Indian J Pure Appl Math (2024) 55:419–424
https://doi.org/10.1007/s13226-023-00402-5

ORIGINAL RESEARCH

The local properties of some subclasses of plurisubharmonic functions

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Received: 20 November 2022 / Accepted: 13 March 2023 / Published online: 26 March 2023
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Abstract The main aim of this note is first to give the definition of the local classes. It also aims at studying the local property of some subclasses of plurisubharmonic functions by using these concepts.

Keywords Local property · Plurisubharmonic function · Complex analysis · Mathematical analysis

Mathematics Subject Classification Primary 32W20 ; Secondary 32Q15

1 Introduction

The classes of subharmonic and plurisubharmonic functions are the main objects in potential and pluripotential theories. We recall here the definitions of these classes. An upper semi-continuous function $u : \Omega \to [-\infty, +\infty)$ was defined on a domain Ω in $\mathbb{R}^n$ which is not identically $-\infty$ is called to be subharmonic if for every ball $B(x, r) \Subset \Omega$

$$u(x) \leq \frac{1}{V_n(B(x,r))} \int\limits_{B(x,r)} u(y) dV_n(y) = \frac{1}{c_n r^n} \int\limits_{B(x,r)} u(y) dV_n(y),$$

where V_n is the Lebesgue measure on $\mathbb{R}^n$ and $c_n = V_n(B(0, 1))$. The set of subharmonic functions (negative subharmonic functions) on Ω is denoted by $SH(\Omega)$ ($SH^-(\Omega)$). An upper semi-continuous function $u : \Omega \to [-\infty, +\infty)$ was defined on a domain Ω in $\mathbb{C}^n$ which is not identically $-\infty$ is called to be plurisubharmonic if $u|_{l\cap\Omega}$ is subharmonic on $l \cap \Omega$ or $u|_{l\cap\Omega} \equiv -\infty$ for every complex line l in $\mathbb{C}^n$. The set of plurisubharmonic functions (negative plurisubharmonic functions) on Ω is denoted by $PSH(\Omega)$ ($PSH^-(\Omega)$). From these definitions, we see that $SH(\Omega)$ and $PSH(\Omega)$ are the local classes, i.e $u \in SH(\Omega)$ if and only if it is subharmonic in a neighbourhood of each point of Ω.

The local property is also mentioned when studying some important subclasses of plurisubharmonic functions. In [8], Cegrell has introduced the class $\mathcal{E}(\Omega)$ of plurisubharmonic functions on which the complex Monge - Ampère operator $(dd^c.)^n$ is well defined and is continuous under decreasing sequences of plurisubharmonic functions. Blocki (by Theorem 1.1 in [6]) has shown that $\mathcal{E}(\Omega)$ is the local class. In [4], the authors have introduced the weighted energy class $\mathcal{E}_\chi(\Omega)$ with Ω is a hyperconvex domain in $\mathbb{C}^n$: assume that $\chi : \mathbb{R}^- \to \mathbb{R}^+$

Communicated by Kaushal Verma.

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is a decreasing function. Then, as in [4], we define

$$\mathcal{E}_\chi(\Omega) = \left\{\varphi \in PSH^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \downarrow \varphi, \sup_{j\geq 1}\int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n < +\infty\right\},$$

where $\mathcal{E}_0(\Omega)$ is the cone of all bounded plurisubharmonic functions φ defined on Ω with finite total Monge - Ampère mass and $\lim_{z\to\xi}\varphi(z) = 0$ for all $\xi \in \partial\Omega$ (see [7]). Then from the proofs of Theorem 1.4 and 1.5 in [1], it follows that if $\varphi \in \mathcal{E}_\chi(\Omega)$ then $\lim_{z\to\xi}\varphi(z) = 0$ for all $\xi \in \partial\Omega$. Hence if $\varphi \in \mathcal{E}_\chi(\Omega)$ then $\varphi \notin \mathcal{E}_\chi(D)$ with D is a relatively compact hyperconvex subdomain in Ω. Thus the class $\mathcal{E}_\chi(\Omega)$ is not the local one. In [10], the authors have introduced the class

$$\mathcal{E}_{\chi,loc}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\ \psi_D \in \mathcal{E}_\chi(\Omega) \text{ such that } \varphi = \psi_D,\ \forall\ D \Subset \Omega\}.$$

Then under some extra assumptions on χ, the class $\mathcal{E}_{\chi,loc}(\Omega)$ is the local one (Theorem 3.4 in [10]). The main aim of this note is first we are going to give the definition of the local classes. Then we use these concepts to study the local property of some subclasses of plurisubharmonic functions on a domain in $\mathbb{C}^n$.

2 Definition of the local classes

Let X be a topological space. $\mathcal{K}$ is called to be a family of functions on open sets in X if $\mathcal{K}(D)$ is a class of functions on D for all open set D in X. We introduce some notions of the local property on a family $\mathcal{K}$ as follows.

Definition 2.1 Let $\mathcal{K}$ be a family of functions on open sets in a topological space X.

(i) $\mathcal{K}$ is called local class of type 1 if

$$\mathcal{K}(U)|_V = \{f|_V : f \in \mathcal{K}(U)\} \subset \mathcal{K}(V),$$

for all open sets $V \subset U \subset X$.

(ii) $\mathcal{K}$ is called the local class of type 2 if a function f on U such that for all $x \in U$ there exists a neighbourhood V of x with $f|_V \in \mathcal{K}(V)$ then $f \in \mathcal{K}(U)$.

(iii) $\mathcal{K}$ is called the local class if $\mathcal{K}$ is a local class of type 1 and 2.

Definition 2.2 Let $\mathcal{K}$ be a family of functions on open sets in a topological space X.

(i) We denote $\mathcal{K}^1_{loc}$ by a family of functions on open sets in a topological space X as follows $\mathcal{K}^1_{loc}(X)$ is the class of functions on X such that for all $x \in X$ there exists a neighbourhood U of x with $f|_U \in \mathcal{K}(X)|_U$.

(ii) We denote $\mathcal{K}^2_{loc}$ by a family of functions on open sets in a topological space X as follows $\mathcal{K}^2_{loc}(X)$ is the class of functions on X such that for all $x \in X$ there exists a neighbourhood U of x with $f|_U \in \mathcal{K}(U)$.

It is well-known that families of all continuous functions, upper semi-continuous functions, smooth functions, holmorphic functions, convex functions, subharmonic functions, plurisuharmonic functions are the local classes. we will establish some basic results.

Proposition 2.3 *Let $\mathcal{K}$ be a family of functions on open sets in a topological space X. Then:*

(i) $\mathcal{K} \subset \mathcal{K}^1_{loc}$.

(ii) If $\mathcal{K}$ is the local class of type 1 then $\mathcal{K}^1_{loc} \subset \mathcal{K}^2_{loc}$.

(iii) $\mathcal{K}$ is the local class of type 2 if and only if $\mathcal{K}^2_{loc} \subset \mathcal{K}$.

(iv) $\mathcal{K}$ is the local class if and only if $\mathcal{K}^1_{loc} = \mathcal{K}^2_{loc} = \mathcal{K}$.

Proof Let D be an open subset of X

(i) We prove that $\mathcal{K}(D) \subset \mathcal{K}^1_{loc}(D)$. Indeed, if $f \in \mathcal{K}(D)$ and $x \in D$ then we can choose $U = D$ as a neighbourhood of x and we have $f|_U = f|_D = f \in \mathcal{K}(D)$. So $f \in \mathcal{K}^1_{loc}(D)$.

(ii) Let $f \in \mathcal{K}^1_{loc}(D)$ and $x \in D$. Then there exists a neighbourhood U of x and a function $g \in \mathcal{K}(D)$ such that $f|_U = g|_U$. By the local property of type 1 of $\mathcal{K}$ we can infer $g|_U \in \mathcal{K}(U)$. Hereby $f|_U \in \mathcal{K}(U)$ and this means that $f \in \mathcal{K}^2_{loc}(D)$.

(iii) Suppose that $\mathcal{K}$ is the local class of type 2. If $f \in \mathcal{K}^2_{loc}(D)$ then for all $x \in D$ there exists a neighbourhood U of x such that $f|_U \in \mathcal{K}(U)$. By the the local property of type 2 of $\mathcal{K}$ we have $f \in \mathcal{K}(D)$.
On the other hand, suppose that $\mathcal{K}^2_{loc} \subset \mathcal{K}$. Let f is a function on D such that for all $x \in D$ there exists a neighbourhood U of x with $f|_U \in K(U)$. Then $f \in \mathcal{K}^2_{loc}(D)$ and so $f \in \mathcal{K}(D)$.

(iv) It follows from i), ii) and iii) above. □

3 The local properties of some subclasses of plurisubharmonic functions

Let Ω be a hyperconvex domain in $\mathbb{C}^n$. We define $\mathcal{B}_0(\Omega)$ and $\overline{\mathcal{B}}_0(\Omega)$ as follows:

$$\mathcal{B}_0(\Omega) = \{\varphi \in PSH^-(\Omega) : \lim_{z\to w} \varphi(z) = 0,\ \forall w \in \partial\Omega\},$$
$$\overline{\mathcal{B}}_0(\Omega) = \{\varphi \in PSH^-(\Omega) : \overline{\lim_{z\to w}} \varphi(z) = 0,\ \forall w \in \partial\Omega\}.$$

Proposition 3.1 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$. Then $\mathcal{B}_0(\Omega)$ and $\overline{\mathcal{B}}_0(\Omega)$ are only the local classes of type 2.*

Proof By the maximum principle of plurisubharmonic functions, we have

$$\mathcal{B}^2_{0,loc}(\Omega) = \overline{\mathcal{B}}^2_{0,loc}(\Omega) = \{0\}.$$

By Proposition 2.3 (iii), we get that $\mathcal{B}_0(\Omega)$ and $\overline{\mathcal{B}}_0(\Omega)$ are the local classes of type 2.

On the other hand, let $f \in PSH^- \cap L^\infty(\Omega)$ and let $D \Subset \Omega$. By Theorem 2.1 in [8] we can choose $\mathcal{E}_0 \cap C(\overline{\Omega}) \ni f_j \searrow f$. Set

$$h^{f_j}_{D,\Omega} = \sup\{\varphi \in PSH^-(\Omega) : \varphi \le f_j \text{ on } D\}$$

Then $h^{f_j}_{D,\Omega} \in \mathcal{E}_0(\Omega)$, $h^{f_j}_{D,\Omega}|_D = f_j|_D$ and $h^{f_j}_{D,\Omega} \searrow h^f_{E,\Omega}$. This infers that $h^f_{D,\Omega} \in \mathcal{B}_0(\Omega)$ and $h^f_{D,\Omega} = f$ on D. Hence $f \in \mathcal{B}^1_{0,loc}(\Omega)$. So we have

$$PSH^- \cap L^\infty(\Omega) \subset \mathcal{B}^1_{0,loc}(\Omega) \subset \overline{\mathcal{B}}^1_{0,loc}(\Omega).$$

These, by Proposition 2.3 (ii), imply that $\mathcal{B}_0(\Omega)$ and $\overline{\mathcal{B}}_0(\Omega)$ are not the local classes of type 1. □

Proposition 3.2 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$. Then $\mathcal{B}_0 \cap L^\infty(\Omega)$ and $\overline{\mathcal{B}}_0 \cap L^\infty(\Omega)$ are only the local classes of type 2.*

Proof The proof is similar to the proof of Proposition 3.1. □

We define $\mathcal{E}_0(\Omega)$ as follows:

$$\mathcal{E}_0(\Omega) = \{\varphi \in \mathcal{B}_0 \cap L^\infty(\Omega) : \int_\Omega (dd^c\varphi)^n < +\infty\}.$$

Proposition 3.3 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$. Then $\mathcal{E}_0(\Omega)$ is only the local class of type 2.*

Proof By the maximum principle of plurisubharmonic functions, we have

$$\mathcal{E}^2_{0,loc}(\Omega) = \{0\}.$$

By Proposition 2.3 (iii), we get that $\mathcal{E}_0(\Omega)$ is the local class of type 2.
Now we prove that

$$\mathcal{E}^1_{0,loc}(\Omega) = PSH^- \cap L^\infty(\Omega).$$

Indeed, for $f \in PSH^- \cap L^\infty(\Omega)$ and $x \in \Omega$, we can choose $r > 0$ small enough such that $B = B(x, r) \Subset \Omega$. Then by [12] we imply that $h^f_{B,\Omega} \in \mathcal{E}_0(\Omega)$ and obviously we also have $h^f_{B,\Omega} = f$ on B. So we have $f \in \mathcal{E}^1_{0,loc}(\Omega)$. Conversely, if $f \in \mathcal{E}^1_{0,loc}(\Omega)$ then by the local property of the class of $PSH^-(\Omega)$, we have $f \in PSH^-(\Omega)$. And by the definition of $\mathcal{E}^1_{0,loc}(\Omega)$ we have $f \in L^\infty(\Omega)$. Since $f \in PSH^- \cap L^\infty(\Omega)$ and the desired conclusion follows.
This, by Proposition 2.3 (ii), implies that $\mathcal{E}_0(\Omega)$ is not the local class of type 1. □

Remark 3.4 (i) We define $\mathcal{K}$ by a family of functions on open sets in Ω as follows $\mathcal{K}(\Omega) = \{0\}$ and $\mathcal{K}(U) = \{0, -1\}$ for all open U of Ω with $U \neq \Omega$. We can check that $\mathcal{K}^1_{loc} = \mathcal{K}$, $\mathcal{K}^2_{loc} = \{0, -1\} \neq \mathcal{K}$ and $\mathcal{K}$ is only the local class of type 1.

(ii) We define $\mathcal{K}$ by a family of functions on open sets in Ω as follows $\mathcal{K}(\Omega) = \{0\}$ and $\mathcal{K}(U) = \{-1\}$ for all open U of Ω with $U \neq \Omega$. We can check that $\mathcal{K}^1_{loc} = \mathcal{K}$, $\mathcal{K}^2_{loc} = \{-1\}$ and $\mathcal{K}$ is not the local class of type 1 and 2.

Theorem 3.5 *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function ($\chi \not\equiv 0$). Assume that E be a complete pluripolar set in Ω. Then, there exists a function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $E = \{\varphi = -\infty\}$.*

Proof We choose a convex, strictly decreasing and smooth function $\theta : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ such that $\theta(t) \geq \chi(t) + |t|$, $\forall t \in \mathbb{R}^-$. We set By Theorem 3.1 in [11], we choose a function $u \in \mathcal{F}_\infty(\Omega)$ such that $E = \{u = -\infty\}$. We set

$$v = \max((-\theta)^{-1} \circ [u - \theta(0)], u).$$

We have $v \in \mathcal{F}_\infty(\Omega)$ and $E = \{v = -\infty\}$. Now, we only need to prove that $v \in \mathcal{E}_\chi(\Omega)$. Indeed, by Theorem 2.1 in [8] we can find $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$. Set

$$v_j = \max((-\theta)^{-1} \circ [u_j - \theta(0)], u_j).$$

Then $v_j \in \mathcal{E}_0(\Omega)$ and $v_j \searrow v$. By Lemma 3.3 in [2], we have

$$\int_\Omega \chi(v_j)(dd^c v_j)^n \leq \int_\Omega [-u_j + \theta(0)](dd^c v_j)^n \leq \int_\Omega [-u_j + \theta(0)](dd^c u_j)^n.$$

Therefore

$$\sup_{j\geq 1} \int_\Omega \chi(v_j)(dd^c v_j)^n \leq \sup_{j\geq 1} \int_\Omega [-u_j + \theta(0)](dd^c u_j)^n < +\infty.$$

□

Theorem 3.6 *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function such that $\chi(t) > 0, \forall t \in \mathbb{R}^-$. Then $\mathcal{E}_\chi(\Omega) \subset \overline{\mathcal{B}}_0(\Omega)$ and $\mathcal{E}_\chi(\Omega) \not\subset \mathcal{B}_0(\Omega)$.*

Proof Let $\{D_j\}_{j\geq 1}$ be a sequence of subsets in Ω such that $D_j \Subset D_{j+1}$ and $\Omega = \cup_{j\geq 1} D_j$. Take $\varphi \in \mathcal{E}_\chi(\Omega)$. By Theorem 3.1 in [9], we get $h^\varphi_{D_j,\Omega} \nearrow 0$ as $j \to \infty$ a.e in dV_{2n}. This implies that $\overline{\lim\limits_{z\to w}} \varphi(z) = 0$ for all $w \in \partial\Omega$. Hence $\varphi \in \overline{\mathcal{B}}_0(\Omega)$.
To show that $\mathcal{E}_\chi(\Omega) \not\subset \mathcal{B}_0(\Omega)$ we consider the following example.
Set $E = \Omega \cap \mathbb{Q}^n$. From E is a pluripolar set and from Theorem 3.5 we can find a function $\varphi \in \mathcal{E}_\chi(\Omega)$ and $E \subset \{z \in \Omega : \varphi(z) = -\infty\}$. We have

$$\underline{\lim_{z\to w}} \varphi(z) \leq \underline{\lim_{z\in\Omega\cap\mathbb{Q}^n, z\to w}} \varphi(z) = -\infty.$$

for all $w \in \partial\Omega$. □

Remark 3.7 We set $t_0 = \sup\{t \in \mathbb{R}^- : \chi(t) > 0\}$. Then $\varphi \in \mathcal{E}_\chi(\Omega)$ for all $\varphi \in PSH^-(\Omega)$ with $\varphi \geq -t_0$. Take $\varphi \in \mathcal{E}_\chi(\Omega)$. By the proof of Theorem 3.1 in [9], we have

$$\underline{\lim_{z\to w}} \varphi(z) \geq -t_0, \ \forall w \in \partial\Omega.$$

Now, we are going to state main theorem as follows.

Theorem 3.8 *Assume that $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ is a decreasing function and*

$$t_0 = \sup\{t \in \mathbb{R}^- : \chi(t) > 0\}.$$

Then

(i) $(\mathcal{E}_\chi)^1_{loc}(\Omega) = \mathcal{E}_{\chi,loc}(\Omega)$ and $(\mathcal{E}_\chi)^2_{loc}(\Omega) = \{\varphi \in PSH^-(\Omega) : \varphi \geq -t_0\}$.
(ii) $\mathcal{E}_\chi(\Omega)$ is only the local class of type 2.

Proof i) By the main result in [10], the class $\mathcal{E}_{\chi,loc}(\Omega)$ is the local one, so we get $\mathcal{E}_{\chi,loc}(\Omega) = (\mathcal{E}_\chi)^1_{loc}(\Omega)$. Take $\varphi \in (\mathcal{E}_\chi)^2_{loc}(\Omega)$. Let $w \in \Omega$. For every $D \Subset \Omega$ such that $w \in \partial D$ we have $\varphi|_D \in \mathcal{E}_\chi(D)$. So we have

$$\varphi(w) \geq \overline{\lim_{z\to w}}\, \varphi(z) \geq \overline{\lim_{z\in D, z\to w}}\, \varphi(z) \geq \underline{\lim_{z\to w}}\, \varphi(z) \geq -t_0.$$

Conversely, let $\varphi \in PSH^-(\Omega)$ be such that $\varphi \geq -t_0$. Let $D \Subset \Omega$, then $\varphi \in PSH^-(D)$. By Theorem 2.1 in [8] we have $\varphi \in \mathcal{E}_\chi(D)$. So $\varphi \in (\mathcal{E}_\chi)^2_{loc}(\Omega)$.
ii) It follows from i) and Remark 3.7. □

Now, we recall the Cegrell's classes.

$$\mathcal{F}(\Omega) = \left\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi, \sup_{j\geq 1}\int_\Omega (dd^c\varphi_j)^n < +\infty\right\}.$$

$$\mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighbourhood } \omega \ni z_0,$$
$$\mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi \text{ on } \omega, \sup_{j\geq 1}\int_\Omega (dd^c\varphi_j)^n < +\infty\Big\}.$$

In [5] and [6], Blocki has proved that $\mathcal{E}(\Omega)$ is the local class.

Remark 3.9 (i) By the remark following Definition 4.6 in [8] we have: for every $u \in \mathcal{E}(\Omega)$ and every $\omega \Subset \Omega$, there is a $u_\omega \in \mathcal{F}(\Omega)$ sucth that $u = u_\omega$ on ω.
(ii) It is known from [3], [4] that if χ is bounded and $\chi(0) > 0$ then $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$.

Corollary 3.10 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$. Then $\mathcal{F}(\Omega)$ is only local class of type 2.*

Proof Assume that χ is bounded and $\chi(0) > 0$. Then $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$ and

$$t_0 = \sup\{t \in \mathbb{R}^- : \chi(t) > 0\} = 0.$$

So by Theorem 3.8(i) we have $\mathcal{F}^2_{loc}(\Omega) = (\mathcal{E}_\chi)^2_{loc}(\Omega) = \{0\}$. Hence $\mathcal{F}(\Omega)$ is the local class of type 2.
On the other hand, we have $\mathcal{F}^1_{loc}(\Omega) = \mathcal{E}(\Omega)$. So by Proposition 2.3(ii), $\mathcal{F}(\Omega)$ is not the local class of type 1. □

Let Ω be a hyperconvex domain and let $\{\Omega_j\}$ be a fundamental sequence of Ω. Let $u \in \mathcal{E}(\Omega)$ be given. For each $j \geq 1$, we set

$$u_j = \sup\{\varphi \in \mathrm{PSH}(\Omega) : \varphi|_{C\Omega_j} \leq u|_{C\Omega_j}\}$$

and

$$\widetilde{u} = (\lim_{j\to\infty} u_j)^*.$$

Then we have

$$\widetilde{u} \geq \lim_{j\to\infty} u_j \geq u_{j+1} \geq u_j \geq u.$$

So $\widetilde{u}, u_j \in \mathcal{E}(\Omega)$ for all $j \geq 1$. Moreover, the definition of $\widetilde{u}$ is independent of the sequence $\{\Omega_j\}$ and $(dd^c\widetilde{u})^n = 0$. We define

$$\mathcal{M}(\Omega) = \{u \in \mathcal{E}(\Omega) : (dd^c u)^n = 0\}, \quad \mathcal{M}^\infty(\Omega) = \mathcal{M} \cap L^\infty(\Omega),$$
$$\mathcal{N}(\Omega) = \{u \in \mathcal{E}(\Omega) : \widetilde{u} = 0\}.$$

Then we have $\widetilde{u}$ which is the smallest maximal plurisubharmonic function above u. So we infer that

$$\mathcal{M}(\Omega) = \{u \in \mathcal{E}(\Omega) : \widetilde{u} = u\}.$$

Theorem 3.11 *The class $\mathcal{M}(\Omega)$ is the local one.*

Proof Let $u \in \mathcal{M}^2_{loc}(\Omega)$. Then for each $z \in \Omega$ there exists a neighbourhood U of z such that $u|_U \in \mathcal{M}(U)$. By the local property of $\mathcal{E}(\Omega)$ we have $u \in \mathcal{E}(\Omega)$. Since $(dd^c u)^n|_U = (dd^c u|_U)^n = 0$ so $(dd^c u)^n = 0$ on Ω. These implies $u \in \mathcal{M}(\Omega)$. By Proposition 2.3(iii), we get that $\mathcal{M}(\Omega)$ is the local class of type 2.
Let $u \in \mathcal{M}^1_{loc}(\Omega)$. Then for each $z \in \Omega$ there exists a neighbourhood U of a and $v \in \mathcal{M}(\Omega)$ such that $u = v$ on U. By the local property of $\mathcal{E}(\Omega)$ we have $u|_U \in \mathcal{E}(U)$. On the other hand, $(dd^c u|_U)^n = (dd^c v|_U)^n = (dd^c v)^n|_U = 0$. These infer that $u|_U \in \mathcal{M}(U)$. So we have $u \in \mathcal{M}^2_{loc}(\Omega)$. To conclude, we have $\mathcal{M}(\Omega) = \mathcal{M}^1_{loc}(\Omega) = \mathcal{M}^2_{loc}(\Omega)$ and thus the class $\mathcal{M}(\Omega)$ is the local one. □

Acknowledgements I am grateful to Professor Pham Hoang Hiep for valuable comments during the preparation of this work.

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