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ORIGINAL RESEARCH

A new generalization of Leonardo hybrid numbers with q-integers

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Abstract In this article, we introduce the q-Leonardo hybrid numbers by using the q-integers. These numbers are a new generalization of the known Leonardo hybrid numbers. Then, we obtain some algebraic characteristics of the q-Leonardo hybrid numbers such as Binet formula, exponential generating function, some summation formulas, Catalan's, Cassini's and d'Ocagne's identities.

Keywords Hybrid numbers · Fibonacci numbers · Leonardo numbers · q-integers

1 Introduction

Many researchers have been interested in two-dimensional number systems during the last century. Complex, hyperbolic, and dual numbers are among the most studied areas. Özdemir [14] introduced a non-commutative number system, which combines all three number systems. This number system called hybrid numbers is defined by

$$\begin{aligned}\mathbb{K} = \{a + b\boldsymbol{i} + c\boldsymbol{\epsilon} + d\boldsymbol{h} : a, b, c, d \in \mathbb{R},\ \boldsymbol{i}^2 = -1,\ \boldsymbol{\epsilon}^2 = 0,\ \boldsymbol{h}^2 = 1,\\ \boldsymbol{i}\boldsymbol{h} = -\boldsymbol{h}\boldsymbol{i} = \boldsymbol{\epsilon} + \boldsymbol{i}\}\end{aligned}$$

Let $x = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{\epsilon} + x_4\boldsymbol{h}$ and $y = y_1 + y_2\boldsymbol{i} + y_3\boldsymbol{\epsilon} + y_4\boldsymbol{h}$ be two hybrid numbers. We provide the following operations and properties related to hybrid numbers:

$$\begin{aligned}&x = y \text{ if and only if } x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3,\ x_4 = y_4 \quad \text{(equality)},\\ &x + y = (x_1 + y_1) + (x_2 + y_2)\,\boldsymbol{i} + (x_3 + y_3)\,\boldsymbol{\epsilon} + (x_4 + y_4)\,\boldsymbol{h} \quad \text{(addition)},\\ &x - y = (x_1 - y_1) + (x_2 - y_2)\,\boldsymbol{i} + (x_3 - y_3)\,\boldsymbol{\epsilon} + (x_4 - y_4)\,\boldsymbol{h} \quad \text{(substraction)},\\ &rx = rx_1 + rx_2\boldsymbol{i} + rx_3\boldsymbol{\epsilon} + rx_4\boldsymbol{h} \quad \text{(multiplication with scalar } r \in \mathbb{R}).\end{aligned}$$

Here, addition operation has both the properties of commutativity and associativity. Zero is the additive identity. In terms of the addition operation, the inverse of $x = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{\epsilon} + x_4\boldsymbol{h}$ is $-x = -x_1 - x_2\boldsymbol{i} - x_3\boldsymbol{\epsilon} - x_4\boldsymbol{h}$. This means that $(\mathbb{K}, +)$ is an Abelian group. Also, the multiplication operation is non-commutative, but it is associative. The multiplication table of hybrid numbers is given in Table 1.

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Table 1 The multiplication table for the basis of $\mathbb{K}$

$\cdot$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{\epsilon}$	$\boldsymbol{h}$
$\mathbf{1}$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{\epsilon}$	$\boldsymbol{h}$
$\boldsymbol{i}$	$\boldsymbol{i}$	$-\mathbf{1}$	$\mathbf{1}-\boldsymbol{h}$	$\boldsymbol{\epsilon}+\boldsymbol{i}$
$\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$\boldsymbol{h}+\mathbf{1}$	$\mathbf{0}$	$-\boldsymbol{\epsilon}$
$\boldsymbol{h}$	$\boldsymbol{h}$	$-\boldsymbol{\epsilon}-\boldsymbol{i}$	$\boldsymbol{\epsilon}$	$\mathbf{1}$

Now, we present some notations with q-integers. Let $s \in \mathbb{N}$ and $q \neq 1$. We denote the q-integer $[s]_q$

$$[s]_q = \frac{1-q^s}{1-q} = 1 + q + q^2 + \cdots + q^{s-1}. \tag{1.1}$$

By using (1.1), we can derive that

$$[s+t]_q = [s]_q + q^s [t]_q .$$

The quantum $(q-)$ calculus theory has been intensively studied in many areas of mathematics, as well as in physics, probability theory, statistics and many other fields. We refer to [3,8] for further information on the quantum q-calculus.

In the literature, integer sequences play an important role. Fibonacci and Lucas sequences are the most well-known of these. Fibonacci and Lucas sequences are defined by the following recurrence relation for $n \geq 2$, respectively,

$$F_n = F_{n-1} + F_{n-2}$$
$$L_n = L_{n-1} + L_{n-2}$$

where $F_0 = 0,\ F_1 = 1$ and $L_0 = 2,\ L_1 = 1$. In this article, we consider the other sequence and it has similar characteristics to Fibonacci. It is called Leonardo sequence. Catarino and Borges [5] defined Leonardo numbers and presented their properties. The Leonardo sequence is defined by the recurrence relation for $n \geq 2$ as follows:

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

where $Le_0 = Le_1 = 1$. There is a relation between Leonardo numbers and the Fibonacci numbers for $n \geq 0$ such that

$$Le_n = 2F_{n+1} - 1.$$

In [10], the generalized Leonardo numbers have been studied. In [9], the hybrid q-Fibonacci numbers and the hybrid q-Lucas numbers were defined by Kızılateş as another generalization of hybrid numbers. The authors [15] considered the two-dimensional recurrence relations of Leonardo numbers derived from their one-dimensional model. The authors [4] introduced incomplete Leonardo numbers and provided some characteristics of these numbers. The authors [2] determined several identities of Leonardo numbers using the matrix form of Leonardo numbers. Alp and Koçer [1] defined the Leonardo hybrid numbers and presented some characteristics of these numbers. They defined the Leonardo hybrid numbers for $n \geq 1$ as follows:

$$HLe_n = Le_n + Le_{n+1}\boldsymbol{i} + Le_{n+2}\boldsymbol{\epsilon} + Le_{n+3}\boldsymbol{h}. \tag{1.2}$$

In view of the foregoing recent studies, we introduce a new family of the Leonardo hybrid numbers by implementing the q-integers. This family is a generalization of hybrid Leonardo numbers. We call these numbers as the q-Leonardo hybrid numbers. The hybrid q-Leonardo numbers are related to hybrid q-Fibonacci and hybrid q-Lucas numbers. We give various results for q-Leonardo hybrid numbers included Binet formula, exponential generating function, summation formulas, Catalan's identity, Cassini's identity and d'Ocagne's identity.

Table 2 Special situations for q-Fibonacci and q-Lucas hybrid numbers (HN)

α	q	q-Fibonacci HN; $HF_n(\alpha;q)$	q-Lucas HN; $HL_n(\alpha;q)$
$\frac{1+\sqrt{5}}{2}$	$-\frac{1}{\alpha^2}$	Fibonacci HN; HF_n [12]	Lucas HN; HL_n [12]
$1+\sqrt{2}$	$-\frac{1}{\alpha^2}$	Pell HN; HP_n [11,12]	Pell-Lucas HN; HQ_n [11,12]
$\frac{k+\sqrt{k^2+4}}{2}$	$-\frac{1}{\alpha^2}$	k-Fibonacci HN; $HF_{k,n}$ [7]	k-Lucas HN; $HL_{k,n}$ [7]
2	$-\frac{1}{2}$	Jacobsthal HN; HJ_n [12,13]	Jacobsthal-Lucas HN; Hj_n [12,13]
$1+\sqrt{1+k}$	$-\frac{k}{\alpha^2}$	k-Pell HN; $HP_{k,n}$ [6]	k-Pell-Lucas HN; $HQ_{k,n}$

2 Preliminaries

In this part, we demonstrate some known definitions and properties with our paper. Throughout this article, we take $\alpha, q \in \mathbb{R}$, $1-q \neq 0$, $n \in \mathbb{N}$, $\hat{\alpha} = 1+\alpha \boldsymbol{i} + \alpha^2 \boldsymbol{\epsilon} + \alpha^3 \boldsymbol{h}$ and $\hat{\beta} = 1+(\alpha q)\boldsymbol{i} + (\alpha q)^2 \boldsymbol{\epsilon} + (\alpha q)^3 \boldsymbol{h}$. Kızılateş [9] defined hybrid q-Fibonacci numbers $HF_n(\alpha;q)$, hybrid q-Lucas numbers $HL_n(\alpha;q)$, and obtained some results related to these numbers as follows:

$$\begin{aligned} HF_n(\alpha;q) &= \alpha^{n-1}[n]_q + \alpha^n [n+1]_q \boldsymbol{i} + \alpha^{n+1}[n+2]_q \boldsymbol{\epsilon} \\ &\quad + \alpha^{n+2}[n+3]_q \boldsymbol{h}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} HL_n(\alpha;q) &= \alpha^n \frac{[2n]_q}{[n]_q} + \alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\boldsymbol{i} + \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\boldsymbol{\epsilon} \\ &\quad + \alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\boldsymbol{h}. \end{aligned} \tag{2.2}$$

The Binet formulas [9] for $HF_n(\alpha;q)$ and $HL_n(\alpha;q)$ are given as follows:

$$HF_n(\alpha;q) = \frac{\alpha^n \hat{\alpha} - (\alpha q)^n \hat{\beta}}{\alpha(1-q)}, \tag{2.3}$$

$$HL_n(\alpha;q) = \alpha^n \hat{\alpha} + (\alpha q)^n \hat{\beta}. \tag{2.4}$$

The Catalan's identities [9] for $HF_n(\alpha;q)$ and $HL_n(\alpha;q)$ are given as follows:

$$\begin{aligned} &HF_{n+r}(\alpha;q)\, HF_{n-r}(\alpha;q) - HF_n^2(\alpha;q) \\ &= \frac{\alpha^{2n-2} q^n (1-q^r)\left(\hat{\beta}\hat{\alpha} - q^{-r}\hat{\alpha}\hat{\beta}\right)}{(1-q)^2}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &HL_{n+r}(\alpha;q)\, HL_{n-r}(\alpha;q) - HL_n^2(\alpha;q) \\ &= \alpha^{2n} q^{n-r}\left(1-q^r\right)\left(\hat{\alpha}\hat{\beta} - q^r \hat{\beta}\hat{\alpha}\right). \end{aligned} \tag{2.6}$$

The d'Ocagne's identities [9] for $HF_n(\alpha;q)$ and $HL_n(\alpha;q)$ are given as follows:

$$\begin{aligned} &HF_m(\alpha;q)\, HF_{n+1}(\alpha;q) - HF_{m+1}(\alpha;q)\, HF_n(\alpha;q) \\ &= \frac{\alpha^{m+n-1}\left(q^n \hat{\alpha}\hat{\beta} - q^m \hat{\beta}\hat{\alpha}\right)}{1-q}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &HL_m(\alpha;q)\, HL_{n+1}(\alpha;q) - HL_{m+1}(\alpha;q)\, HL_n(\alpha;q) \\ &= \alpha^{m+n-1}(q-1)\left(q^n \hat{\alpha}\hat{\beta} - q^m \hat{\beta}\hat{\alpha}\right). \end{aligned} \tag{2.8}$$

Many researchers have investigated some special situations of hybrid q-Fibonacci numbers $HF_n(\alpha;q)$ and hybrid q-Lucas numbers $HL_n(\alpha;q)$. These situations are given in Table 2.

3 The hybrid q-Leonardo numbers

In this part, we introduce a new generalization of the known hybrid Leonardo numbers. We call such numbers as hybrid q-Leonardo numbers. We provide Binet formula and some other identities for the hybrid q-Leonardo numbers. Throughout this article, let $\hat{1} = 1 + \boldsymbol{i} + \boldsymbol{\epsilon} + \boldsymbol{h}$.

Definition 31 The q-Leonardo hybrid numbers are defined by

$$\begin{aligned} HLe_n(\alpha; q) = &\left(2\alpha^n [n+1]_q - 1\right) + \left(2\alpha^{n+1} [n+2]_q - 1\right) \boldsymbol{i} \\ &+ \left(2\alpha^{n+2} [n+3]_q - 1\right) \boldsymbol{\epsilon} + \left(2\alpha^{n+3} [n+4]_q - 1\right) \boldsymbol{h}. \end{aligned} \tag{3.1}$$

If we take $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = -\frac{1}{\alpha^2}$ in (3.1), then we have $HLe_n(\alpha; q) = HLe_n$. So, the hybrid q-Leonardo numbers are a generalization of the known Leonardo hybrid numbers.

Moreover, by (2.1) and (3.1), we can easily derive the following relationship between hybrid q-Leonardo numbers and hybrid q-Fibonacci numbers.

Corollary 31 *For $n \in \mathbb{N}$, we have*

$$HLe_n(\alpha; q) = 2HF_{n+1}(\alpha; q) - \hat{1}.$$

Theorem 31 *The Binet formula for the hybrid q-Leonardo numbers $HLe_n(\alpha; q)$ is*

$$HLe_n(\alpha; q) = 2\left(\frac{\alpha^{n+1}\hat{\alpha} - (\alpha q)^{n+1}\hat{\beta}}{\alpha(1-q)}\right) - \hat{1}. \tag{3.2}$$

Proof By (3.1) and (1.1), we obtain

$$\begin{aligned} HLe_n(\alpha; q) = &\left(2\alpha^n [n+1]_q - 1\right) + \left(2\alpha^{n+1} [n+2]_q - 1\right) \boldsymbol{i} \\ &+ \left(2\alpha^{n+2} [n+3]_q - 1\right) \boldsymbol{\epsilon} + \left(2\alpha^{n+3} [n+4]_q - 1\right) \boldsymbol{h} \\ = &\left(2\alpha^{n+1}\frac{1-q^{n+1}}{\alpha(1-q)} - 1\right) + \left(2\alpha^{n+2}\frac{1-q^{n+2}}{\alpha(1-q)} - 1\right) \boldsymbol{i} \\ &+ \left(2\alpha^{n+3}\frac{1-q^{n+3}}{\alpha(1-q)} - 1\right) \boldsymbol{\epsilon} + \left(2\alpha^{n+4}\frac{1-q^{n+4}}{\alpha(1-q)} - 1\right) \boldsymbol{h} \\ = &\, 2\left(\frac{\alpha^{n+1}\left(1 + \alpha\boldsymbol{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\boldsymbol{h}\right)}{\alpha(1-q)}\right) \\ &- 2\left(\frac{(\alpha q)^{n+1}\left(1 + (\alpha q)\,\boldsymbol{i} + (\alpha q)^2\,\boldsymbol{\epsilon} + (\alpha q)^3\,\boldsymbol{h}\right)}{\alpha(1-q)}\right) \\ &- (1 + \boldsymbol{i} + \boldsymbol{\epsilon} + \boldsymbol{h}) \\ = &\, 2\left(\frac{\alpha^{n+1}\hat{\alpha} - (\alpha q)^{n+1}\hat{\beta}}{\alpha(1-q)}\right) - \hat{1}. \end{aligned}$$

Theorem 32 *The exponential generating function for the hybrid q-Leonardo numbers $HLe_n(\alpha; q)$ is*

$$\sum_{n=0}^{\infty} HLe_n(\alpha; q)\frac{t^n}{n!} = 2\left(\frac{\hat{\alpha}e^{\alpha t} - q\hat{\beta}e^{(\alpha q)t}}{1-q}\right) - \hat{1}\cdot e^t.$$

Proof From Binet formula for the hybrid q-Leonardo numbers in (3.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty} HLe_n(\alpha;q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left[2\left(\frac{\alpha^{n+1}\hat{\alpha}-(\alpha q)^{n+1}\hat{\beta}}{\alpha(1-q)}\right)-\hat{1}\right]\frac{t^n}{n!}\\ &= 2\frac{\alpha\hat{\alpha}}{\alpha(1-q)}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}-2\frac{\alpha q\hat{\beta}}{\alpha(1-q)}\sum_{n=0}^{\infty}\frac{(\alpha q t)^n}{n!}-\hat{1}\cdot\sum_{n=0}^{\infty}\frac{t^n}{n!}\\ &= 2\left(\frac{\hat{\alpha}e^{\alpha t}-q\hat{\beta}e^{(\alpha q)t}}{1-q}\right)-\hat{1}\cdot e^t.\end{aligned}$$

Theorem 33 *Let n and k be non-negative integer numbers. Then we get*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2 q\right)^{n-i} HLe_{2i+k}(\alpha;q)\\ &= \begin{cases}2\Delta^{\frac{n}{2}} HF_{n+k+1}(\alpha;q)-\left(1-\alpha^2 q\right)^n\cdot\hat{1}, & if\ n\ is\ even\\ 2\Delta^{\frac{n-1}{2}} HL_{n+k+1}(\alpha;q)-\left(1-\alpha^2 q\right)^n\cdot\hat{1}, & if\ n\ \ is\ odd\end{cases},\\ &\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(-\alpha^2 q\right)^{n-i} HLe_{2i+k}(\alpha;q)\\ &= 2\left(-\alpha[2]_q\right)^n HF_{n+k+1}(\alpha;q)+(-1)^{n+1}\left(1+\alpha^2 q\right)^n\cdot\hat{1},\end{aligned}$$

where $\Delta=(\alpha-\alpha q)^2$.

Proof From Binet formula for the hybrid q-Leonardo numbers in (3.2), we find

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2 q\right)^{n-i} HLe_{2i+k}(\alpha;q)\\ &= \sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2 q\right)^{n-i}\left[2\left(\frac{\alpha^{2i+k+1}\hat{\alpha}-(\alpha q)^{2i+k+1}\hat{\beta}}{\alpha(1-q)}\right)-\hat{1}\right]\\ &= \frac{2}{\alpha(1-q)}\left[\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2 q\right)^{n-i}\left(\alpha^2\right)^i\alpha^{k+1}\hat{\alpha}\right]\\ &\quad+\frac{2}{\alpha(1-q)}\left[-\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2 q\right)^{n-i}\left((\alpha q)^2\right)^i(\alpha q)^{k+1}\hat{\beta}\right]\\ &\quad-\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2 q\right)^{n-i}\cdot\hat{1}\\ &= 2\left[\frac{\left(\alpha^2-\alpha^2 q\right)^n\alpha^{k+1}\hat{\alpha}-\left((\alpha q)^2-\alpha^2 q\right)^n(\alpha q)^{k+1}\hat{\beta}}{\alpha(1-q)}\right]-\left(1-\alpha^2 q\right)^n\cdot\hat{1}\\ &= 2\left[\frac{\left(\alpha\sqrt{\Delta}\right)^n\alpha^{k+1}\hat{\alpha}-\left(-\alpha q\sqrt{\Delta}\right)^n(\alpha q)^{k+1}\hat{\beta}}{\alpha(1-q)}\right]-\left(1-\alpha^2 q\right)^n\cdot\hat{1}\\ &= \left(\sqrt{\Delta}\right)^n 2\left[\frac{\alpha^{n+k+1}\hat{\alpha}+(-1)^{n+1}(\alpha q)^{n+k+1}\hat{\beta}}{\alpha(1-q)}\right]-\left(1-\alpha^2 q\right)^n\cdot\hat{1}.\end{aligned}\tag{3.3}$$

By (3.3), (3.2) and (2.3), if n is even, then we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2q\right)^{n-i}HLe_{2i+k}(\alpha;q) &= \left(\sqrt{\Delta}\right)^n 2\left[\frac{\alpha^{n+k+1}\hat{\alpha}-(\alpha q)^{n+k+1}\hat{\beta}}{\alpha(1-q)}\right]\\ &\quad-\left(1-\alpha^2q\right)^n\cdot\hat{1}\\ &= 2\Delta^{\frac{n}{2}}HF_{n+k+1}(\alpha;q)-\left(1-\alpha^2q\right)^n\cdot\hat{1}.\end{aligned}$$

By (3.3), (3.2) and (2.4), if n is odd, then we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2q\right)^{n-i}HLe_{2i+k}(\alpha;q) &= \left(\sqrt{\Delta}\right)^n 2\left[\frac{\alpha^{n+k+1}\hat{\alpha}+(\alpha q)^{n+k+1}\hat{\beta}}{\sqrt{\Delta}}\right]\\ &\quad-\left(1-\alpha^2q\right)^n\cdot\hat{1}\\ &= \left(\sqrt{\Delta}\right)^{n-1}2\left(\alpha^{n+k+1}\hat{\alpha}+(\alpha q)^{n+k+1}\hat{\beta}\right)\\ &\quad-\left(1-\alpha^2q\right)^n\cdot\hat{1}\\ &= 2\Delta^{\frac{n-1}{2}}HL_{n+k+1}(\alpha;q)-\left(1-\alpha^2q\right)^n\cdot\hat{1}.\end{aligned}$$

On the other hand, from (3.2), (1.1) and (2.3), we get

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(-\alpha^2q\right)^{n-i}HLe_{2i+k}(\alpha;q)\\ &=\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2q\right)^{n-i}(-1)^i\left[2\left(\frac{\alpha^{2i+k+1}\hat{\alpha}-(\alpha q)^{2i+k+1}\hat{\beta}}{\alpha(1-q)}\right)-\hat{1}\right]\\ &=\frac{2}{\alpha(1-q)}\left[\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2q\right)^{n-i}\left(-\alpha^2\right)^i\alpha^{k+1}\hat{\alpha}\right]\\ &\quad+\frac{2}{\alpha(1-q)}\left[-\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2q\right)^{n-i}\left(-(\alpha q)^2\right)^i(\alpha q)^{k+1}\hat{\beta}\right]\\ &\quad-\sum_{i=0}^{n}\binom{n}{i}\left(-\alpha^2q\right)^{n-i}(-1)^i\cdot\hat{1}\\ &=2\left[\frac{\left(-\alpha^2-\alpha^2q\right)^n\alpha^{k+1}\hat{\alpha}-\left(-(\alpha q)^2-\alpha^2q\right)^n(\alpha q)^{k+1}\hat{\beta}}{\alpha(1-q)}\right]-\left(-1-\alpha^2q\right)^n\cdot\hat{1}\\ &=(-\alpha)^n(1+q)^n2\left[\frac{\alpha^{n+k+1}\hat{\alpha}-(\alpha q)^{n+k+1}\hat{\beta}}{\alpha(1-q)}\right]-\left(-1-\alpha^2q\right)^n\cdot\hat{1}\\ &=2\left(-\alpha[2]_q\right)^nHF_{n+k+1}(\alpha;q)+(-1)^{n+1}\left(1+\alpha^2q\right)^n\cdot\hat{1}.\end{aligned}$$

Theorem 34 *For $n\in\mathbb{N}$, we have*

$$\begin{aligned}&\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i}\left(-\alpha^2q\right)^{n-2i}HLe_{4i}(\alpha;q)\\ &=\begin{cases}\left(\Delta^{\frac{n}{2}}+\left(\alpha[2]_q\right)^n\right)HF_{n+1}(\alpha;q)\\ -\frac{1}{2}\left(\left(1-\alpha^2q\right)^n+\left(1+\alpha^2q\right)^n\right)\cdot\hat{1}, & if\ n\ is\ even\\ \Delta^{\frac{n-1}{2}}HL_{n+1}(\alpha;q)-\left(\alpha[2]_q\right)^nHF_{n+1}(\alpha;q)\\ -\frac{1}{2}\left(\left(1-\alpha^2q\right)^n-\left(1+\alpha^2q\right)^n\right)\cdot\hat{1}, & if\ n\ \ is\ odd\end{cases},\end{aligned}$$

$$\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} \left(-\alpha^2 q\right)^{n-(2i+1)} HLe_{4i+1}(\alpha;q) = \begin{cases} \left(\Delta^{\frac{n}{2}} - \left(\alpha [2]_q\right)^n\right) HF_n(\alpha;q) \\ -\frac{1}{2}\left(\left(1-\alpha^2 q\right)^n - \left(1+\alpha^2 q\right)^n\right)\cdot \hat{1}, & if\ n\ is\ even \\ \Delta^{\frac{n-1}{2}} HL_n(\alpha;q) + \left(\alpha [2]_q\right)^n HF_n(\alpha;q) \\ -\frac{1}{2}\left(\left(1-\alpha^2 q\right)^n + \left(1+\alpha^2 q\right)^n\right)\cdot \hat{1}, & if\ n\ \ is\ odd \end{cases},$$

where $\Delta = (\alpha - \alpha q)^2$.

Proof By Theorem 33, we get

$$\begin{aligned} &\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2i} \left(-\alpha^2 q\right)^{n-2i} HLe_{4i}(\alpha;q) \\ &= \frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \left(1+(-1)^i\right) \left(-\alpha^2 q\right)^{n-i} HLe_{2i}(\alpha;q) \\ &= \frac{1}{2}\left[\sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} HLe_{2i}(\alpha;q)\right] \\ &\quad + \frac{1}{2}\left[\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(-\alpha^2 q\right)^{n-i} HLe_{2i}(\alpha;q)\right] \\ &= \begin{cases} \left(\Delta^{\frac{n}{2}} + \left(\alpha [2]_q\right)^n\right) HF_{n+1}(\alpha;q) \\ -\frac{1}{2}\left(\left(1-\alpha^2 q\right)^n + \left(1+\alpha^2 q\right)^n\right)\cdot \hat{1}, & if\ n\ is\ even \\ \Delta^{\frac{n-1}{2}} HL_{n+1}(\alpha;q) - \left(\alpha [2]_q\right)^n HF_{n+1}(\alpha;q) \\ -\frac{1}{2}\left(\left(1-\alpha^2 q\right)^n - \left(1+\alpha^2 q\right)^n\right)\cdot \hat{1}, & if\ n\ \ is\ odd \end{cases}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} \left(-\alpha^2 q\right)^{n-(2i+1)} HLe_{4i+1}(\alpha;q) \\ &= \frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \left(1+(-1)^{i+1}\right) \left(-\alpha^2 q\right)^{n-i} HLe_{2i-1}(\alpha;q) \\ &= \frac{1}{2}\left[\sum_{i=0}^{n} \binom{n}{i} \left(-\alpha^2 q\right)^{n-i} HLe_{2i-1}(\alpha;q)\right] \\ &\quad + \frac{1}{2}\left[-\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(-\alpha^2 q\right)^{n-i} HLe_{2i-1}(\alpha;q)\right] \\ &= \begin{cases} \left(\Delta^{\frac{n}{2}} - \left(\alpha [2]_q\right)^n\right) HF_n(\alpha;q) \\ -\frac{1}{2}\left(\left(1-\alpha^2 q\right)^n - \left(1+\alpha^2 q\right)^n\right)\cdot \hat{1}, & if\ n\ is\ even \\ \Delta^{\frac{n-1}{2}} HL_n(\alpha;q) + \left(\alpha [2]_q\right)^n HF_n(\alpha;q) \\ -\frac{1}{2}\left(\left(1-\alpha^2 q\right)^n + \left(1+\alpha^2 q\right)^n\right)\cdot \hat{1}, & if\ n\ \ is\ odd \end{cases}. \end{aligned}$$

Theorem 35 *For $n \in \mathbb{N}$, we get*

$$\sum_{i=0}^{n} \binom{n}{i} (\alpha (1+q))^i \left(-\alpha^2 q\right)^{n-i} HLe_i(\alpha; q) = HLe_{2n}(\alpha; q) + \hat{1} - \left(\alpha (1+q) - \alpha^2 q\right)^n \cdot \hat{1}.$$

Proof By Binet formula for the hybrid q-Leonardo numbers in (3.2), we obtain

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (\alpha (1+q))^i \left(-\alpha^2 q\right)^{n-i} HLe_i(\alpha; q) \\
&= \sum_{i=0}^{n} \binom{n}{i} (\alpha (1+q))^i \left(-\alpha^2 q\right)^{n-i} \left[2\left(\frac{\alpha^{i+1}\hat{\alpha} - (\alpha q)^{i+1}\hat{\beta}}{\alpha (1-q)}\right) - \hat{1}\right] \\
&= \frac{2}{\alpha (1-q)} \left[\sum_{i=0}^{n} \binom{n}{i} \left(\alpha^2 (1+q)\right)^i \left(-\alpha^2 q\right)^{n-i} \alpha\hat{\alpha}\right] \\
&\quad + \frac{2}{\alpha (1-q)} \left[-\sum_{i=0}^{n} \binom{n}{i} \left(\alpha^2 q (1+q)\right)^i \left(-\alpha^2 q\right)^{n-i} (\alpha q)\hat{\beta}\right] \\
&\quad - \sum_{i=0}^{n} \binom{n}{i} (\alpha (1+q))^i \left(-\alpha^2 q\right)^{n-i} \cdot \hat{1} \\
&= \frac{2}{\alpha (1-q)} \left(\alpha^{2n+1}\hat{\alpha} - (\alpha q)^{2n+1}\hat{\beta}\right) - \left(\alpha (1+q) - \alpha^2 q\right)^n \cdot \hat{1} \\
&= HLe_{2n}(\alpha; q) + \hat{1} - \left(\alpha (1+q) - \alpha^2 q\right)^n \cdot \hat{1}.
\end{aligned}$$

Theorem 36 ***(Catalan's Identity)*** *Let $n, r \in \mathbb{Z}^+$ be such that $n \geq r$. Then we get*

$$\begin{aligned}
&HLe_{n+r}(\alpha; q)\, HLe_{n-r}(\alpha; q) - HLe_n^2(\alpha; q) \\
&= 4\left(\frac{\alpha^{2n} q^{n+1} (1-q^r)\left(\hat{\beta}\hat{\alpha} - q^{-r}\hat{\alpha}\hat{\beta}\right)}{(1-q)^2}\right) + (HLe_n(\alpha; q) - HLe_{n+r}(\alpha; q)) \cdot \hat{1} \\
&\quad + \hat{1} \cdot (HLe_n(\alpha; q) - HLe_{n-r}(\alpha; q)).
\end{aligned}$$

Proof By Corollary 31 and (2.5), we find

$$\begin{aligned}
&HLe_{n+r}(\alpha; q)\, HLe_{n-r}(\alpha; q) - HLe_n^2(\alpha; q) \\
&= \left(2HF_{n+r+1}(\alpha; q) - \hat{1}\right)\left(2HF_{n-r+1}(\alpha; q) - \hat{1}\right) - \left(2HF_{n+1}(\alpha; q) - \hat{1}\right)^2 \\
&= 4\left(HF_{n+r+1}(\alpha; q)\, HF_{n-r+1}(\alpha; q) - HF_{n+1}^2(\alpha; q)\right) \\
&\quad + (2HF_{n+1}(\alpha; q) - 2HF_{n+r+1}(\alpha; q)) \cdot \hat{1} \\
&\quad + \hat{1} \cdot (2HF_{n+1}(\alpha; q) - 2HF_{n-r+1}(\alpha; q)) \\
&= 4\left(\frac{\alpha^{2n} q^{n+1} (1-q^r)\left(\hat{\beta}\hat{\alpha} - q^{-r}\hat{\alpha}\hat{\beta}\right)}{(1-q)^2}\right) + (HLe_n(\alpha; q) - HLe_{n+r}(\alpha; q)) \cdot \hat{1} \\
&\quad + \hat{1} \cdot (HLe_n(\alpha; q) - HLe_{n-r}(\alpha; q)).
\end{aligned}$$

In Theorem 36, if we take $r = 1$, then we can derive Cassini's identity for q-Leonardo hybrid numbers in the following result.

Corollary 32 ***(Cassini's Identity)*** *For $n \geq 1$, the following equality holds:*

$$\begin{aligned} &HLe_{n+1}(\alpha;q)\,HLe_{n-1}(\alpha;q) - HLe_n^2(\alpha;q) \\ &= 4\left(\frac{\alpha^{2n}q^{n+1}\left(\hat{\beta}\hat{\alpha} - q^{-1}\hat{\alpha}\hat{\beta}\right)}{1-q}\right) + (HLe_n(\alpha;q) - HLe_{n+1}(\alpha;q))\cdot\hat{1} \\ &\quad +\hat{1}\cdot(HLe_n(\alpha;q) - HLe_{n-1}(\alpha;q)). \end{aligned}$$

Theorem 37 ***(d'Ocagne's Identity)*** *For a non-negative integer n and a positive integer m such that $m > n+1$, we get*

$$\begin{aligned} &HLe_m(\alpha;q)\,HLe_{n+1}(\alpha;q) - HLe_{m+1}(\alpha;q)\,HLe_n(\alpha;q) \\ &= 4\left(\frac{\alpha^{m+n+1}\left(q^{n+1}\hat{\alpha}\hat{\beta} - q^{m+1}\hat{\beta}\hat{\alpha}\right)}{1-q}\right) - (HLe_m(\alpha;q) - HLe_{m+1}(\alpha;q))\cdot\hat{1} \\ &\quad +\hat{1}\cdot(HLe_n(\alpha;q) - HLe_{n+1}(\alpha;q)). \end{aligned}$$

Proof By Corollary 31 and (2.7), we obtain

$$\begin{aligned} &HLe_m(\alpha;q)\,HLe_{n+1}(\alpha;q) - HLe_{m+1}(\alpha;q)\,HLe_n(\alpha;q) \\ &= \left(2HF_{m+1}(\alpha;q) - \hat{1}\right)\left(2HF_{n+2}(\alpha;q) - \hat{1}\right) \\ &\quad - \left(2HF_{m+2}(\alpha;q) - \hat{1}\right)\left(2HF_{n+1}(\alpha;q) - \hat{1}\right) \\ &= 4\left(HF_{m+1}(\alpha;q)\,HF_{n+2}(\alpha;q) - HF_{m+2}(\alpha;q)\,HF_{n+1}(\alpha;q)\right) \\ &\quad -2\left(HF_{m+1}(\alpha;q) - HF_{m+2}(\alpha;q)\right)\cdot\hat{1} - \hat{1}\cdot 2\left(HF_{n+2}(\alpha;q) - HF_{n+1}(\alpha;q)\right) \\ &= 4\left(\frac{\alpha^{m+n+1}\left(q^{n+1}\hat{\alpha}\hat{\beta} - q^{m+1}\hat{\beta}\hat{\alpha}\right)}{1-q}\right) - (HLe_m(\alpha;q) - HLe_{m+1}(\alpha;q))\cdot\hat{1} \\ &\quad +\hat{1}\cdot(HLe_n(\alpha;q) - HLe_{n+1}(\alpha;q)). \end{aligned}$$

4 Conclusions

Alp and Koçer defined the hybrid Leonardo numbers in [1]. By means of the q-calculus, we generalize the hybrid Leonardo numbers, and call such numbers as the q-Leonardo hybrid numbers. Moreover, Kızılateş introduced the hybrid q-Fibonacci numbers and the hybrid q-Lucas numbers in [9]. We present some relations among the hybrid q-Fibonacci, the hybrid q-Lucas and the hybrid q-Leonardo numbers. Then, we demonstrate Binet formula, exponential generating function, summation formulas, Catalan's identity, Cassini's identity and d'Ocagne's identity for the hybrid q-Leonardo numbers.

References

1. Alp, Y., Koçer, E.G.: Hybrid Leonardo numbers. *Chaos, Solitons and Fractals* **150**, 111-128 (2021)
2. Alp, Y., Koçer, E.G.: Some properties of Leonardo numbers. *Konuralp Journal of Mathematics* **9**(1), 183-189 (2021)
3. Andrews, G.E., Askey. R., Roy, R.: *Special functions*. New York: Cambridge Univ. Press, (1999)
4. Catarino. P., Borges. A.: A note on incomplete Leonardo numbers. *Integers* **20**(7), (2020)
5. Catarino, P., Borges, A.: On Leonardo numbers. *Acta Mathematica Universitatis Comenianae* **89**(1), 75-86 (2019)
6. Catarino, P.: On k-Pell hybrid numbers. *Journal of Discrete Mathematical Sciences and Cryptography* **22**(1), 83-89 (2019)
7. Erkan, E., Dağdeviren, A.: k-Fibonacci and k-Lucas Hybrid Numbers. *Tamap Journal of Mathematics and Statistics* (2021). https://doi.org/10.29371/2021.16.125
8. Kac, V., Cheung, P.: *Quantum Calculus Universitext*. New York: Springer-Verlag, (2002)
9. Kızılateş, C.: A new generalization of Fibonacci hybrid and Lucas hybrid numbers. *Chaos, Solitons and Fractals* **130**, 109449 (2020)

10. Shannon, A.G.: A note on generalized Leonardo numbers. *Note on Number Theory and Discrete Mathematics* **25**(3), 97-101 (2019)
11. Szynal-Liana, A., Wloch, I.: On Pell and Pell-Lucas hybrid numbers. *Commentationes Mathematicae* **58**, 11-17 (2018)
12. Szynal-Liana, A.: The horadam hybrid numbers. *Discuss. Math. Gen. Algebra Appl.* **38**(1), 91-98 (2018)
13. Szynal-Liana A, Wloch I. On Jacobsthal and Jacobsthal-Lucas hybrid numbers. *Annales Mathematicae Silesianae* **33**(1), 276-283 (2019)
14. Özdemir, M.: Introduction to hybrid numbers. *Adv. Appl. Clifford Algebras* **28**:11 (2018) https://doi.org/10.1007/s00006-018-0833-3
15. Vieira, R.P.M., Alves, F.R.V., Catarino, P.M.M.C.: *Relacoes bidimensionais e identidades da sequencia de Leonardo. Revista Sergipana de Matematica e Educacao Matematica* **4**(2), 156-173 (2019)

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