha(t)g(\cdot) + \int_0^t \mathcal{I}_\alpha(t-s)F(u(s,\cdot))ds$$

is a strict contraction on $X_{M,T}$.

By the nonlinearity assumption (1.1) and Theorem 1.1, it follows that

$$\|\mathcal{H}_\alpha(u)\|_{X_{M,T}} \leq C\left(\|\mathcal{I}_\alpha(\cdot)g\|_{X_{M,T}} + \|u\|^{p+1}_{L^{(p+1)q_1'}\left((-T,T);L_\alpha^{(p+1)r_1'}\left(\mathbb{R}_+^{d+1}\right)\right)}\right)$$

with (q_1, r_1) is a $\frac{d+2\alpha+2}{2}$-admissible pair.

On the other hand, we use the condition $p \in \left(0, \frac{4}{d+2\alpha+2}\right)$, we take $r_1 = p+2$ and $(q_1 = q, p+2)$ a $\frac{d+2\alpha+2}{2}$ admissible pair such that $q > p+2$. Then applying Corollary 3.1 and Hölder's inequality in time, we obtain

$$\|\mathcal{H}_\alpha(u)\|_{X_{M,T}} \leq C\|g\|_{\alpha,2} + CT^{\frac{q-p-2}{q}}\|u\|^{p+1}_{L^q\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)}. \tag{3.12}$$

Hence, for every $u \in X_{M,T}$ one has

$$\|\mathcal{H}_\alpha(u)\|_{X_{M,T}} \leq C\|g\|_{\alpha,2} + CT^{\frac{q-p-2}{q}}M^{p+1}.$$

Choosing $M = 2C\|g\|_{\alpha,2}$, we see that if T is sufficiently small (depending on $\|g\|_{\alpha,2}$) then $\mathcal{H}_\alpha(u) \in X_{M,T}$ for all $u \in X_{M,T}$. Moreover, arguing as above, we obtain

$$d\left(\mathcal{H}_\alpha(u), \mathcal{H}_\alpha(v)\right) \leq CT^{\frac{q-p-2}{q}}M^p d(u,v)$$

for all $u, v \in X_{M,T}$. Thus $\mathcal{H}_\alpha$ is a contraction in $X_{M,T}$ provided T is small enough, more precisely if $T < \left(\frac{1}{2CM^p}\right)^{\frac{q}{q-p-2}}$. Hence $\mathcal{H}_\alpha$ has a fixed point u, which is the unique solution of (SW) in $X_{M,T}$, and there exist $T^*, T_* \in (0,\infty]$ such that u belongs to

$$C\left(\left(-T_*, T^*\right); L_\alpha^2\left(\mathbb{R}_+^{d+1}\right)\right) \cap L^q_{loc}\left(\left(-T_*, T^*\right); L_\alpha^r\left(\mathbb{R}_+^{d+1}\right)\right)$$

for the $\frac{d+2\alpha+2}{2}$-admissible pair $(q, p+2)$, with

$T^* = \sup\{T > 0$, there exists a solution of (SW) on $[0, T]\}$

$T_* = \sup\{T > 0$, there exists a solution of (SW) on $[-T, 0]\}$

Moreover, from Theorem 1.1 and by the argument we use to prove (3.12), we have $u \in L^{q_1}_{loc}\left((-T_*, T^*); L_\alpha^{r_1}\left(\mathbb{R}_+^{d+1}\right)\right)$ for every $\frac{d+2\alpha+2}{2}$-admissible pair (q_1, r_1).

• If $p = \frac{4}{d+2\alpha+2}$, let $g \in L_\alpha^2\left(\mathbb{R}_+^{d+1}\right)$. Since $\mathcal{I}_\alpha(\cdot)g \in L^{p+2}\left(\mathbb{R}; L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)$, by Corollary 3.1, we have

$$\|\mathcal{I}_\alpha(.)g\|_{L^{p+2}\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)} \to 0 \quad \text{as} \quad T \downarrow 0. \tag{3.13}$$

Therefore for $\epsilon > 0$, there is $T > 0$, such that

$$\|\mathcal{I}_\alpha(.)g\|_{L^{p+2}\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)} < \epsilon. \tag{3.14}$$

Let us consider the set

$$Y_\epsilon := \left\{u \in L^{p+2}\left((-T,T); L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right) : \|u\|_{L^{p+2}\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)} \leq 2\epsilon\right\}.$$

We can verify that Y_ϵ is a complete metric space when it is equipped with the distance

$$d(u,v) = \|u-v\|_{L^{p+2}\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)}.$$

As above, by Theorem 1.1 , the following estimate holds

$$\begin{aligned}&\|\mathcal{H}_\alpha(u)\|_{L^{p+2}\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)}\\ &\leq C\left(\|\mathcal{I}_\alpha(\cdot)g\|_{L^{p+2}\left((-T,T);L_\alpha^{p+2}\left(\mathbb{R}_+^{d+1}\right)\right)} + \|u\|^{p+1}_{L^{(p+2)}\left((-T,T);L_\alpha^{(p+2)}\left(\mathbb{R}_+^{d+1}\right)\right)}\right),\end{aligned}$$

where we have taken $q = q_1 = r = r_1 = p + 2$. Hence, for every $u \in Y_\epsilon$:

$$\|\mathcal{H}_\alpha(u)\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \leq C\,\|\mathcal{I}_\alpha(\cdot)g\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + C\epsilon^{p+1}.$$

From relations (3.13) and (3.14), we see that if T is small enough, then we can choose M such that $\mathcal{H}_\alpha(u)$ belongs to Y_ϵ for all $u \in Y_\epsilon$. As above we prove also that $\mathcal{H}_\alpha$ is a contraction on the space Y_ϵ provided T is sufficiently small. Thus $\mathcal{H}_\alpha$ has a fixed point u, which is the unique solution of (SW) in Y_ϵ. Moreover, from Theorem 1.1, there exists T^*, $T_* \in (0, \infty]$ such that

$$u \in C\left(\left(-T_*, T^*\right); L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) \cap L^{q_1}_{loc}\left(\left(-T_*, T^*\right); L^{r_1}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$$

for every $\frac{d+2\alpha+2}{2}$-admissible pair (q_1, r_1).

Step 2: Uniqueness. We first note that the uniqueness is a local property, so that we need only to establish it on possibly small intervals. To see this, we argue for positive times, the case for negative times being the same. Suppose that $u_1, u_2 \in C\left([0,T]; L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) \cap L^q\left([0,T]; L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$ are two solutions of (SW) and assume that $u_1(t) \neq u_2(t)$ for some $t \in [0,T]$. Let $t_0 = \inf\{t \in [0,T], u_1(t) \neq u_2(t)\}$. Since both u_1 and u_2 are continuous into $L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$, this definition makes sense and $u_1(t_0) = u_2(t_0) = \chi$. Moreover, the curves $U_1(t) = u_1(t + t_0)$ and $U_2(t) = u_2(t + t_0)$ both satisfy the equation $w = \mathcal{I}_\alpha(\cdot)\chi + \Phi_\alpha(F(w))$ on $[0, T - t_0]$ As above we apply Theorem 1.1 and the argument of proof of (3.12), to obtain that for all $t \in [t_0, T]$

$$\begin{aligned}
&\|u_1 - u_2\|_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \\
&\leq C\,(t - t_0)^{\frac{4-(d+2\alpha+2)p}{4}} \sum_{i=1}^{2} \|u_i\|^p_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \|u_1 - u_2\|_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)},
\end{aligned}$$

where $\left(q = \frac{4(p+2)}{p(d+2\alpha+2)}, p+2\right)$ is a $\frac{d+2\alpha+2}{2}$-admissible pair. For $t > t_0$, but sufficiently close to t_0, it follows that

$$C\,(t - t_0)^{\frac{4-(d+2\alpha+2)p}{4}} \sum_{i=1}^{2} \|u_i\|^p_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} < 1$$

and so that

$$\|u_1 - u_2\|_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} = 0.$$

This contradicts the choice of t_0, and thus proves that $u_1(t) = u_2(t)$ for all $t \in [0,T]$. □

In the proof of local existence with $p = \frac{4}{d+2\alpha+2}$, we used T is small enough, we can change this condition by $\|g\|_{\alpha,2}$ small enoug. This is what we are going to prove in the following corollary.

Corollary 3.2 *If $p = \frac{4}{d+2\alpha+2}$, there exists $\delta > 0$ such that if $g \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ satisfies $\|g\|_{\alpha,2} \leq \delta$, (i.e. $\|g\|_{\alpha,2}$ small enough), for $T \in (0, \infty[$ there exist a unique solution u of* (SW) *belonging to*

$$C\left((-T,T); L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right) \cap L^q\left((-T,T); L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$$

for every $\frac{d+2\alpha+2}{2}$-admissible pair (q, r).

Proof For $T > 0$, by Theorem 1.1 the following Strichartz estimate holds

$$\|\mathcal{H}_\alpha(u)\|_{X_T} \leq C\left(\|g\|_{\alpha,2} + \|F(u)\|_{L^{q'}\left([-T,T];L^{r'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\right) \tag{3.15}$$

with $(q = p+2, r = p+2)$ a $\frac{d+2\alpha+2}{2}$-admissible and $X_T = C\left([-T,T];L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)\cap L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$ the Banach space with norm

$$\|v\|_{X_T} = \|v\|_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + \|v\|_{C\left([-T,T];L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}$$

By our nonlinearity assumption (1.1) and Hölder inequality, we have

$$\|\mathcal{H}_\alpha(u)\|_{X_T} \leq C\left(\|g\|_{\alpha,2} + \|u\|^{p+1}_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\right). \tag{3.16}$$

As above, we prove that $\mathcal{H}_\alpha$ maps the Banach space X_T into itself, and moreover the ball $X_{M,T}$ into itself, provided M and $\|g\|_{\alpha,2}$ are small enough, where

$$X_{M,T} = \left\{u \in X_T : \|u\|_{X_T} \leq M\right\}.$$

Let δ small enough, and also that $\|g\|_{\alpha,2} < \delta$. By (3.16), we cansee that

$$\|\mathcal{H}_\alpha(u)\|_{X_T} \leq C\delta + CM^{p+1} < M$$

provided δ is such that $2C\delta = M$ and $CM^p \leq \frac{1}{2}$, and we assume now that $u_i \in X_{M,T}$, i=1,2. We have also

$$\begin{aligned}
&\|\mathcal{H}_\alpha(u_1) - \mathcal{H}_\alpha(u_2)\|_{X_T} \leq C\left\| F(u_1) - F(u_2)\right\|_{L^{q'}\left([-T,T];L^{r'}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\\
&\leq C\|u_1-u_2\|_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\left(\|u_1\|^p_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} + \|u_2\|^p_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\right)\\
&\leq \|u_1-u_2\|_{X_{M,T}}\, 2CM^p \leq \frac{1}{2}\|u_1-u_2\|_{X_{M,T}}
\end{aligned}$$

provided M is so small that $2CM^p \leq \frac{1}{2}$. Thus, if initial data are small enough i.e. $\|g\|_{\alpha,2} < \delta$, then the map $\mathcal{H}_\alpha$ is a contraction which implies that there exists a unique solution $u(t,x)$ of the Cauchy problem (SW) such that $u(t,x) \in L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$. As observed above one can see that this is the unique solution in $u(t,x) \in C\left([-T,T],L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$, i.e. $u \in X_T$, with small initial data in L^2_α. □

Proof of Theorem 1.3 As Corollary 3.2, we assume that initial data are small enough i.e. $\|g\|_{\alpha,2} < \delta$.

For $n \in \mathbb{N}$, there is a unique solution $u_n \in X_n$.

Similarly, there is a unique solution $u_{n+1} \in X_{n+1}$, but

$$u_n = u_{n+1}, \text{ for all } t \text{ in } [-n,n].$$

Thus,

$$u = \lim_{n\to\infty} u_n$$

is a unique solution of (SW) in $C\left(\mathbb{R};L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)\cap L^{p+2}_{loc}\left(\mathbb{R};L^{p+2}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)$, which achieved the proof of the global existence . □

Proposition 3.3 *Assume that F is as in Theorem* 1.2, *$g \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ and u is the maximal solution of* (SW), *then we have*

(i) If $p \in \left(0, \frac{4}{d+2\alpha+2}\right)$ and $T^* < \infty$ (respectively, $T_* < \infty$), then

$$\|u(t)\|_{\alpha,2} \to \infty \text{ as } t \uparrow T^* (\text{respectively, as } t \downarrow T_*).$$

(ii) If $p = \frac{4}{d+2\alpha+2}$ *and* $T^* < \infty$ *(respectively,* $T_* < \infty$ *), then*

$$\|u\|_{L^q\left((0,T^*);L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} = \infty \text{ (respectively, } \|u\|_{L^q\left((-T_*,0);L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} = \infty\Big)$$

for every $\frac{d+2\alpha+2}{2}$*-admissible pair* (q,r) *with* $r \geq p+2$.

Proof i) If $p \in \left(0, \frac{4}{d+2\alpha+2}\right)$, it follows from Step 1 of the proof of Theorem 1.2 and the uniqueness property that

$$T^* - t \geq \left(\frac{1}{4C^2\|u(t,\cdot)\|^p_{\alpha,2}}\right)^{\frac{q}{q-p-2}}.$$

Suppose now that $T^* < \infty$, then

$$\|u(t)\|_{\alpha,2} \geq \left(\frac{1}{4C^2\,(T^*-t)^{\frac{q-p-2}{q}}}\right)^{\frac{1}{p}}.$$

As $q > p+2$, we obtain $\|u(t)\|_{\alpha,2} \to \infty$ as $t \uparrow T^*$. One shows by the same argument that if $T_* < \infty$, then $\|u(t)\|_{\alpha,2} \to \infty$ as $t \downarrow T_*$.

ii) If $p = \frac{4}{d+2\alpha+2}$, we show the blowup alternative by contradiction. Suppose that $T^* < \infty$ and that $\|u\|_{L^{(p+2)}\left((0,T^*);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} < \infty$. Let $0 \leq t \leq t+\tau < T^*$. It follows that

$$\mathcal{I}_\alpha(\tau)u(t,\cdot) = u(t+\tau,\cdot) - \int_0^\tau \mathcal{I}_\alpha(\tau-s)F(u(t+s,\cdot))ds.$$

By Theorem 1.1 we deduce that there exists $C \geq 0$ such that

$$\begin{aligned}\|\mathcal{I}_\alpha(\cdot)u(t)\|_{L^{(p+2)}\left((0,T^*-t);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} &\leq \|u\|_{L^{(p+2)}\left((t,T^*);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \\ &\quad + C\|u\|^{p+1}_{L^{(p+2)}\left((t,T^*);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\end{aligned}$$

Therefore for t close enough to T^*,

$$\|\mathcal{I}_\alpha(\cdot)u(t)\|_{L^{(p+2)}\left((0,T^*-t);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \leq \frac{M}{2}.$$

By Step 1 in the proof of Theorem 1.2, u can be extended beyong T^*, which is a contradiction. This shows that $\|u\|_{L^{(p+2)}\left((0,T^*);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} = \infty$. Let now (q,r) be a $\frac{d+2\alpha+2}{2}$-admissible pair such that $r > p+2$. It follows from Hölder's inequality that for any $T < T^*$,

$$\|u\|_{L^{(p+2)}\left((0,T);L^{(p+2)}_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} \leq \|u\|^{\lambda}_{L^\infty\left((0,T);L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)}\|u\|^{1-\lambda}_{L^q\left((0,T);L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)},$$

with $\lambda = \frac{2(r-p-2)}{(p+2)(r-2)}$. Letting $T \uparrow T^*$, we obtain $\|u\|_{L^q\left((0,T^*);L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} = \infty$. One shows by the same argument that if $T_* < \infty$, then $\|u\|_{L^q\left((-T_*,0);L^r_\alpha\left(\mathbb{R}^{d+1}_+\right)\right)} = \infty$ □

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