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ORIGINAL RESEARCH

Reciprocal distance signless Laplacian spread of connected graphs

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Abstract Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Recall that the reciprocal distance signless Laplacian matrix of G is defined to be $RQ(G) = RT(G) + RD(G)$, where $RD(G)$ is the reciprocal distance matrix, and RT_i is the reciprocal distance degree of vertex v_i for $i = 1, 2, \ldots, n$, $RT(G) = \mathrm{diag}(RT_1, RT_2, \ldots, RT_n)$. Denote by $\mu_1(RQ(G))$ and $\mu_n(RQ(G))$ the largest eigenvalue and the least eigenvalue of $RQ(G)$, respectively. The reciprocal distance signless Laplacian spread of G is defined as $S_{RQ}(G) = \mu_1(RQ(G)) - \mu_n(RQ(G))$. In this paper, we obtain some bounds on reciprocal distance signless Laplacian spread of a graph.

Keywords Graph · Reciprocal distance matrix · Reciprocal Distance signless Laplacian matrix · Reciprocal distance signless Laplacian spread

Mathematics Subject Classification 05C50 · 05C12 · 15A18

1 Introduction

In this paper, we considered finite, simple and connected graphs. Let G be a such graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|E(G)| = m$. Let d_{v_i} denote the degree of vertex v_i, $N(v_i)$ denote the neighbor set of v_i, and $\Delta(G)$ be the maximum degree of all vertices in G. The distance between vertices v_i and v_j in G is the length of the shortest path connecting v_i to v_j, which is denoted as $d(v_i, v_j)$ (or d_{ij} for short). The Harary matrix of G, denoted by $RD(G)$, which is also called as the reciprocal distance matrix, is an $n \times n$ matrix defined as [1]

$$RD_{i,j} = \begin{cases} \frac{1}{d(v_i, v_j)}, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

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The transmission of vertex v_i, denoted by $Tr_G(v_i)$ or Tr_i, is defined to be the sum of the distances from v_i to all vertices in G, that is, $Tr_G(v_i) = Tr_i = \sum_{u \in V(G)} d(v_i, u)$. The transmission of v_i is also called the distance degree or the first distance degree of v_i. A graph G is said to be k-transmission regular if $Tr_G(v_i) = k$ for all $i = 1, 2, \ldots, n$.

Definition 1.1 Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The reciprocal distance degree of vertex v_i, denoted by $RTr_G(v_i)$, is given by

$$RTr_G(v_i) = \sum_{\substack{v_j \in V(G) \\ j \neq i}} \frac{1}{d(v_i, v_j)}.$$

Let $RT(G)$ be the $n \times n$ diagonal matrix defined by $RT_{i,i} = RTr_G(v_i)$, that is, $RT(G) = \mathrm{diag}(RTr_G(v_1), RTr_G(v_2), \ldots, RTr_G(v_n))$.

Sometimes we use the notation RT_i instead of $RTr_G(v_i)$ for $i = 1, \ldots, n$.

Definition 1.2 A connected graph G is called a k-reciprocal distance degree regular graph if $RT_i = k$ for all $i = 1, 2, \ldots, n$.

Definition 1.3 Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $v_i \in V(G)$, we call $t_{v_i} = \frac{\sum_{v_i \in V(G)} RT_i}{d_{v_i}}$ the average reciprocal distance degree of v_i.

The Harary index of a graph G, denoted by $H(G)$, is defined in [1] as

$$H(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} RD_{i,j} = \frac{1}{2} \sum_{\substack{v_i, v_j \in V(G) \\ i \neq j}} \frac{1}{d(v_i, v_j)}.$$

Clearly,

$$H(G) = \frac{1}{2} \sum_{v_i \in V(G)} RT_i.$$

In [2,3], the authors supplied lower and upper bounds for the largest eigenvalue of Harary matrix. Furthermore, they obtained Nordhaus-Gaddum-type results for Harary matrix.

In [4], the authors characterized the graphs with maximum reciprocal distance spectral radius from three different classes of graphs which are graphs with fixed matching number, bipartite graphs with fixed matching number and graphs with given number of cut edges.

In [5], Bapat and Panda defined the reciprocal distance Laplacian matrix as $RL(G) = RT(G) - RD(G)$. In [6], Alhevaz et al. defined the reciprocal distance signless Laplacian matrix as $RQ(G) = RT(G) + RD(G)$.

Since $RD(G)$ and $RQ(G)$ are real symmetric matrices, we can denote

$$\lambda_1(RD(G)) \geq \lambda_2(RD(G)) \geq \ldots \geq \lambda_{n-1}(RD(G)) \geq \lambda_n(RD(G))$$

and

$$\mu_1(RQ(G)) \geq \mu_2(RQ(G)) \geq \ldots \geq \mu_{n-1}(RQ(G)) \geq \mu_n(RQ(G))$$

to the eigenvalues of $RD(G)$ and $RQ(G)$, respectively.

Let M be a symmetric matrix of order n having eigenvalues w_i, $i = 1, 2, \ldots, n$. If w_1 and w_n are the largest and least eigenvalues of M respectively, then the spread of M is defined as $S(M) = w_1 - w_n$. From [7,8], we know that the spread of a matrix is a very interesting topic. Let $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$ be the distance eigenvalues of the graph G. The distance spread of G was considered in [9] and is defined as $S_D(G) = \rho_1 - \rho_n$. Various lower and upper bounds for the parameter $S_D(G)$ can be found in [9].

Motivated by the results obtained for the distance spread, You et al. [10] put forward concept of the distance signless Laplacian spread of G as $S_{DQ}(G) = q_1(DQ(G)) - q_n(DQ(G))$, where $q_1(DQ(G))$ and $q_n(DQ(G))$

are the largest and least distance signless Laplacian eigenvalues of G respectively, and obtained various lower and upper bounds for the parameter $S_{DQ}(G)$ in terms of different graph parameters.

In [11], author defined reciprocal distance laplacian spread of the connected graph and attained several upper and lower bounds for reciprocal distance Laplacian spread in terms of various graph parameters.

We define the reciprocal distance spread and reciprocal distance signless Laplacian spread of graph G as the different between the largest and least eigenvalues of $RD(G)$ and $RQ(G)$, respectively, that is

$$S_{RD}(G) = \lambda_1(RD(G)) - \lambda_n(RD(G))$$

and

$$S_{RQ}(G) = \mu_1(RQ(G)) - \mu_n(RQ(G)).$$

This paper is organized as follows. In Sect. 2, we obtain some bounds for $S_{RQ}(G)$, in terms of various graph parameters. We also establish a relation between the reciprocal distance signless Laplacian spread and the reciprocal distance spread of a graph G. In Sect. 3, we obtain lower bounds for $S_{RQ(G)}$ in terms of the maximum degree Δ, the order n, the reciprocal distance degree RT_i, the average reciprocal distance degree t_{v_i} and the Harary index of a bipartite graph G.

2 Bounds of $S_{RQ}(G)$ for a graph G

In this section, we obtain some bounds for $S_{RQ}(G)$, in terms of various graph parameters. We also establish a relation between the reciprocal distance signless Laplacian spread and the reciprocal distance spread of a graph G.

Lemma 2.1 *([12]) Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be Hermitian matrices of order n such that* $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$. *Then*

$$\lambda_k(\mathbf{Z}) \leq \lambda_j(\mathbf{X}) + \lambda_{k-j+1}(\mathbf{Y}), \quad n \geq k \geq j \geq 1,$$
$$\lambda_k(\mathbf{Z}) \geq \lambda_l(\mathbf{X}) + \lambda_{k-j+n}(\mathbf{Y}), \quad n \geq l \geq k \geq 1,$$

where $\lambda_i(M)$ *is the* i*-th largest eigenvalue of the matrix* M. *In either of these inequalities, equality holds if and only if there exists a unit vector that is an eigenvector to each of the three eigenvalues involved.*

Lemma 2.2 *([13])Let G be a graph of order* $n \geq 2$. *The graph G has exactly two distinct reciprocal distance signless Laplacian eigenvalues if and only if G is the complete graph* K_n. *In particular,* $\mu_1(RQ(K_n)) = 2n - 2$, *and* $\mu_i(RQ(K_n)) = n - 2$ *for* $i = 2, 3, \ldots, n$.

Lemma 2.3 *([6])Let G ba a graph of order n. Then*

$$\mu_1(RQ(G)) \geq \frac{4H(G)}{n}.$$

The equality holds if and only if G is a reciprocal distance degree regular graph.

If G is a k-reciprocal distance degree regular graph. Then $RT(G) = kI_n$ (where I_n is the identity matrix of order n) and so

$$RQ(G) = RT(G) + RD(G) = kI_n + RD(G).$$

Therefore, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the reciprocal distance eigenvalues of k-reciprocal distance degree regular graph G, its reciprocal distance signless Laplacian eigenvalues are $\mu_i = k + \lambda_i$, $1 \leq i \leq n$, Thus, for a k-reciprocal distance degree regular graph G, we have

$$S_{RQ}(G) = \mu_1 - \mu_n = (k + \lambda_1) - (k + \lambda_n) = S_{RD}(G).$$

Hence, for a k-reciprocal distance degree regular graph G, the study of the reciprocal distance signless Laplacian spread $S_{RQ}(G)$ is same as the reciprocal distance spread $S_{RD}(G)$.

The following result gives a relation between the reciprocal distance signless Laplacian spread $S_{RQ}(G)$ and the reciprocal distance spread $S_{RD}(G)$ of a connected graph G.

Theorem 2.4 *Let G be a graph of order n having reciprocal distance degree sequence $\{RT_1, RT_2, \ldots, RT_n\}$. If $RT_{max} = \max\limits_{1\le i\le n} RT_i$ and $RT_{min} = \min\limits_{1\le i\le n} RT_i$, then*

$$|(RT_{max} - RT_{min}) - S_{RD}(G)| \le S_{RQ}(G) \le (RT_{max} + RT_{min}) + S_{RD}(G). \tag{2.1}$$

Equality occurs on the both sides if and only if G is a reciprocal distance degree regular graph.

Proof Let G be a graph of order n having reciprocal distance signless laplacian matrix $RQ(G)$. Since $RQ(G) = RT(G) + RD(G)$, by taking $k = 1, j = 1$ in the first inequality and $k = 1, l = n$ in second inequality of Lemma 2.1, it follows that

$$\lambda_n(RT(G)) + \lambda_1(RD(G)) \le \lambda_1(RQ(G)) \le \lambda_1(RT(G)) + \lambda_1(RD(G)). \tag{2.2}$$

Again taking $k = n$, $j = 1$ in first inequality and $k = n, l = n$ in second inequality of Lemma 2.1, we get

$$\lambda_n(RT(G)) + \lambda_n(RD(G)) \le \lambda_n(RQ(G)) \le \lambda_1(RT(G)) + \lambda_n(RD(G)). \tag{2.3}$$

Form (2.2) and (2.3), we have

$$\lambda_1(RQ(G)) - \lambda_n(RQ(G)) \le (\lambda_1(RT(G)) - \lambda_n(RT(G))) + (\lambda_1(RD(G)) - \lambda_n(RD(G))),$$

and

$$\lambda_1(RQ(G)) - \lambda_n(RQ(G)) \ge |(\lambda_1(RT(G)) - \lambda_n(RT(G))) - (\lambda_1(RD(G)) - \lambda_n(RD(G)))|.$$

The result now follows form these inequalities. Equality occurs in (2.1) if and only if equality occurs in (2.2) and (2.3). Suppose that equality occurs on the right of (2.2), then by Lemma 2.1, the eigenvalues $\lambda_1(RQ(G))$, $\lambda_1(RT(G))$ and $\lambda_1(RD(G))$ of the matrices $RQ(G)$, $RT(G)$ and $RD(G)$ have the same unit eigenvector $\mathbf{x}$. Similarly, if equality occurs on the left of (2.2), then again by Lemma 2.1, the eigenvalues $\lambda_1(RQ(G))$, $\lambda_n(RT(G))$ and $\lambda_1(RD(G))$ of the matrices $RQ(G)$, $RT(G)$ and $RD(G)$ have the same unit eigenvector $\mathbf{x}$. Form this it follows that the eigenvalues $\lambda_1(RT(G))$ and $\lambda_n(RT(G))$ of matrix $RT(G)$ have the same eigenvectors, which is only possible if $\lambda_1(RT(G)) = \lambda_n(RT(G))$. This gives that $RT(G) = \lambda_1(RT(G))I_n$, this is, G is a $\lambda_1(RT(G))$-reciprocal distance degree regular graph. In a similar way we can discuss the equality in (2.3).

Conversely, if G is a k-reciprocal distance degree regular graph, then $RQ(G) = kI_n + RD(G)$ and so equality holds in (2.2) and (2.3). □

Form inequality (2.2), we obtain the bounds for the reciprocal distance signless Laplacian spectral radius μ_1 as follows

$$RT_{min} + \lambda_1(RD(G)) \le \mu_1 \le RT_{max} + \lambda_1(RD(G)),$$

with equality occurs on both sides if and only if G is a reciprocal distance degree regular graph.

Similarly, form inequality (2.3), we obtain the bounds for the smallest reciprocal distance signless laplacian eigenvalues μ_n as follows

$$RT_{min} + \lambda_n(RD(G)) \le \mu_1 \le RT_{max} + \lambda_n(RD(G)),$$

with equality occurs on both sides if and only if G is a reciprocal distance degree regular graph.

Definition 2.5 The Frobenius norm of an $n \times n$ matrix $M = (m_{i,j})$ is

$$\|M\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |m_{i,j}|^2}.$$

If M is a normal matrix, then $\|M\|_F^2 = \sum\limits_{i=1}^{n} |\lambda_i(M)|^2$, where $\lambda_1(M), \ldots, \lambda_n(M)$ are the eigenvalues of M.

In particular, this property is satisfied to matrices $RD(G)$ and $RQ(G)$.

Lemma 2.6 *([14]) Let G be a graph of order n. Then*

$$\sum_{i=1}^{n}\mu_i(RQ(G)) = \sum_{i=1}^{n} RT_i = 2H(G),$$
$$\sum_{i=1}^{n}(\mu_i(RQ(G)))^2 = \|RD(G)\|_F^2 + \sum_{i=1}^{n} RT_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2.$$

Theorem 2.7 *Let G be a graph of order n. Then*

$$S_{RQ}(G) \geq \frac{n}{n-1}\mu_1 - \frac{2H(G)}{n-1}, \tag{2.4}$$

equality holds if and only if $G \cong K_n$.

Proof Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ be the reciprocal distance signless Laplacian eigenvalues of G. By Lemma 2.6, $\sum_{i=1}^{n}\mu_i = 2H(G)$. Then $2H(G) \geq \mu_1 + (n-1)\mu_n$, and so $\mu_n \leq \frac{2H(G)-\mu_1}{n-1}$. Thus

$$S_{RQ}(G) \geq \mu_1 - \frac{2H(G)-\mu_1}{n-1} = \frac{n}{n-1}\mu_1 - \frac{2H(G)}{n-1}.$$

Equality in (2.4) holds if and only if $\mu_2 = \ldots = \mu_n$, That is, if and only if G has exactly two distinct eigenvalues. Using Lemma 2.2, it follows that G is the complete graph K_n.

Conversely, if $G \cong K_n$, it can be seen by direct computations that equality occurs in (2.4). □

Corollary 2.8 *Let G be a graph of order n. Then*

$$S_{RQ}(G) \geq \frac{2H(G)}{n-1},$$

with equality hold if and only if $G \cong K_n$.

Theorem 2.9 *Let G be a graph of order n. Then*

$$S_{RQ}(G) \geq \mu_1 - \sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2 - \mu_1^2}{n-1}},$$

equality holds if and only if $G \cong K_n$.

Proof By Lemma 2.6,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2 = \sum_{i=1}^{n}\mu_i^2 \geq \mu_1^2 + (n-1)\mu_n^2. \tag{2.5}$$

Then

$$\mu_n \leq \sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2 - \mu_i^2}{n-1}}.$$

So

$$S_{RQ}(G) \geq \mu_1 - \sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2 - \mu_i^2}{n-1}}.$$

Equality in (2.5) holds if and only if $\mu_2 = \ldots = \mu_n$, that is, G has exactly two distinct eigenvalues. Using Lemma 2.2, it follows that G is the complete graph K_n.

Conversely, if $G \cong K_n$, it can be seen by direct computations that equality occurs in (2.5). □

Corollary 2.10 *Let G be a graph of order n. Then*

$$S_{RQ}(G) \geq \frac{4H(G)}{n} - \sqrt{\frac{n^2\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum\limits_{i=1}^{n} RT_i^2\right) - 16H^2(G)}{(n-1)n^2}},$$

with equality hold if and only if $G \cong K_n$.

Theorem 2.11 *Let G be a graph of order n having Harary index $H(G)$. Then*

$$S_{RQ}(G) \geq \frac{2}{n}\sqrt{n\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2\right) - 4H^2(G)}.$$

Proof Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ be the reciprocal singless laplacian distance eigenvalues of G. We assume that the non-negative real numbers $a_2, a_3, \ldots, a_{n-1}, b_2, b_3, \ldots, b_{n-1}$ are defined by the equations $\mu_i^2 = a_i\mu_1^2 + b_i\mu_n^2$, and $a_i + b_i = 1$, $i = 2, 3, \ldots, n-1$. Putting $a = 1 + \sum\limits_{i=2}^{n-1} a_i$ and $b = 1 + \sum\limits_{i=2}^{n-1} b_i$, then $a + b = n$ and $\sum\limits_{i=1}^{n} \mu_i^2 = a\mu_1^2 + b\mu_n^2$. On the other hand, we have $\mu_i^2 = (a_i\mu_1^2 + b_i\mu_n^2)(a_i + b_i)$, and so $\mu_i \geq a_i\mu_1 + b_i\mu_n$, $i = 2, 3, \ldots, n-1$. Therefor $\sum\limits_{i=1}^{n} \mu_i \geq a\mu_1 + b\mu_n$. Now, observing that $ab \leq \frac{(a+b)^2}{4} = \frac{n^2}{4}$, we obtain

$$\begin{aligned} n\sum_{i=1}^{n}\mu_i^2 - \left(\sum_{i=1}^{n}\mu_i\right)^2 &\leq (a\mu_1^2 + b\mu_n^2)(a+b) - (a\mu_1 + b\mu_n)^2 \\ &= ab(\mu_1 - \mu_n)^2 \\ &\leq \frac{n^2}{4}(\mu_1 - \mu_n)^2. \end{aligned}$$

Hence, we can get

$$S_{RQ}(G) \geq \frac{2}{n}\sqrt{n\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2\right) - 4H^2(G)}.$$

The theorem follows. □

Lemma 2.12 *([15]) Let M be an $n \times n$ normal matrix. Then*

$$S(M) \leq \left(2\|M\|_F^2 - \frac{2}{n}tr^2(M)\right)^{\frac{1}{2}},$$

equality occurs if and only if the eigenvalues $\omega_1, \omega_2, \ldots, \omega_n$ of M satisfy the $\omega_2 = \omega_3 = \ldots = \omega_{n-1} = \frac{\omega_1+\omega_n}{2}$.

Theorem 2.13 *Let G be a graph of order n. Then*

$$S_{RQ}(G) \leq \sqrt{2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n} RT_i^2\right) - \frac{8H^2(G)}{n}},$$

with equality if and only if $\mu_2 = \mu_3 = \ldots = \mu_{n-1} = \frac{\mu_1+\mu_n}{2}$.

Proof Since the matrix $RQ(G)$ is a normal matrix and by Lemma 2.6, we have $\|RQ(G)\|_F^2 = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum\limits_{i=1}^{n} RT_i^2$, and $tr(RQ(G)) = 2H(G)$. Then the result follows form Lemma 2.12 by taking $M = RQ(G)$. □

Lemma 2.14 *([16]) If M is an $n \times n$ nonzero Hermitian matrix, then*

$$S(M) \geq \left| \frac{tr(M^3)}{tr(M^2)} - \frac{tr(M)}{n} \right|.$$

Lemma 2.15 *([17]) Let $t_i \geq 0$, $T = \sum_{i=1}^{n} t_i$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for $i = 1, 2, \ldots, n$. Then*

$$\left| \frac{1}{T} \sum_{i=1}^{n} t_i a_i b_i - \frac{1}{T^2} \sum_{i=1}^{n} t_i a_i \sum_{i=1}^{n} t_i b_i \right| \leq \frac{1}{4}(A-a)(B-b).$$

Lemma 2.16 *([18]) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for $i = 1, 2, \ldots, n$. Then*

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leq \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) (A-a)(B-b),$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Theorem 2.17 *Let G be a graph of order n, and the reciprocal distance degree sequence of G be $\{RT_1, RT_2, \ldots, RT_n\}$. Then*

$$S_{RQ}(G) \geq \left| \frac{a}{b} - \frac{2H(G)}{n} \right|,$$

where

$$a = \sum_{i=1}^{n} RT_i^3 + 3 \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{d_{ij}} \right)^2 RT_i + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^{n} \frac{1}{d_{ij}} \frac{1}{d_{im}} \frac{1}{d_{jm}},$$

$$b = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{d_{ij}} \right)^2 + \sum_{i=1}^{n} RT_i^2.$$

Proof Denote $RQ(G) = RT(G) + RD(G)$. Then

$$\begin{aligned} tr(RQ^3(G)) =& tr(RT^3(G)) + tr(RT(G)RD(G)RT(G)) + tr(RD^2(G)RT(G)) \\ &+ tr(RD(G)RT^2(G)) + tr(RT^2(G)RD(G)) + tr(RT(G)RD^2(G)) \\ &+ tr(RD^3(G)) + tr(RD(G)RT(G)RD(G)) \\ =& \sum_{i=1}^{n} RT_i^3 + 3 \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{d_{ij}} \right)^2 RT_i + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^{n} \frac{1}{d_{ij}} \frac{1}{d_{im}} \frac{1}{d_{jm}}. \end{aligned}$$

By Lemma 2.6, we have

$$tr(RQ(G)) = 2H(G), \quad tr(RQ^2(G)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{d_{ij}} \right)^2 + \sum_{i=1}^{n} RT_i^2.$$

By Lemma 2.14, the theorem follows. □

Theorem 2.18 *Let G be a graph of order n, and the reciprocal distance degree sequence of graph G be $\{RT_1, RT_2, \ldots, RT_n\}$. Then*

$$S_{RQ}(G) \geq \sqrt{\left| \frac{2a}{H(G)} - \frac{b^2}{H^2(G)} \right|},$$

where

$$a=\sum_{i=1}^{n}RT_i^3+3\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2RT_i+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{m=j+1}^{n}\frac{1}{d_{ij}}\frac{1}{d_{im}}\frac{1}{d_{jm}},$$
$$b=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}RT_i^2.$$

Proof By Theorem 2.17, we have

$$\sum_{i=1}^{n}\mu_i^3(RQ(G))=\sum_{i=1}^{n}RT_i^3+3\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2RT_i+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{m=j+1}^{n}\frac{1}{d_{ij}}\frac{1}{d_{im}}\frac{1}{d_{jm}},$$
$$\sum_{i=1}^{n}\mu_i^2(RQ(G))=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}RT_i^2,$$
$$\sum_{i=1}^{n}\mu_i(RQ(G))=2H(G).$$

Let $\mu_1(RQ(G))\geq\mu_2(RQ(G))\geq\ldots\geq\mu_n(RQ(G))$ be the eigenvalues of the reciprocal distance signless Laplacian matrix $RQ(G)$ of G. Take $a_i=b_i=t_i=\mu_i(RQ(G))$ for $1\leq i\leq n$, $A=B=\mu_1(RQ(G))$, and $a=b=\mu_n(RQ(G))$. By Lemmas 2.15 and 2.6, we have

$$\left|\frac{1}{2H(G)}\sum_{i=1}^{n}\mu_i^3(RQ(G))-\frac{1}{4H^2(G)}\sum_{i=1}^{n}\mu_i^2(RQ(G))\sum_{i=1}^{n}\mu_i^2(RQ(G))\right|\leq\frac{1}{4}S_{RQ}^2(G).$$

Then

$$\left|\frac{a}{2H(G)}-\frac{b^2}{4H^2(G)}\right|\leq\frac{1}{4}S_{RQ}^2(G),$$

where

$$a=\sum_{i=1}^{n}RT_i^3+3\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2RT_i+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{m=j+1}^{n}\frac{1}{d_{ij}}\frac{1}{d_{im}}\frac{1}{d_{jm}}$$
$$b=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}RT_i^2.$$

This completes the proof. □

Theorem 2.19 *Let G be a graph of order n, and the reciprocal distance degree sequence of graph G be $\{RT_1, RT_2, \ldots, RT_n\}$. Then*

$$S_{RQ}(G)\geq\frac{1}{n\sqrt{\xi(n)}}\sqrt{n\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2+\sum_{i=1}^{n}RT_i^2\right)-4H^2(G)},$$

where

$$\xi(n)=\frac{1}{4}\left(1-\frac{1+(-1)^{n+1}}{2n^2}\right).$$

Proof In this proof we use Lemma 2.16. Let $\mu_1(RQ(G)) \geq \mu_2(RQ(G)) \geq \ldots \geq \mu_n(RQ(G))$ be the eigenvalues of the reciprocal distance signless Laplacian matrix $RQ(G)$ of G. Take $a_i = b_i = \mu_i(RQ(G))$ for $1 \leq i \leq n$, $A = B = \mu_1(RQ(G))$, and $a = b = \mu_n(RQ(G))$. We have

$$\left| n\sum_{i=1}^{n}\mu_i^2 - \left(\sum_{i=1}^{n}\mu_i\right)^2 \right| \leq \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)n^2S_{RQ}^2(G).$$

Since

$$\xi(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{1+(-1)^{n+1}}{2n^2}\right),$$

and by Lemma 2.6, we have

$$\sum_{i=1}^{n}\mu_i^2(RQ(G)) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n}RT_i^2,$$

$$\sum_{i=1}^{n}\mu_i(RQ(G)) = 2H(G).$$

Then

$$S_{RQ}(G) \geq \frac{1}{n\sqrt{\xi(n)}}\sqrt{n\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{d_{ij}}\right)^2 + \sum_{i=1}^{n}RT_i^2\right) - 4H^2(G)}.$$

The proof is completed. □

3 Bounds of $S_{RQ}(G)$ for a bipartite graph G

In this section, we obtain lower bounds for $S_{RQ}(G)$ in terms of the maximum degree Δ, the order n, the reciprocal distance degree RT_i, the average reciprocal distance degree t_{v_i} and the Harary index of a bipartite graph G.

Definition 3.1 ([19]) Let M be a real matrix of order n described in the following block form

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix}, \tag{3.1}$$

where the diagonal blocks M_{ii} are $n_i \times n_i$ matrices for any $i \in \{1, 2, \ldots, t\}$ and $n = n_1 + \ldots + n_t$. For any $i, j \in \{1, 2, \ldots, t\}$, let b_{ij} denote the average row sum of M_{ij}, that is, b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $B(M) = (b_{ij})$ is called the quotient matrix of M. In addition, if M_{ij} has constant row sum for each pair i and j, then $B(M)$ is called the equitable quotient matrix of M.

Lemma 3.2 *([20]) Let M be a symmetric matrix having the form (3.1), and B be the quotient matrix of M. Then the eigenvalues of B interlace the eigenvalues of M.*

Lemma 3.3 *(Cauchy alternating theorem) Let A be a real symmetric matrix of order n and B be a principal submatrix of order m of A. Suppose A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and B has eigenvalues $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_m$. Then for $k = 1, 2, \ldots, m$, $\lambda_k \geq \beta_k \geq \lambda_{n-m+k}$.*

Lemma 3.4 *Let G be a graph of order n and S be a subset of vertex set $V(G)$ such that $N(x) = N(y)$ for any $x, y \in S$, where $|s| = t$.*

(1) *If S is an independent set, then RT_i of vertex v_i is a constant h for each $v \in S$, and $RQ(G)$ has $(h - \frac{1}{2})$ as an eigenvalue with multiplicity at least $t - 1$.*

(2) *If S is a clique, then RT_i of vertex v_i is a constant h' for each $v \in S$, and $RQ(G)$ has $(h' - \frac{1}{2})$ as an eigenvalue with multiplicity at least $t - 1$.*

Proof The proof of this lemma is similar to that of theorem 2.1 of references [21] and is omitted here. □

Lemma 3.5 *The reciprocal distance signless Laplacian spectrum of complete bipartite graph $K_{r,s}$ consists of the eigenvalue $s + \frac{1}{2}r - 1$ with multiplicity $r - 1$, the eigenvalue $r + \frac{1}{2}s - 1$ with multiplicity $s - 1$ and the remaining two eigenvalues as $x_1 \geq x_2$, where $x_1, x_2 = s + r - 1 \pm \sqrt{sr}$.*

Proof Let V_1 and V_2 be the partite sets of $K_{r,s}$ such that the degree of each vertex in V_1 in S, and the degree of each vertex in V_2 is r. It is clear that $R_i = s + \frac{1}{2}(r-1)$ of vertex v_i, for all $V_i \in V_1$ and $R_j = r + \frac{1}{2}(s-1)$ of vertex v_j, for all $V_j \in V_2$. Now, using Lemma 3.4 with $S = V_1$ and then with $S = V_2$. we get the eigenvalue $s + \frac{1}{2}r - 1$ with multiplicity $r - 1$, and the eigenvalue $r + \frac{1}{2}s - 1$ with multiplicity $s - 1$. The remaining two eigenvalues (as the quotient matrix for the graph $K_{r,s}$ is an equitable quotient matrix) can be obtained from the quotient matrix of the matrix $RQ(K_{r,s})$ give by $B = \begin{pmatrix} s + r - 1 & s \\ r & r + s - 1 \end{pmatrix}$. □

For a bipartite graph G, a lower bound for $S_{RQ}(G)$ in terms of the maximum degree Δ, the order n, the reciprocal distance degree RT_i, the average reciprocal distance degree t_{v_i} and the Harary index is as follows.

Theorem 3.6 *Let G be a bipartite graph of order $n \geq 3$ with maximum degree Δ. Suppose $d_{v_1} = d_{v_2} = \ldots = d_{v_k} = \Delta$.*

(1) *If $\Delta \leq n - 2$, then*

$$S_{RQ}(G) \geq \max_{1 \leq i \leq k} \left\{ \frac{\sqrt{a_i^2 - 4b_i(\Delta+1)(n-\Delta+1)}}{(\Delta+1)(n-\Delta-1)} \right\},$$

where $a_i = (\Delta+1)(4H(G) - 4RT_{v_i} - 4t_{v_i}\Delta) + n(\frac{1}{2}\Delta^2 + \frac{3}{2}\Delta + t_{v_i}\Delta + 4RT_{v_i})$, and $b_i = 2H(G)(3\Delta + \Delta^2 + 2t_{v_i}\Delta + 2RT_{v_i}) - RT_{v_i}^2 - 4t_{v_i}\Delta(t_{v_i}\Delta + 2RT_{v_i})$.

(2) *If $\Delta = n - 1$, then*

$$S_{RQ}(G) = \begin{cases} \frac{1}{2}n - \frac{1}{2} + \sqrt{n-1}, & \text{if } n > 5, \\ 2\sqrt{n-1}, & \text{if } n \leq 5, \end{cases}$$

Proof (1) Let $\Delta \leq n-2$. For $1 \leq i \leq k$, let $N(v_i) = \{v_{i1}, v_{i2}, \ldots, v_{i\Delta}\}$, and $N[v_i] = N(v_i) \cup \{v_i\}$. Then $V(G)$ is partitioned into two parts which are $N[v_i]$ and $V(G) \setminus N[v_i]$. Corresponding to this partition, $RD(G)$ can be presented as

$$RD(G) = \begin{pmatrix} RT_{v_i} & 1 & 1 & \cdots & 1 & \\ 1 & RT_{v_{i1}} & \frac{1}{2} & \cdots & \frac{1}{2} & * \\ 1 & \frac{1}{2} & RT_{v_{i2}} & \cdots & \frac{1}{2} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & RT_{v_{i\Delta}} & \\ & * & & & & * \end{pmatrix},$$

and the quotient matrix B is as follow

$$B = \begin{pmatrix} \frac{\frac{3}{2}\Delta + \frac{1}{2}\Delta^2 + t_{v_i}\Delta + RT_{v_i}}{\Delta+1} & \frac{t_{v_i}\Delta + RT_i - (\frac{3}{2}\Delta + \frac{1}{2}\Delta^2)}{\Delta+1} \\ \frac{t_{v_i}\Delta + RT_{v_i} - (\frac{3}{2}\Delta + \frac{1}{2}\Delta^2)}{n-\Delta-1} & \frac{4H(G) - 3t_{v_i}\Delta - 3RT_{v_i} + \frac{3}{2}\Delta + \frac{1}{2}\Delta^2}{n-\Delta-1} \end{pmatrix}.$$

Solving $|\lambda I - B| = 0$, we get the eigenvalues of B:

$$\lambda_{1,2} = \frac{a_i \pm \sqrt{a_i^2 - 4b_i(\Delta+1)(n-\Delta-1)}}{2(n-\Delta-1)(\Delta+1)},$$

where $a_i = (\Delta+1)(4H(G) - 4RT_{v_i} - 4t_{v_i}\Delta) + n(\frac{1}{2}\Delta^2 + \frac{3}{2}\Delta + t_{v_i}\Delta + 4RT_{v_i})$, and $b_i = 2H(G)(3\Delta + \Delta^2 + 2t_{v_i}\Delta + 3RT_{v_i}) - RT_{v_i}^2 - 4t_{v_i}\Delta(t_{v_i}\Delta + 2RT_{v_i})$. By Lemma 3.2, (1) follows.

(2) Let $\Delta = n-1$. Then G being a bipartite graph implies that $G \cong K_{1,n-1}$. The result now follows form Lemma 3.4, by taking $r = 1$ and $s = n-1$ This completes the proof. □

Taking $r = a$ and $s = n-a$ with $s \geq r$ in Lemma 3.5, it follows that the reciprocal distance signless Laplacian spectrum of $K_{a,n-a}$ is

$$\left\{n - \frac{1}{2}a - 1^{[a-1]}, \frac{1}{2}n + \frac{1}{2}a - 1^{[n-a-1]}, n-1 \pm \sqrt{a(n-a)}\right\}.$$

We have $n-1+\sqrt{a(n-a)} \geq n - \frac{1}{2}a - 1$, implying that $\frac{1}{2}a + \sqrt{a(n-a)} \geq 0$. It is easy to prove that the inequality, so it follows that $\mu_1(K_{a,n-a}) = n-1+\sqrt{a(n-a)}$. And then compare the eigenvalues $\frac{1}{2}n + \frac{1}{2}a - 1$ and $n-1-\sqrt{a(n-a)}$, we get, when $n > 5a$, $\mu_n(K_{a,n-a}) = \frac{1}{2}n + \frac{1}{2}a - 1$, else $\mu_n(K_{a,n-a}) = n-1-\sqrt{a(n-a)}$.

If $3 \leq n \leq 5$, then $\mu_n(K_{a,n-a}) = n-1-\sqrt{a(n-a)}$, and so

$$S_{RQ}(K_{a,n-a}) = 2\sqrt{a(n-a)}.$$

It is easy to see that $S_{RQ}(K_{a,n-a})$ is a increasing function on a for all $a \in [1, \frac{n}{2}]$. Then

$$S_{RQ}(K_{1,n-1}) \leq S_{RQ}(K_{2,n-2}) \leq \ldots \leq S_{RQ}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).$$

If $5 < n \leq 10$, then $\mu_n(K_{1,n-1}) = \frac{1}{2}n - \frac{1}{2}$, and $\mu_n(K_{a,n-a}) = 2\sqrt{a(n-a)}$ for all $a \in [2, \frac{n}{2}]$. Then

$$S_{RQ}(K_{1,n-1}) = \frac{1}{2}n - \frac{1}{2} + \sqrt{(n-1)},\ S_{RQ}(K_{a,n-a}) = 2\sqrt{a(n-a)},\ a \in [2, \frac{n}{2}].$$

It is easy to see that $S_{RQ}(K_{a,n-a})$ is a increasing function on a for all $a \in [2, \frac{n}{2}]$. Then

$$S_{RQ}(K_{2,n-2}) \leq S_{RQ}(K_{3,n-3}) \leq \ldots \leq S_{RQ}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).$$

By direct computation, it can be seen that

$$\frac{1}{2}n - \frac{1}{2} + \sqrt{(n-1)} \leq 2\sqrt{2(n-2)},$$

which implies that $S_{RQ}(K_{1,n-1}) \leq S_{RQ}(K_{2,n-2})$.

If $10 < n \leq 15$, then $\mu_n(K_{1,n-1}) = \frac{1}{2}n - \frac{1}{2}$, $\mu_n(K_{2,n-2}) = \frac{1}{2}n$, and $\mu_n(K_{a,n-a}) = 2\sqrt{a(n-a)}$ for all $a \in [3, \frac{n}{2}]$, and so

$$\begin{aligned} S_{RQ}(K_{1,n-1}) &= \frac{1}{2}n - \frac{1}{2} + \sqrt{(n-1)}, \\ S_{RQ}(K_{2,n-2}) &= \frac{1}{2}n - 1 + \sqrt{2(n-2)}, \\ S_{RQ}(K_{a,n-a}) &= 2\sqrt{a(n-a)}, a \in [3, \frac{n}{2}]. \end{aligned}$$

It is easy to see that $S_{RQ}(K_{a,n-a})$ is a increasing function on a for all $a \in [3, \frac{n}{2}]$. Then

$$S_{RQ}(K_{3,n-3}) \leq S_{RQ}(K_{4,n-4}) \leq \ldots \leq S_{RQ}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).$$

By direct computation, it can be seen that

$$\frac{1}{2}n - \frac{1}{2} + \sqrt{(n-1)} \leq \frac{1}{2}n - 1 + \sqrt{2(n-2)} \leq 2\sqrt{3(n-3)},$$

which implies that $S_{RQ}(K_{1,n-1}) \leq S_{RQ}(K_{2,n-2}) \leq S_{RQ}(K_{3,n-3})$. And so the proof goes on, it follows that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the complete bipartite graph of order n having maximum reciprocal distance signless Laplacian spread, and $K_{1,n-1}$ has the minimum reciprocal distance signless Laplacian spread. Thus we have proved the following.

Theorem 3.7 *Among all complete bipartite graphs $K_{a,n-a}$ of order n with $a \leq n-a$, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has the maximum reciprocal distance signless Laplacian spread, and $K_{1,n-1}$ has the minimum reciprocal distance signless Laplacian spread.*

Corollary 3.8 *Let $K_{a,n-a}$ be a complete bipartite graph of order n with $a \leq n-a$. Then*

$$S_{RQ}(K_{a,n-a}) \geq \begin{cases} \frac{1}{2}n - \frac{1}{2} + \sqrt{n-1}, & \text{if } n > 5a, \\ 2\sqrt{n-1}, & \text{if } n \leq 5a, \end{cases}$$

and

$$S_{RQ}(K_{a,n-a}) \leq \begin{cases} \frac{1}{2} - \lfloor \frac{n}{4} \rfloor + \sqrt{(n - \lfloor \frac{n}{2} \rfloor)\lfloor \frac{n}{2} \rfloor}, & \text{if } n > 5a, \\ 2\sqrt{(n - \lfloor \frac{n}{2} \rfloor)\lfloor \frac{n}{2} \rfloor}, & \text{if } n \leq 5a. \end{cases}$$

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