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ORIGINAL RESEARCH

On generalized weighted dynamic inequalities for diamond $-\propto$ integral on time scales calculus

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Abstract In this paper, we will prove a new version of generalized exponentially weighted dynamic inequalities of Hardy's type for diamond $-\propto$ integral on time scales calculus. Our results are a new version of some published works earlier on generalized Hardy's type integral inequalities.

Keywords Hardy type inequality · Time scales · Diamond integral

Mathematics Subject Classification 47B38 · 34N05 · 35A23 · 26A15

1 Introduction

The discrete Hardy inequality was proved by G.H. Hardy in [1].

$$\sum_{d=1}^{\infty}\left(\frac{c_1+c_2+c_3+\ldots+c_d}{d}\right)^q<\left(\frac{q}{q-1}\right)^q\sum_{d=1}^{\infty}c_d^q \tag{1}$$

where $q>1$, $d\geq 1$, $q\in\mathbb{R}$ and (c_d) is a sequence of non-negative real numbers.

The classical Hardy inequality was proved by G.H. Hardy in [2].

$$\int_0^{\infty}\left(\frac{1}{s}\int_0^{s}g(\tau)d\tau\right)^q ds\leq\left(\frac{q}{q-1}\right)^q\int_0^{\infty}g^q(s)ds \tag{2}$$

where $g(\tau)$ is a positive integrable function over any finite interval $(0,s)$, g^q is an integrable convergent function over $\mathbb{R}^+$, and $q>1$. In [2], Hardy generalized inequality (1) and proved that if $q>1, g(m)\geq 0, v(m)>0$ and $W(m)=\sum_{c=1}^{m}v(c)$, then

$$\sum_{m=1}^{\infty}v(m)\left(\frac{1}{W(m)}\sum_{c=1}^{m}v(c)g(c)\right)^q\leq\left(\frac{q}{q-1}\right)^q\sum_{m=1}^{m}v(m)g^q(m). \tag{3}$$

For further generalizations, extensions, and applications of these inequalities, we refer the reader to the monographs [3–12].

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In [13], Copson generalized inequality (3) and proved that if $1<c\leq q$, $v(m)>0$, $m\geq 1$ and $g(m)\geq 0$, then

$$\sum_{m=1}^{\infty}\frac{v(m)}{W^{c}(m)}\left(\sum_{k=1}^{m}v(k)g(k)\right)^{q}\leq\left(\frac{q}{c-1}\right)^{q}\sum_{m=1}^{m}v(m)g^{q}(m)W^{q-c}(m). \tag{4}$$

In [14], Copson proved a continuous version of inequality (4) and proved that if $c>1, q\geq 1$, then

$$\int_{0}^{a}\frac{v(s)K^{q}(s)}{W^{c}(s)}ds\leq\left(\frac{q}{c-1}\right)^{q}\int_{0}^{a}v(s)g^{q}(s)W^{q-c}(s)ds, \tag{5}$$

where $W(s)=\int_{0}^{s}v(x)dx$, and $K(s)=\int_{0}^{s}v(x)g(x)dx$.

In [15], Mohapatra and Vajravelu created new extensions of inequality (5). They proved that if $v(s)$ is positive and continuous in $[0,\infty)$ and there exist positive constants I_1 and I_2 such that

$$\begin{cases}\left|v^{'}(s)\right|\leq I_1 v(s)s^{-1}\\ v(s)\leq I_2 W(s)s^{-1}\end{cases} \tag{6}$$

for all $s>0$, then

$$\frac{1}{(I_1+I_2)^{q}}\int_{0}^{\infty}\left(\frac{K(s)}{W(s)}\right)^{q}v(s)ds\leq\int_{0}^{\infty}\left(\frac{1}{s}\int_{0}^{s}v^{1/q}(x)g(x)dx\right)^{q}ds \tag{7}$$

for $q>1$ and $g(s)\geq 0$. Also, they proved that if (6) holds and $v^{1/q}(x)g(x)x^{-1}\in L^{1}(s,\infty)$ for all $s>0$, then

$$\frac{1}{(qI+I_2)^{q}}\int_{0}^{\infty}\left(\int_{s}^{\infty}\frac{v(x)g(x)}{W(x)}\right)^{q}v(s)ds\leq\int_{0}^{\infty}\left(\int_{s}^{\infty}v^{1/q}(x)g(x)x^{-1}dx\right)^{q}ds, \tag{8}$$

where $I=I_1+I_2+1$.

Recently, the study of dynamic integral inequalities on time scales has received much attention. Dynamic inequalities and equations have played an important role in differential calculus, difference calculus, and quantum calculus. It has also gained an important place in pure and applied mathematics. One reason for this increased interest is that discrete operators can even behave differently from their counterparts (see [16, 17]). Some authors have tried to find some discrete results in different areas of analysis. In this study, we will focus on finding the discrete versions of inequalities (7) and (8). For more details on dynamic integral inequalities on time scales, we refer the reader to the monographs [18–25, 38–41]. Although there are many studies on dynamic integral inequalities on time scales, the number of studies on Hardy type is very few. Therefore, we will try to prove a new version of generalized cases of weighted dynamic inequalities on time scales via diamond $-\propto$ integral that include discrete versions of inequalities (7) and (8) in particular cases.

In [26], Saker et al. proved that if $1<c\leq q$, $\mu,h\in C_{rd}([b,\infty)\cap\mathbb{T},(0,\infty))$, $x\in([b,\infty)\cap\mathbb{T})$ and $b\in[0,\infty)$, then

$$\left(\frac{c-1}{q}\right)^{q}\int_{b}^{\infty}\frac{(\Gamma^{\sigma}(x))^{q}}{(\mathrm{W}^{\sigma}(x))^{c}}\mu(x)\Delta x\leq\int_{b}^{\infty}\frac{(\mathrm{W}^{\sigma}(x))^{c(q-1)}}{(W(x))^{q(c-1)}}h^{q}(x)\mu(x)\Delta x,$$

where $W(x)=\int_{b}^{x}\mu(t)\Delta t$, and $\Gamma(x)=\int_{b}^{x}\mu(t)h(t)\Delta t$.

In [12], authors proved that if $q>1$, $0\leq c<1$, $b\in[0,\infty)$, $x\in[b,\infty)\cap\mathbb{T}$ and $\mu,h\in C_{rd}([b,\infty)\cap\mathbb{T},(0,\infty))$, then

$$\left(\frac{1-c}{q}\right)^{q}\int_{b}^{\infty}\frac{\left(\widetilde{\Gamma}(x)\right)^{q}}{(\mathrm{W}^{\sigma}(x))^{c}}\mu(x)\Delta x\leq\int_{b}^{\infty}(\mathrm{W}^{\sigma}(x))^{(q-c)}h^{q}(x)\mu(x)\Delta x,$$

where $W(x)=\int_{b}^{x}\mu(t)\Delta t$, and $\widetilde{\Gamma}(x)=\int_{x}^{\infty}\mu(t)h(t)\Delta t$.

In [27], Saker et al. proved a new version more generalized weighted dynamic integral inequalities of Hardy's type on time scales.

Our main purpose in this article is to prove a new version of the generalized exponential-weighted dynamic inequalities of Hardy type on time scale $\mathbb{T}$ with the diamond- $\propto$ integral, and to derive discrete versions of inequalities (7) and (8) in certain cases. The results obtained in this paper have numerous applications in examining the oscillation of dynamical classes of equations on time scales. For details, we refer the reader to the monographs [28, 29].

The remainder of the article is organized as follows. In Sect. 2, we will present some preliminary information on the theory of time scales and illustrate the key propositions that will be needed to prove the main conclusions. In Sect. 3, we will examine the discrete versions of inequalities (7) and (8) in certain cases by means of the diamond- $\propto$ integral on time scale $\mathbb{T}$ and prove the main theorems. In the last section, we will discuss the results.

2 Mathematical background and preliminaries

The pioneer of the development of time scales theory was Stefan Hilger in [30]. Since then, many authors have studied dynamic integral inequalities over time scales, with very important findings. These studies attracted a lot of attention. It has become an important field of study not only in pure and applied mathematics but also in other fields of science. For more detailed information, we refer the reader to the monographs [20, 31, 32, 36, 37, 42].

A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Let $[a, b]$ be an arbitrary closed interval on time scale $\mathbb{T}$. The time scale interval $[a, b]_{\mathbb{T}}$ is denoted by $[a, b] \cap \mathbb{T}$.

Definition 1 [31] The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) = sup\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$. $\sigma(t)$ is forward jump operator, and $\rho(t)$ is backward jump operator, respectively. If $\sigma(t) > t$, then t is right-scattered, and if $\sigma(t) = t$, then t is called right-dense. If $\rho(t) < t$, then t is left-scattered, and if $\rho(t) = t$, then t is called left-dense.

Definition 2 [31] Let two mappings $\mu, \vartheta : \mathbb{T} \to \mathbb{R}^+$ such that $\mu(t) = \sigma(t) - t, \vartheta(t) = t - \rho(t)$. $\mu(t)$ and $\vartheta(t)$ are called graininess mappings.

If time scale $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$. In [20, 31, 32], $\mathbb{T}^k$ is defined as follows

$$\mathbb{T}^k = \begin{cases} T \setminus (\rho sup\mathbb{T}, sup\mathbb{T}], if sup\mathbb{T} < \infty \\ T, if sup\mathbb{T} = \infty, \end{cases}$$

and

$$\mathbb{T}_k = \begin{cases} T \setminus [inf\mathbb{T}, \sigma(inf\mathbb{T})], |inf\mathbb{T}| < \infty \\ T, inf\mathbb{T} = -\infty. \end{cases}$$

Assume that $h : \mathbb{T} \to \mathbb{R}$ is a function. Let t be right-dense.

(i) If h is $\Delta-$ differentiable at point t ($t \in \mathbb{T}^k (t \neq min\mathbb{T})$), then h is continuous at point t.

(ii) If h is left continuous at point t, and t is right-scattered, then h is $\Delta-$ differentiable at point t,

$$h^{\Delta}(t) = \frac{h^{\sigma}(t) - h(t)}{\mu(t)}$$

(iii) If h is $\Delta-$ differentiable at point t and $\lim_{s \to t} \frac{h(t)-h(s)}{t-s}$, then

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s}.$$

(iv) If h is $\Delta-$ differentiable at point t, then $h^{\sigma}(t) = h(t) + \mu(t)h^{\Delta}(t)$.

Remark 1 [20] If $\mathbb{T} = \mathbb{R}$, then $h^{\Delta}(t) = h'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $h^{\Delta}(t)$ reduces to $\Delta h(t)$.

Definition 3 [32] If $H:\mathbb{T}\to\mathbb{R}$ is defined a $\Delta-$(Cauchy) antiderivative of $h:\mathbb{T}\to\mathbb{R}$, then $H^{\Delta}=h(t)$ holds for all $t\in\mathbb{T}$, and we define the $\Delta-$ integral of h by.

$$\int_s^t h(\tau)\Delta\tau = H(t)-H(s),$$

for all $s,t\in\mathbb{T}$.

We now give similar definitions for the $\nabla-$(nabla) integral.

Definition 4 [32] Let $h:\mathbb{T}_k\to\mathbb{R}$ is called $\nabla-$ differentiable at $t\in\mathbb{T}_k$. If $\varepsilon>0$, then there exists a neighborhood V of t such that the following inequality is provided.

$$\left|h(\rho(t))-h(s)-h^{\nabla}(t)(\rho(t)-s)\right|\le\varepsilon|\rho(t)-s|,$$

for all $s\in V$.

Definition 5 [32] Let $H:\mathbb{T}\to\mathbb{R}$ is called a $\nabla-$ antiderivative of $h:\mathbb{T}\to\mathbb{R}$, then we define.

$$\int_s^t h(\tau)\nabla\tau = H(t)-H(s),$$

for all $s,t\in\mathbb{T}$. If $f(t)$ is $\diamond_{\alpha}-$ differentiable on $\mathbb{T}$ for all $\alpha,t\in\mathbb{T}$, then we can define $f^{\diamond_{\alpha}}(t)$ by

$$f^{\diamond_{\alpha}}(t)=\alpha f^{\Delta}(t)+(1-\alpha)f^{\nabla}(t)$$

for $0\le\alpha\le1$.

Proposition 1 *[32] If $f,h:\mathbb{T}\to\mathbb{R}$ are $\diamond_{\alpha}-$ differentiable for all $\alpha,t\in\mathbb{T}$ with $0\le\alpha\le1$, then.*

(i) *$(f+h):\mathbb{T}\to\mathbb{R}$ is $\diamond_{\alpha}-$ differentiable for all $t\in\mathbb{T}$ with*

$$(f+h)^{\diamond_{\alpha}}(t)=f^{\diamond_{\alpha}}(t)+h^{\diamond_{\alpha}}(t).$$

(ii) *$kf:\mathbb{T}\to\mathbb{R}$ is $\diamond_{\alpha}-$ differentiable for all $\alpha,t\in\mathbb{T}$ with*

$$(kf)^{\diamond_{\alpha}}(t)=kf^{\diamond_{\alpha}}(t),$$

2.1 *where t, $k\in\mathbb{R}$.*

(iii) *$f,h:\mathbb{T}\to\mathbb{R}$ is $\diamond_{\alpha}-$ differentiable for all $\alpha,t\in\mathbb{T}$ with*

$$(fh)^{\diamond_{\alpha}}(t)=f^{\diamond_{\alpha}}(t)h(t)+\alpha f^{\sigma}(t)h^{\Delta}(t)+(1-\alpha)f^{\rho}(t)h^{\nabla}(t).$$

Definition 6 [32] If $f:\mathbb{T}\to\mathbb{R}$ is $\diamond_{\alpha}-$ integrable for all $\alpha,b,t\in\mathbb{T}$, then.

$$\int_b^t f(\delta)\diamond_{\alpha}\delta=\alpha\int_b^t f(\delta)\Delta\delta+(1-\alpha)\int_b^t f(\delta)\nabla\delta$$

for $0\le\alpha\le1$.

Definition 7 [33, 34] If $f\in C_{rd}(\mathbb{T},\mathbb{R})$, $t\in\mathbb{T}^k$ and $f:\mathbb{T}\to\mathbb{R}$ is $\diamond_{\alpha}-$ integrable, then.

$$\int_t^{\sigma(t)} f(\tau)\diamond_{\alpha}\tau=\mu(t)f(t).$$

The partial integration formula on the time scale is given by

$$\int_x^y u(s)w^{\diamond_\propto}(s)\diamond_\propto s = u(s)w(s)\Gamma_x^y - \int_x^y u^{\diamond_\propto}(s)w^{\sigma}(s)\diamond_\propto s$$

for $0 \leq \propto \leq 1$. In [27] Hölder inequality on time scale is given by

$$\int_x^y |u(s)w(s)|\diamond_\propto s \leq \left(\int_x^y |u(s)|^p \diamond_\propto s\right)^{\frac{1}{p}} . \left(\int_x^y |w(s)|^{p'} \diamond_\propto s\right)^{\frac{1}{p'}} \tag{9}$$

where $0 \leq \propto \leq 1$, $1/p + 1/p' = 1$, $p > 1$ and $u, w \in C_{rd}([x,y] \cap \mathbb{T}, \mathbb{R})$.

In [27], Minkowski's inequality on time scales is given by

$$\left(\int_x^y |h(s)||u(s)+w(s)|^p \diamond_\propto s\right)^{\frac{1}{p}} \leq \left(\int_x^y |h(s)||u(s)|^p \diamond_\propto s\right)^{\frac{1}{p}} + \left(\int_x^y |h(s)||w(s)|^p \diamond_\propto s\right)^{\frac{1}{p}} \tag{10}$$

where $0 \leq \propto \leq 1, p > 1$ and $h, u, w \in C_{rd}([x,y] \cap \mathbb{T}, \mathbb{R})$.

Lemma 1 [11] *Let $b, t \in \mathbb{T}$ such that $b \leq t$ and $0 \leq \propto \leq 1$. If $q \geq 1$, then.*

$$\left(\int_b^{\sigma(t)} h(s)\diamond_\propto s\right)^q \leq q \int_b^{\sigma(t)} h(s)\left(\int_b^{\sigma(s)} h(x)\diamond_\propto x\right)^{q-1} \diamond_\propto s.$$

Lemma 2 [11] *Let $b, t \in \mathbb{T}$ such that $t \leq b$ and $0 \leq \propto \leq 1$. If $q \geq 1$, then.*

$$\left(\int_t^b h(s)\diamond_\propto s\right)^q \leq q \int_b^b h(s)\left(\int_s^b h(x)\diamond_\propto x\right)^{q-1} \diamond_\propto s.$$

Lemma 3 [35] *Let non-negative and $rd-$ continuous functions f, F be defined on $[x,y] \cap \mathbb{T}$. If $x, y \in \mathbb{T}$, then.*

$$\int_x^y F(s)\left(\int_s^y f(t)\diamond_\propto s\right)\diamond_\propto t = \int_x^y f(s)\left(\int_x^{\sigma(s)} F(t)\diamond_\propto t\right)\diamond_\propto s,$$

where $0 \leq \propto \leq 1$.

3 Main Results

In this section, we will state and prove our theorems that assert the validity of the generalized exponential-weighted Hardy-type inequality on time scale calculus. Before coming to the proof of our results, we will assume that the functions in the statements of the theorems are $rd-$ continuous non-negative functions defined on $J = [b, \infty) \cap \mathbb{T}$, and the integrals considered are assumed to be finite.

Theorem 1 *Let $\mathbb{T}$ be a time scale with $b \in [0, \infty) \cap \mathbb{T}$, and let $\Gamma(s) = \frac{r}{(1-r)} \int_0^s \varphi(x)\diamond_\propto x$ be positive operator for $\Gamma : J \to (0, \infty), \varphi : J \to (0, \infty)$, $0 < r < 1$. Let $\varphi(s)$ be a positive function for all $s \in J$ and $0 \leq \propto \leq 1$ with I_1 and I_2 positive constants.*

$$\begin{cases} \frac{\sigma(s)|\varphi^{\diamond\propto}(s)|}{\varphi(s)} \leq I_1 \\ \frac{\sigma(s)\varphi^{\sigma}(s)}{\Gamma^{\sigma}(s)} \leq I_2. \end{cases} \tag{11}$$

Then the following inequality is valid

$$\frac{\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} \varphi(x)g(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s}{\int_b^\infty \left(\frac{1}{\sigma(s)}\int_b^{\sigma(s)} \varphi^{\frac{1}{q}}(x)g(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s} \le (I_1 + I_2)^q, \tag{12}$$

where $q > 1$ *with* $1/q + 1/q' = 1$.

Proof Let $h(s) = \int_b^s \varphi^{\frac{1}{q}}(x)g(x)\diamondsuit_\propto x$. In this case, $\diamondsuit_\propto-$ derivative of the function $h(s)$ will be $h^{\diamondsuit_\propto}(s) = \varphi^{\frac{1}{q}}(s)g(s)$. The left-hand side of (12) can be written in terms of h in the form.

$$\begin{aligned}
&\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} \varphi(x)g(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s \\
&= \int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} \varphi(x)h^{\diamondsuit_\propto}(x)\varphi^{-\frac{1}{q}}(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s \\
&= \int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} h^{\diamondsuit_\propto}(x)\varphi^{1/q'}(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s,
\end{aligned} \tag{13}$$

where $q > 1$ *with* $1/q + 1/q' = 1$.

If we integrate the term $\int_b^{\sigma(s)} h^{\diamondsuit_\propto}(x)\varphi^{1/q'}(x)\diamondsuit_\propto x$ by parts with

$$\begin{cases} u(x) = \varphi^{1/q'}(x) \\ w^{\diamondsuit_\propto}(x) = h^{\diamondsuit_\propto}(x), \end{cases}$$

we obtain

$$\begin{aligned}
\int_b^{\sigma(s)} h^{\diamondsuit_\propto}(x)\varphi^{1/q'}(x)\diamondsuit_\propto x &= h(x)\varphi^{1/q'}(x)\mathrm{I}_b^{\sigma(s)} - \int_b^{\sigma(s)} h^\sigma(x)\left(\varphi^{1/q'}(x)\right)^{\diamondsuit_\propto}\diamondsuit_\propto x \\
&= h^\sigma(s)(\varphi^\sigma(s))^{1/q'} - \int_b^{\sigma(s)} h^\sigma(x)\left(\varphi^{1/q'}(x)\right)^{\diamondsuit_\propto}\diamondsuit_\propto x.
\end{aligned} \tag{14}$$

If we consider the inequalities (13) and (14), then we get the following inequality

$$\begin{aligned}
&\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} \varphi(x)g(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s \\
&= \int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\left[h^\sigma(s)(\varphi^\sigma(s))^{1/q'} - \int_b^{\sigma(s)} h^\sigma(x)\left(\varphi^{1/q'}(x)\right)^{\diamondsuit_\propto}\diamondsuit_\propto x\right]\right)^q \diamondsuit_\propto s.
\end{aligned} \tag{15}$$

If we apply the inequality of (10) to the above inequality, then we get the following inequality

$$\begin{aligned}
\left(\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} \varphi(x)g(x)\diamondsuit_\propto x\right)^q \diamondsuit_\propto s\right)^{\frac{1}{q}} &\le \left(\int_b^\infty \varphi(s)\left(\frac{h^\sigma(s)(\varphi^\sigma(s))^{1/q'}}{\Gamma^\sigma(s)}\right)^q \diamondsuit_\propto s\right)^{\frac{1}{q}} \\
&+ \left(\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} h^\sigma(x)\left|\left(\varphi^{1/q'}(x)\right)^{\diamondsuit_\propto}\right|\diamondsuit_\propto x\right)^q \diamondsuit_\propto s\right)^{\frac{1}{q}}.
\end{aligned}$$

Applying the chain rule with $1/q + 1/q' = 1$, we obtain

$$\left(\varphi^{1/q'}(x)\right)^{\diamondsuit_\propto} \le \frac{1}{q'}\varphi^{\frac{1}{q'}-1}(x)\varphi^{\diamondsuit_\propto}(x) = \frac{1}{q'}\varphi^{-\frac{1}{q}}(x)\varphi^{\diamondsuit_\propto}(x). \tag{16}$$

By using (15), we obtain

$$\left(\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} \varphi(x)g(x)\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s\right)^{\frac{1}{q}} \le \left(\int_b^\infty \varphi(s)\left(\frac{h^\sigma(s)(\varphi^\sigma(s))^{1/q'}}{\Gamma^\sigma(s)}\right)^q \diamondsuit_\alpha s\right)^{\frac{1}{q}} + \frac{1}{q'}\left(\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} h^\sigma(x)\varphi^{-\frac{1}{q}}(x)\left|\varphi^{\diamondsuit_\alpha}(x)\right|\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s\right)^{\frac{1}{q}}. \tag{17}$$

Now by using (11), we have

$$\int_b^\infty \varphi(s)\left(\frac{h^\sigma(s)(\varphi^\sigma(s))^{1/q'}}{\Gamma^\sigma(s)}\right)^q \diamondsuit_\alpha s \le \int_b^\infty \varphi^\sigma(s)\left(\frac{I_2 h^\sigma(s)(\varphi^\sigma(s))^{-\frac{1}{q}}}{\sigma(s)}\right)^q \diamondsuit_\alpha s = (I_2)^q \int_b^\infty \left(\frac{h^\sigma(s)}{\sigma(s)}\right)^q \diamondsuit_\alpha s. \tag{18}$$

Let $v(x) = h^\sigma(x)\varphi^{-\frac{1}{q}}(x)\left|\varphi^{\diamondsuit_\alpha}(x)\right|$ be a positive function, and let $\Psi(s) = \int_0^s v(x)\diamondsuit_\alpha x$ be a positive operator. Using Lemma 1 and Lemma 3 we get the following inequality

$$\begin{aligned}
&\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} h^\sigma(x)\varphi^{-\frac{1}{q}}(x)\left|\varphi^{\diamondsuit_\alpha}(x)\right|\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s \\
&= \int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} v(x)\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s \\
&\le q\int_b^\infty \frac{\varphi(s)}{(\Gamma^\sigma(s))^q}\left(\int_b^{\sigma(s)} v(x)\left(\int_b^{\sigma(x)} v(\tau)\diamondsuit_\alpha \tau\right)^{q-1}\diamondsuit_\alpha x\right)\diamondsuit_\alpha s \\
&= q\int_b^\infty \frac{\varphi(s)}{(\Gamma^\sigma(s))^q}\left(\int_b^{\sigma(s)} v(x)(\Psi(\sigma(x)))^{q-1}\diamondsuit_\alpha x\right)\diamondsuit_\alpha s \\
&\le q\int_b^\infty v(x)(\Psi(\sigma(x)))^{q-1}\left(\int_x^\infty \frac{\varphi(s)}{(\Gamma^\sigma(s))^q}\diamondsuit_\alpha s\right)\diamondsuit_\alpha x.
\end{aligned} \tag{19}$$

Since $\Gamma(s) \le \Gamma^\sigma(s)$, then we have

$$((1-f)\Gamma(s) + f\Gamma^\sigma(s))^q \le ((1-f)\Gamma^\sigma(s) + f\Gamma^\sigma(s))^q.$$

Since $q > 1$, then we have

$$(1-q)((1-f)\Gamma(s) + f\Gamma^\sigma(s))^q \ge (1-q)((1-f)\Gamma^\sigma(s) + f\Gamma^\sigma(s))^q.$$

Applying the chain rule, we get the following inequality

$$\left(\Gamma^{(1-q)}(s)\right)^{\diamondsuit_\alpha} \le \frac{(1-q)\varphi(s)}{(\Gamma^\sigma(s))^q}.$$

Integrating the last inequality from x to ∞, we get the following inequality

$$\int_x^\infty \frac{(1-q)\varphi(s)}{(\Gamma^\sigma(s))^q}\diamondsuit_\alpha s \le \frac{(\Gamma(x))^{1-q}}{(q-1)}.$$

Substituting into (19), we get the following inequality

$$\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)} h^\sigma(x)\varphi^{-\frac{1}{q}}(x)\left|\varphi^{\diamondsuit_\alpha}(x)\right|\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s$$

$$\le \frac{q}{(q-1)}\int_b^\infty v(x)(\Psi(\sigma(x)))^{q-1}(\Gamma(x))^{1-q}\diamondsuit_\alpha x = \frac{q}{(q-1)}\int_b^\infty v(x)\left(\frac{\Psi(\sigma(x))}{\Gamma(x)}\right)^{q-1}\diamondsuit_\alpha x.$$

Hence, we obtain

$$\int_b^\infty \varphi(s)\left(\frac{\Psi(\sigma(s))}{\Gamma(s)}\right)^q \diamondsuit_\propto s \le \frac{q}{(q-1)}\int_b^\infty v(x)\left(\frac{\Psi(\sigma(x))}{\Gamma(x)}\right)^{q-1}\diamondsuit_\propto x. \tag{20}$$

Now $v(x) = \left(v(x)\varphi^{-\frac{1}{q}}(x)\right)\varphi^{\frac{1}{q}}(x)$, and applying Hölder's inequality with $1/q + 1/q' = 1$, we get the following inequality

$$\begin{aligned}&\int_b^\infty v(x)\left(\frac{\Psi(\sigma(x))}{\Gamma(x)}\right)^{q-1}\diamondsuit_\propto x\\ &\le \left(\int_b^\infty \left[v(x)\varphi^{-\frac{1}{q}}(x)\right]^q \diamondsuit_\propto x\right)^{\frac{1}{q}}\left(\int_b^\infty \varphi(x)\left(\frac{\Psi(\sigma(x))}{\Gamma(x)}\right)^q\diamondsuit_\propto x\right)^{\frac{1}{q'}}.\end{aligned} \tag{21}$$

Thus, we can see from (20) and (21) that

$$\begin{aligned}&\int_b^\infty \varphi(s)\left(\frac{\Psi(\sigma(s))}{\Gamma(s)}\right)^q \diamondsuit_\propto s\\ &\le \frac{q}{(q-1)}\left(\int_b^\infty \left[v(x)\varphi^{-\frac{1}{q}}(x)\right]^q \diamondsuit_\propto x\right)^{\frac{1}{q}}\left(\int_b^\infty \varphi(x)\left(\frac{\Psi(\sigma(x))}{\Gamma(x)}\right)^q\diamondsuit_\propto x\right)^{\frac{1}{q'}}.\end{aligned} \tag{22}$$

From (11), we have

$$\frac{v(x)}{\varphi^{\frac{1}{q'}}(x)} = \frac{h^\sigma(x)\left|\varphi^{\diamondsuit_\propto}\right|}{\varphi^{\frac{1}{q}}(x)\varphi^{\frac{1}{q'}}(x)} = \frac{h^\sigma(x)\left|\varphi^{\diamondsuit_\propto}\right|}{\varphi(x)} \le I_1\frac{h^\sigma(x)}{\sigma(x)}.$$

Substituting into (22), we get the following inequality

$$\left(\int_b^\infty \varphi(s)\left(\frac{\Psi(\sigma(s))}{\Gamma(s)}\right)^q\diamondsuit_\propto s\right)^{\frac{1}{q}} \le \frac{q}{(q-1)}\left(\int_b^\infty\left(I_1\frac{h^\sigma(x)}{\sigma(x)}\right)^q\diamondsuit_\propto x\right)^{\frac{1}{q}}.$$

Hence, we have

$$\int_b^\infty \varphi(s)\left(\frac{\Psi(\sigma(s))}{\Gamma(s)}\right)^q\diamondsuit_\propto s \le \left(I_1\frac{q}{(q-1)}\right)^q\int_b^\infty\left(\frac{h^\sigma(x)}{\sigma(x)}\right)^q\diamondsuit_\propto x. \tag{23}$$

Substituting (23) and (18) into (17), we get the following inequality

$$\begin{aligned}&\left(\int_b^\infty \varphi(s)\left(\frac{1}{\Gamma^\sigma(s)}\int_b^{\sigma(s)}\varphi(x)g(x)\diamondsuit_\propto x\right)^q\diamondsuit_\propto s\right)^{\frac{1}{q}}\\ &\le \left((I_2)^q\int_b^\infty\left(\frac{h^\sigma(s)}{\sigma(s)}\right)^q\diamondsuit_\propto s\right)^{\frac{1}{q}} + \frac{1}{q'}\left(\left(I_1\frac{q}{(q-1)}\right)^q\int_b^\infty\left(\frac{h^\sigma(s)}{\sigma(s)}\right)^q\diamondsuit_\propto s\right)^{\frac{1}{q}}\\ &= (I_1+I_2)\left(\int_b^\infty\left(\frac{h^\sigma(s)}{\sigma(s)}\right)^q\diamondsuit_\propto s\right)^{\frac{1}{q}}.\end{aligned}$$

Herewith, the proof of Theorem 1 is complete.

Remark 1 As a special case of Theorem 1 when $\mathbb{T} = \mathbb{R}$ and $\propto = 1$, we obtain inequality (7). In the case of $\mathbb{T} = \mathbb{N}$ and $\propto = 1$, we formulate the discrete version of inequality (7).

If $q > 1$, $\frac{(m+1)\left|\varphi^{\diamondsuit_\propto}(m)\right|}{\varphi(m)} \le I_1$ and $\frac{(m+1)\varphi(m+1)}{\Gamma(m+1)} \le I_2$, then we have

$$\sum_{m=1}^\infty \varphi(m)\left(\frac{1}{\Gamma(m+1)}\sum_{c=1}^m \varphi(c)g(c)\right)^q \le (I_1+I_2)^q\sum_{m=1}^\infty\left(\frac{1}{(m+1)}\sum_{c=1}^m \varphi^{\frac{1}{q}}(c)g(c)\right)^q.$$

Corollary 1 *Let* $\Gamma : J \to (0,\infty)$, $\varphi : J \to (0,\infty)$, *and let* $g(s), \varphi(x) = \frac{1}{x+1}$ *be non-negative functions on*

$s \geq 0$ *for* $q > 1$, $b = 0$. *If* $\Gamma(s) = \frac{r}{(1-r)} \int_b^s \varphi(x) \diamondsuit_\propto x$ *is positive operator for* $0 < r < 1$, *then we get the following inequality.*

$$\int_0^\infty \frac{1}{s+1} \left(\frac{1}{\Gamma^\sigma(s)} \int_0^{\sigma(s)} \frac{g(x)}{x+1} \diamondsuit_\propto x \right)^q \diamondsuit_\propto s \leq (I_1 + I_2)^q \int_0^\infty \left(\frac{1}{\sigma(s)} \int_0^{\sigma(s)} \left(\frac{1}{x+1} \right)^{\frac{1}{q}} g(x) \diamondsuit_\propto x \right)^q \diamondsuit_\propto s,$$

where I_1 *and* I_2 *positive constants.*

Example 1 Let $\Gamma : J \to (0, \infty), \varphi : J \to (0, \infty)$, and let $g(s) = s^2, \varphi(x) = \frac{1}{x+1}$ be non-negative functions on $s \geq 0$ for $q = 2$, $b = 0$. If $\Gamma(s) = \int_0^s \frac{1}{x+1} \diamondsuit_\propto x$ is positive operator, then we obtain.

$$\int_0^\infty \frac{1}{s+1} \left(\frac{1}{\Gamma^\sigma(s)} \int_0^{\sigma(s)} \frac{x^2}{x+1} \diamondsuit_\propto x \right)^2 \diamondsuit_\propto s \leq (I_1 + I_2)^2 \int_0^\infty \left(\frac{1}{\sigma(s)} \int_0^{\sigma(s)} \left(\frac{x^4}{x+1} \right)^{\frac{1}{2}} \diamondsuit_\propto x \right)^2 \diamondsuit_\propto s.$$

Theorem 2 *Let* $\varphi(s)$ *be a non-decreasing and positive function with* $q > 1$, *and let* $b \in [0, \infty) \cap \mathbb{T}$. *If there exists* $\frac{\Gamma^\sigma(x)}{\Gamma(x)} \leq d$ *with a constant* $d \geq 1$, *then the following inequality is valid.*

$$\frac{\int_b^\infty \varphi(s) \left(\int_{\sigma(s)}^\infty \frac{\varphi(x) g(x)}{\Gamma(x)} \diamondsuit_\propto x \right)^q \diamondsuit_\propto s}{\int_b^\infty \left(\int_s^\infty \frac{\varphi^{1/q}(x) g(x)}{x} \diamondsuit_\propto x \right)^q \diamondsuit_\propto s} \leq (I_2 + qdJ)^q, \tag{24}$$

where $J = I_1 + I_2 + 1$.

Proof Let $W(s)$ be a positive operator with.

$$\begin{cases} W(s) = \int_s^\infty \frac{\varphi^{1/q}(x) g(x)}{x} \diamondsuit_\propto x \\ W^{\diamondsuit_\propto}(s) = \frac{\varphi^{1/q}(s) g(s)}{s}. \end{cases}$$

This implies that

$$\int_{\sigma(s)}^\infty \frac{\varphi(x) g(x)}{\Gamma(x)} \diamondsuit_\propto x = - \int_{\sigma(s)}^\infty \frac{W^{\diamondsuit_\propto}(x) x \varphi^{\frac{1}{q}}(x)}{\Gamma(x)} \diamondsuit_\propto x.$$

If we integrate the term by parts with $w^{\diamondsuit_\propto}(x) = W^{\diamondsuit_\propto}(x)$ and $u(x) = \frac{x \varphi^{\frac{1}{q}}(x)}{\Gamma(x)}$, then we get the following inequality

$$\begin{aligned} \int_{\sigma(s)}^\infty \frac{\varphi(x) g(x)}{\Gamma(x)} \diamondsuit_\propto x &= \frac{W^\sigma(s) \sigma(s) \varphi^{\frac{1}{q}}(\sigma(s))}{\Gamma^\sigma(s)} + \int_{\sigma(s)}^\infty \left(\frac{x \varphi^{\frac{1}{q}}(x)}{\Gamma(x)} \right)^{\diamondsuit_\propto} W^\sigma(x) \diamondsuit_\propto x \\ &\leq \frac{W^\sigma(s) \sigma(s) \varphi^{\frac{1}{q}}(\sigma(s))}{\Gamma^\sigma(s)} + \int_{\sigma(s)}^\infty W^\sigma(x) \left| \left(\frac{x \varphi^{\frac{1}{q}}(x)}{\Gamma(x)} \right)^{\diamondsuit_\propto} \right| \diamondsuit_\propto x. \end{aligned}$$

Hence, we have

$$\int_b^\infty \varphi(s)\left(\int_{\sigma(s)}^\infty \frac{\varphi(x)g(x)}{\Gamma(x)}\diamondsuit_\propto x\right)^q \diamondsuit_\propto s$$
$$\leq \int_b^\infty \varphi(s)\left(\frac{W^\sigma(s)\sigma(s)\varphi^{\frac{1}{q}}(\sigma(s))}{\Gamma^\sigma(s)} + \int_{\sigma(s)}^\infty W^\sigma(x)\left|\left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right)^{\diamondsuit_\propto}\right|\diamondsuit_\propto x\right)^q \diamondsuit_\propto s.$$

Applying the inequality (10), we get the following inequality

$$\left(\int_b^\infty \varphi(s)\left(\int_{\sigma(s)}^\infty \frac{\varphi(x)g(x)}{\Gamma(x)}\diamondsuit_\propto x\right)^q \diamondsuit_\propto s\right)^{\frac{1}{q}}$$
$$\leq \left(\int_b^\infty \varphi(s)\left(\frac{W^\sigma(s)\sigma(s)\varphi^{\frac{1}{q}}(\sigma(s))}{\Gamma^\sigma(s)}\right)^q \diamondsuit_\propto s\right)^{\frac{1}{q}} \tag{25}$$
$$+\left(\int_b^\infty \varphi(s)\left(\int_{\sigma(s)}^\infty W^\sigma(x)\left|\left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right)^{\diamondsuit_\propto}\right|\diamondsuit_\propto x\right)^q \diamondsuit_\propto s\right)^{\frac{1}{q}}.$$

Let $W(s)$ be a non-increasing function, and let $\varphi(s)$ be non-decreasing function. Since $W(s) \geq W^{\diamondsuit_\propto}(s)$, we see that

$$\int_b^\infty \varphi(s)\left(\frac{W^\sigma(s)\sigma(s)\varphi^{\frac{1}{q}}(\sigma(s))}{\Gamma^\sigma(s)}\right)^q \diamondsuit_\propto s$$
$$\leq \int_b^\infty \varphi^\sigma(s)\left(\frac{W^\sigma(s)(I_2)\sigma(s)\varphi^{\frac{1}{q}}(\sigma(s))}{\varphi^\sigma(s)\sigma(s)}\right)^q \diamondsuit_\propto s$$
$$= \int_b^\infty \varphi^\sigma(s)\left(W^\sigma(s)(I_2)\sigma(s)\varphi^{-\frac{1}{q}}(\sigma(s))\right)^q \diamondsuit_\propto s \tag{26}$$
$$= (I_2)^q \int_b^\infty (W^\sigma(s))^q \diamondsuit_\propto s \leq (I_2)^q \int_b^\infty (W(s))^q \diamondsuit_\propto s.$$

Using quotient and product rules in derivatives, we get the following inequality

$$\left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right)^{\diamondsuit_\propto} = \frac{\Gamma(x)\left(x\varphi^{\frac{1}{q}}(x)\right)^{\diamondsuit_\propto} - x\varphi^{\frac{1}{q}}(x)\Gamma^{\diamondsuit_\propto}(x)}{\Gamma^\sigma(x)\Gamma(x)}$$
$$= (I_2)^q \int_b^\infty (W^\sigma(s))^q \diamondsuit_\propto s \leq (I_2)^q \int_b^\infty (W(s))^q \diamondsuit_\propto s.$$
$$\leq \frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma^\sigma(x)}\left(\frac{1}{x} + \frac{\sigma(x)\varphi^{\diamondsuit_\propto}(x)}{x\varphi(x)} - \frac{\varphi(x)}{\Gamma(x)}\right).$$

Hence, we obtain

$$\left|\left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right)^{\diamondsuit_\propto}\right| \leq \frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma^\sigma(x)}\left(\frac{1}{x} + \frac{\sigma(x)\left|\varphi^{\diamondsuit_\propto}(x)\right|}{q'x\varphi(x)} - \frac{\varphi(x)}{\Gamma(x)}\right)$$
$$\leq \frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma^\sigma(x)}\left(\frac{1}{x} + \frac{\sigma(x)\left|\varphi^{\diamondsuit_\propto}(x)\right|}{q'x\varphi(x)} - \frac{\varphi^\sigma(x)}{\Gamma(x)}\right).$$

Since $\varphi(x)$ is a non-decreasing function and considering (11), we have

$$\left|\left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right)^{\diamondsuit_\alpha}\right| \le \frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma^\sigma(x)}\left(\frac{I_1}{q'x}+\frac{1+I_2}{x}\right) < J\left(\frac{\varphi^{\frac{1}{q}}(x)}{\Gamma^\sigma(x)}\right) \le J\left(\frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right),$$

where $J = 1 + I_1 + I_2$. Since $W(x)$ is a non-increasing function, we have

$$\begin{aligned}\int_b^\infty \varphi(s)\left(\int_{\sigma(s)}^\infty \left|\left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}\right)^{\diamondsuit_\alpha}\right| W^\sigma(x)\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s \\ \le J\int_b^\infty \varphi(s)\left(\int_{\sigma(s)}^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)} W(x)\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s.\end{aligned} \tag{27}$$

Let $\Pi(x) = \int_b^\infty \frac{\varphi^{\frac{1}{q}}(\tau)}{\Gamma(\tau)}\diamondsuit_\alpha \tau$ be a positive operator. Then $\Pi^{\diamondsuit_\alpha}(x) = \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)} > 0$.
Since $\Pi(x)$ is non-increasing, and since $\Pi(\sigma(x)) \le \Pi(x)$, we get

$$\int_b^\infty \varphi(s)\left(\int_{\sigma(s)}^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)} W(x)\diamondsuit_\alpha x\right)^q \diamondsuit_\alpha s = \int_b^\infty \varphi(s)(\Pi(\sigma(x)))^q \diamondsuit_\alpha s \le \int_b^\infty \varphi(s)(\Pi(x))^q \diamondsuit_\alpha s.$$

Applying Lemma 2, we get the following inequality

$$\int_b^\infty \varphi(s)(\Pi(x))^q \diamondsuit_\alpha s \le q\int_b^\infty \varphi(s)\left(\int_s^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}(\Pi(x))^{q-1}\diamondsuit_\alpha x\right)\diamondsuit_\alpha s.$$

Since $\Gamma^\sigma(s) \le d\Gamma(s)$, if we integrate the term by parts with $v^{\diamondsuit_\alpha}(s) = \varphi(s)$, $w(s) = \int_s^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}(\Pi(x))^{q-1}\diamondsuit_\alpha x$, we get the following inequality

$$\begin{aligned}&\int_b^\infty \varphi(s)\left(\int_s^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}(\Pi(x))^{q-1}\diamondsuit_\alpha x\right)\diamondsuit_\alpha s \\ &= \left(\Gamma(s)\int_s^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}(\Pi(x))^{q-1}\diamondsuit_\alpha x\right)I_b^\infty + \int_b^\infty \Gamma^\sigma(s)\left(-\int_s^\infty \frac{\varphi^{\frac{1}{q}}(x)}{\Gamma(x)}(\Pi(x))^{q-1}\diamondsuit_\alpha x\right)^{\diamondsuit_\alpha}\diamondsuit_\alpha s \\ &= \int_b^\infty \Gamma^\sigma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}(\Pi(s))^{q-1}\diamondsuit_\alpha s \le d\int_b^\infty \Gamma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}(\Pi(s))^{q-1}\diamondsuit_\alpha s. \\ &= \int_b^\infty \Gamma^\sigma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}(\Pi(s))^{q-1}\diamondsuit_\alpha s \le d\int_b^\infty \Gamma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}(\Pi(s))^{q-1}\diamondsuit_\alpha s.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\int_b^\infty \varphi(s)(\Pi(s))^q \diamondsuit_\alpha s &\le qd\int_b^\infty \Gamma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}(\Pi(s))^{q-1}\diamondsuit_\alpha s \\ &= qd\int_b^\infty \Gamma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}\varphi^{-\frac{1}{q}}(s)\varphi^{\frac{1}{q}}(s)(\Pi(s))^{q-1}\diamondsuit_\alpha s.\end{aligned}$$

Using Hölder inequality (9), we get the following inequality

$$\int_b^\infty \varphi(s)(\Pi(s))^q \diamondsuit_\alpha s \le qd\left(\int_b^\infty \left(\Gamma(s)\frac{\varphi^{\frac{1}{q}}(s)}{\Gamma(s)}\varphi^{-\frac{1}{q}}(s)\right)^q \diamondsuit_\alpha s\right)^{\frac{1}{q}}\left(\int_b^\infty \varphi(s)(\Pi(s))^q \diamondsuit_\alpha s\right)^{\frac{1}{q'}}.$$

Thus, we obtain

$$\int_b^\infty \varphi(s)(\Pi(s))^q \diamondsuit_\propto s \le (qd)^q \int_b^\infty (\mathrm{W}(s))^q \diamondsuit_\propto s. \tag{28}$$

Substituting (27) into (28), we have

$$\int_b^\infty \varphi(s) \left(\int_{\sigma(s)}^\infty \left| \left(\frac{x\varphi^{\frac{1}{q}}(x)}{\Gamma(x)} \right)^{\diamondsuit_\propto} \right| W^\sigma(x) \diamondsuit_\propto x \right)^q \diamondsuit_\propto s \le (Jqd)^q \int_b^\infty (\mathrm{W}(s))^q \diamondsuit_\propto s. \tag{29}$$

Substituting (26) and (29) into (25), we obtain

$$\left(\int_b^\infty \varphi(s) \left(\int_{\sigma(s)}^\infty \frac{\varphi(x)g(x)}{\Gamma(x)} \diamondsuit_\propto x \right)^q \diamondsuit_\propto s \right)^{\frac{1}{q}}$$

$$\le \left((I_2)^q \int_b^\infty (\mathrm{W}(s))^q \diamondsuit_\propto s \right)^{\frac{1}{q}} + \left((Jqd)^q \int_b^\infty (\mathrm{W}(s))^q \diamondsuit_\propto s \right)^{\frac{1}{q}} = (I_2 + Jqd) \int_b^\infty (\mathrm{W}(s))^q \diamondsuit_\propto s.$$

Herewith, the proof of Theorem 2 is complete.

Remark 2 As a special case of Theorem 2, when $\mathbb{T} = \mathbb{R}$, $d = 1$ and $\propto = 1$, we obtain inequality (8). If we consider the case $\mathbb{T} = \mathbb{N}$ and $\propto = 1$, then we formulate the discrete version of inequality (8).

If $q > 1$, $\frac{(m+1)\left|\varphi^{\diamondsuit_\propto}(m)\right|}{\varphi(m)} \le I_1$, and $\frac{(m+1)\varphi(m+1)}{\Gamma(m)} \le I_2$, then

$$\sum_{m=1}^\infty \varphi(m) \left(\sum_{c=m+1}^\infty \frac{\varphi(c)g(c)}{\Gamma(c)} \right)^q \le (Jqd + I_2)^q \sum_{m=1}^\infty \left(\sum_{c=m}^\infty \frac{\varphi^{\frac{1}{q}}(c)g(c)}{c} \right)^q,$$

where $J = I_1 + I_2 + 1$.

Corollary 2 *Let $q > 1$, $b = 0$, $\Gamma, \varphi : J \to (0, \infty)$, $\varphi(s) = \frac{1}{s+1}$, and let $g(s), \varphi(s)$ be non-negative functions on $s \ge 0$. If there exists $\frac{\Gamma^\sigma(x)}{\Gamma(x)} \le 3$, then we get the following inequality.*

$$\int_0^\infty \frac{1}{s+1} \left(\int_{\sigma(s)}^\infty \frac{g(x)}{(x+1)\Gamma(x)} \diamondsuit_\propto x \right)^q \diamondsuit_\propto s \le (I_2 + q3J)^q \int_0^\infty \left(\int_s^\infty \frac{g(x)}{(x+1)^{1/q}_x} \diamondsuit_\propto x \right)^q \diamondsuit_\propto s,$$

where I_1, I_2 positive constants and $J = I_1 + I_2 + 1$.

Example 2 Let $\Gamma : J \to (0, \infty), \varphi : J \to (0, \infty)$, and let $g(s) = s^2, \varphi(s) = \frac{1}{s+1}$ be non-negative functions on $s \ge 0$ for $q = 2, b = 0$. If there exists $\frac{\Gamma^\sigma(x)}{\Gamma(x)} \le 3$, then we get the following inequality.

$$\int_0^\infty \frac{1}{s+1} \left(\int_{\sigma(s)}^\infty \frac{x^2}{(x+1)\Gamma(x)} \diamondsuit_\propto x \right)^2 \diamondsuit_\propto s \le (I_2 + 6J)^2 \int_0^\infty \left(\int_s^\infty \frac{x}{\sqrt{x+1}} \diamondsuit_\propto x \right)^2 \diamondsuit_\propto s,$$

where I_1, I_2 positive constants and $J = I_1 + I_2 + 1$.

4 Conclusion

In this work, we establish a new approximation of generalizations and the discrete versions of weighted inequalities of Hardy-type on time scales $\mathbb{T}$ via the diamond $-\propto$ integral. From these inequalities, we derive some integral and new discrete inequalities. The method is based on applications of well-known inequalities from the time scale. Using fractional integrals and fractional derivatives on time scales, we can exploit such inequalities for future work.

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Declarations

Data availability I declare that the Indian journal of Pure and Applied Mathematics may use the data in this article.

Conflict of interest The corresponding author states that there is no conflict of interest.

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