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ORIGINAL RESEARCH

Congruences with q-harmonic numbers and q-binomial coefficients

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Abstract In this paper, we show the congruences $\sum_{k=m}^{p-1} q^k \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_{k,2}(q) \pmod{[p]_q^2}$ and $\sum_{k=m}^{p-1} q^k \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_k^2(q) \pmod{[p]_q^2}$ with a prime number $p > 2$ and $m = 0, 1, 2, .., p-1$.

Keywords Congruence · q-Analog · q-Harmonic number

Mathematics Subject Classification 11A07, 11B65

1 Introduction

The Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } Re(s) > 1.$$

The harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n \in \mathbb{Z}^+.$$

Wolstenholme [14] discovered that for any prime number $p \geq 5$,

$$H_{p-1} \equiv 0 \pmod{p^2}.$$

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For $r \in \mathbb{Z}^+$, harmonic numbers of order r are those rational numbers

$$H_{0,r} = 0 \text{ and } H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r} \text{ for } n \in \mathbb{Z}^+.$$

For non-negative integer n, the analog of n is

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}.$$

It is seen that $\lim_{q\to 1} [n]_q = n$.

The q-harmonic number of order r is given by

$$\widetilde{H}_{n,r}(q) = \sum_{k=1}^{n} \frac{q^k}{[k]_q^r}.$$

When $r = 1$, $\widetilde{H}_{n,1}(q) = \widetilde{H}_n(q)$ is q-harmonic number. The q-Pochhammer symbol is given by

$$(x;q)_0 = 1 \text{ and } (x;q)_n = \prod_{k=0}^{n-1} \left(1 - xq^k\right).$$

For any $m, n \in \mathbb{N}$, the q-binomial coefficients (or Gauss coefficients) are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & \text{if } n \geq m, \\ 0, & \text{if } n < m. \end{cases}$$

It is clear that

$$\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m},$$

where $\binom{n}{m}$ is the usual binomial coefficient. The q-binomial coefficients satisfy the recurrence relation

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q, \tag{1}$$

$$\begin{bmatrix} k+1 \\ m \end{bmatrix}_q = \frac{[k+1]_q}{[k+1-m]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q, \text{ for } k \geq m. \tag{2}$$

Also, if $k \geq m$, then

$$[k-m]_q - [k]_q = -q^{k-m} [m]_q . \tag{3}$$

In [8], Shi et al. showed that for any prime number $p \geq 5$,

$$\widetilde{H}_{p-1}(q) \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2 [p]_q \pmod{[p]_q^2}, \tag{4}$$

and

$$\widetilde{H}_{p-1,2}(q) \equiv -\frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q}. \tag{5}$$

In [3], for any prime number p, integer α and any integers $k = 0, 1, ..., \alpha p - 1$, it is seen that

$$\begin{aligned} &\begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\ &\equiv (-1)^k q^{\alpha pk - \binom{k+1}{2}} \left(1 - q^{-\alpha p} [p]_q [\alpha]_{q^p} \widetilde{H}_k(q) + \frac{1}{2} q^{-2\alpha p} [p]_q^2 [\alpha]_{q^p}^2 \right. \\ &\left. \times \left(\widetilde{H}_k^2(q) - q^k \widetilde{H}_{k,2}(q) - (1-q)\left([k]_q \widetilde{H}_{k,2}(q) - \widetilde{H}_k(q)\right)\right)\right) \pmod{[p]_q^3}. \end{aligned} \tag{6}$$

In [7], Pan established that for any prime number p,

$$q^{kp} \equiv 1 - k\,(1-q)\,[p]_q + \binom{k}{2}(1-q)^2\,[p]_q^2 \pmod{[p]_q^3}. \tag{7}$$

In [11], Sved gave the q-analog of the well-known binomial identity as follows: For any integers n, p and m with $m \le n - p$,

$$\sum_{k=m}^{n-p} q^{(k-m)(p+1)} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} n-k \\ p \end{bmatrix}_q = \begin{bmatrix} n+1 \\ m+p+1 \end{bmatrix}_q. \tag{8}$$

In [2], Graham et al. gave that for $n, m \in \mathbb{N}$,

$$\sum_{k=1}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} H_{n+1} - \frac{1}{m+1}.$$

In [9], Spieß showed that for $n, m \in \mathbb{N}$,

$$\sum_{k=1}^{n} \binom{k}{m} H_{k,2} = \binom{n+1}{m+1} H_{n,2} - \frac{1}{(m+1)!} \sum_{k=1}^{m+1} s(m+1,k) H_{n,2-k},$$

where $s(n,k)$ are Stirling numbers of the first kind.

In [10], Sun proved that for any prime number $p > 3$,

$$\sum_{k=1}^{p-1} H_k^2 \equiv 2p - 2 \pmod{p^2},$$

$$-2\sum_{k=1}^{p-1} \frac{H_{k,2}^2}{k^2} \equiv \sum_{\substack{0 \le i,j \le p-3 \\ i+j=2p-6}} \binom{p-2}{i}\binom{p-2}{j} B_i B_j \pmod{p},$$

where B_n is the nth Bernoulli number.

In [5], Meštrović established that for any prime number $p > 5$,

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p^2(2) + \frac{2}{3} p q_p^3(2) + \frac{p}{12} B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} \equiv -\frac{1}{3} q_p^3(2) - \frac{25}{24} B_{p-3} \pmod{p^3},$$

where $q_p(2) = \left(2^{p-1} - 1\right)/p$ is the Fermat quotient of p to base 2.

In [6], Meštrović et al. obtained that for each prime number $p > 3$ and all integers $m = 0, 1, ..., p-2$,

$$\sum_{k=m}^{p-1} \binom{k}{m} H_k \equiv \frac{(-1)^m}{m+1}\left(1 - pH_{m+1} + \frac{p^2}{2}\left(H_{m+1}^2 - H_{m+1,2}\right)\right) \pmod{p^3}.$$

In [13], Wang et al. showed congruences with the form $\sum_{k=1}^{p-1} k^m H_k^n \pmod{p^2}$ for $m = 1, 2, ..., p-2$ and $n = 1, 2, 3$. For any prime number $p > 3$,

$$\sum_{k=1}^{p-1} k^m H_k \equiv B_m - \frac{p}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k H_{m-k} \pmod{p^2},$$

and in [12], Wang et al. gave the kind of congruences involving $\sum_{k=1}^{p-1} k^m H_{k,r} \pmod{p^2}$ and $\sum_{k=1}^{p-1} k^{p-2} \left(H_{k,2}\right)^2 \pmod{p^2}$. Let $p > 3$ be a prime number and m, r be two positive integers less than $p(p-1)$ such that $m \not\equiv -1 \pmod{p}$ and $p - 1 \nmid m, r$. If $m \leq r$ and $\delta_{m,r}$ is the Kronecker delta, then

$$(m+1)\sum_{k=1}^{p-1} k^m H_{k,r} \equiv \frac{p}{2}\delta_{m,r} - \sum_{k=\delta_{m,r}}^{m} \binom{m+1}{k} B_k H_{p-1,r-m+k-1} \pmod{p^2}.$$

In [4], Medžiūnas investigated congruence relationships of particular finite sums involving generalized harmonic numbers. Let m, r be non-negative integers and p any prime number such that $\max(m, r) + 3 \leq p$. For $0 < r \leq m$, then

$$\begin{aligned}\sum_{k=1}^{p-1} k^m H_{k,r} \equiv\ & \frac{1}{m+1}\binom{m+1}{r} B_{m-r+1} + \frac{p}{m+1}\sum_{k=1}^{m-r+1}\binom{m+1}{k} B_{m-r-k+1} B_k \\ & - \frac{p}{m+1}\sum_{\substack{k=m-r+2 \\ r\geq 2}}^{m}\binom{m+1}{k}\frac{r-m+k-1}{r-m+k} B_{p-r+m-k} B_k \pmod{p^2}.\end{aligned}$$

In [1], Abel's partial summation formula asserts that for every pair of families $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ of complex numbers, there is the relation

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1}\left[(a_k - a_{k+1})\left(\sum_{j=1}^{k} b_j\right)\right] + a_n\left(\sum_{j=1}^{n} b_j\right). \tag{9}$$

2 On congruences with q-harmonic numbers

In this section, for the proof of Theorem 1, we will need the following auxiliary results.

Lemma 1 *Let $n \geq 0$ be integer. Then for each $m = 0, 1, 2, .., n-2$,*

$$\sum_{k=m}^{n} q^{2k}\begin{bmatrix} k \\ m \end{bmatrix}_q = (1-q)q^{2m}\begin{bmatrix} n+1 \\ m+2 \end{bmatrix}_q + q^{n+m}\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q.$$

Proof Setting $a_k = q^k$ and $b_k = q^k\begin{bmatrix} k \\ m \end{bmatrix}_q$ in (9), by (8), the proof is complete. □

Lemma 2 *Let $n \geq 0$ be integer. Then for each $m = 1, 2, .., n$,*

$$\begin{aligned}\sum_{k=m}^{n}\frac{q^k}{[k+1]_q}\begin{bmatrix} k \\ m \end{bmatrix}_q = & \frac{q^{-1}}{[m]_q}\begin{bmatrix} n+1 \\ m \end{bmatrix}_q + (-1)^m q^{-\binom{m+1}{2}-1}\widetilde{H}_{n+1}(q) \\ & + (-1)^m q^{-\binom{m+1}{2}-1}\sum_{k=1}^{m-1}\frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q,\end{aligned}$$

and for each $m = 0, 1, 2, .., n$,

$$\begin{aligned}&\sum_{k=m}^{n}\frac{q^{2k}}{[k+1]_q}\begin{bmatrix} k \\ m \end{bmatrix}_q \\ &= (-1)^m (1-q)q^{-\binom{m+1}{2}-2}\sum_{k=1}^{m}(-1)^k\frac{q^{\binom{k+1}{2}+k}}{[k]_q}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \\ &\quad + (-1)^m q^{n-1-\binom{m+1}{2}}\sum_{k=1}^{m}(-1)^k\frac{q^{\binom{k+1}{2}}}{[k]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q + (-1)^m q^{-\binom{m+1}{2}}\sum_{k=0}^{n}\frac{q^{2k}}{[k+1]_q}.\end{aligned}$$

Proof Let $L(m,n)=\sum_{k=m}^{n}\frac{q^k}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q$. By (1) and (8), we write

$$\begin{aligned}q^m L(m,n)+L(m-1,n)&=\sum_{k=m}^{n}\frac{q^{k+m}}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q+\sum_{k=m-1}^{n}\frac{q^k}{[k+1]_q}\begin{bmatrix}k\\m-1\end{bmatrix}_q\\&=\sum_{k=m}^{n}\frac{q^k}{[k+1]_q}\left(q^m\begin{bmatrix}k\\m\end{bmatrix}_q+\begin{bmatrix}k\\m-1\end{bmatrix}_q\right)+\frac{q^{m-1}}{[m]_q}\\&=\frac{1}{[m]_q}\sum_{k=m}^{n}q^k\begin{bmatrix}k\\m-1\end{bmatrix}_q+\frac{q^{m-1}}{[m]_q}\\&=\frac{q^{m-1}}{[m]_q}\begin{bmatrix}n+1\\m\end{bmatrix}_q.\end{aligned}$$

Then, from $L(0,n)=\frac{1}{q}\widetilde{H}_{n+1}(q)$, we have

$$\begin{aligned}L(m,n)&=\frac{q^{-1}}{[m]_q}\begin{bmatrix}n+1\\m\end{bmatrix}_q-q^{-m}L(m-1,n)\\&=\frac{q^{-1}}{[m]_q}\begin{bmatrix}n+1\\m\end{bmatrix}_q-q^{-m}\left(\frac{q^{-1}}{[m-1]_q}\begin{bmatrix}n+1\\m-1\end{bmatrix}_q-q^{-m+1}L(m-2,n)\right)\\&=\frac{q^{-1}}{[m]_q}\begin{bmatrix}n+1\\m\end{bmatrix}_q-\frac{q^{-m-1}}{[m-1]_q}\begin{bmatrix}n+1\\m-1\end{bmatrix}_q+q^{-2m+1}L(m-2,n)\\&=\frac{q^{-1}}{[m]_q}\begin{bmatrix}n+1\\m\end{bmatrix}_q-\frac{q^{-m-1}}{[m-1]_q}\begin{bmatrix}n+1\\m-1\end{bmatrix}_q+\frac{q^{-2m}}{[m-2]_q}\begin{bmatrix}n+1\\m-2\end{bmatrix}_q\\&\quad-\frac{q^{-3m+2}}{[m-3]_q}\begin{bmatrix}n+1\\m-3\end{bmatrix}_q+\frac{q^{-4m+5}}{[m-4]_q}\begin{bmatrix}n+1\\m-4\end{bmatrix}_q-q^{-5m+10}L(m-5,n)\\&\quad\vdots\\&=\frac{q^{-1}}{[m]_q}\begin{bmatrix}n+1\\m\end{bmatrix}_q+(-1)^m q^{-\binom{m+1}{2}-1}\widetilde{H}_{n+1}(q)\\&\quad+(-1)^m q^{-\binom{m+1}{2}-1}\sum_{k=1}^{m-1}\frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q,\end{aligned}$$

as claimed. Similarly, from (1) and Lemma 1, we have the proof of other identity. □

Lemma 3 *Let $n\geq 0$ and $r\geq 1$ be integers. Then for each $m=1,2,..,n-1$,*

$$[m+1]_q\sum_{k=m}^{n-1}q^{k-m}\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)=[n-m]_q\begin{bmatrix}n\\m\end{bmatrix}_q\widetilde{H}_{n,r}(q)-q\sum_{k=m}^{n-1}\frac{q^k}{[k+1]_q^{r-1}}\begin{bmatrix}k\\m\end{bmatrix}_q,$$

and $m=0,1,2,..,n-1$,

$$\begin{aligned}&[m+1]_q\sum_{k=m}^{n-1}q^{k-m}\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_k^2(q)\\&=[n-m]_q\begin{bmatrix}n\\m\end{bmatrix}_q\widetilde{H}_n^2(q)-2q\sum_{k=m}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_k(q)-\sum_{k=m}^{n-1}\frac{q^{2k+2}}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q.\end{aligned}$$

Proof Consider that

$$\sum_{k=m}^{n}[k-m]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)=\sum_{k=m-1}^{n-1}[k+1-m]_q\begin{bmatrix}k+1\\m\end{bmatrix}_q\widetilde{H}_{k+1,r}(q)$$
$$=\widetilde{H}_{m,r}(q)+\sum_{k=m}^{n-1}[k+1-m]_q\begin{bmatrix}k+1\\m\end{bmatrix}_q\widetilde{H}_{k+1,r}(q).$$

From (2), we have

$$\sum_{k=m}^{n}[k-m]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)$$
$$=\widetilde{H}_{m,r}(q)+\sum_{k=m}^{n-1}[k+1-m]_q\frac{[k+1]_q}{[k+1-m]_q}\begin{bmatrix}k\\m\end{bmatrix}_q\left(\widetilde{H}_{k,r}(q)+\frac{q^{k+1}}{[k+1]_q^r}\right)$$
$$=\widetilde{H}_{m,r}(q)+\sum_{k=m}^{n-1}[k+1]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+\sum_{k=m}^{n-1}[k+1]_q\frac{q^{k+1}}{[k+1]_q^r}\begin{bmatrix}k\\m\end{bmatrix}_q$$
$$=\widetilde{H}_{m,r}(q)+\sum_{k=m}^{n-1}\left([k]_q+q^k\right)\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+q\sum_{k=m}^{n-1}\frac{q^k}{[k+1]_q^{r-1}}\begin{bmatrix}k\\m\end{bmatrix}_q$$
$$=\widetilde{H}_{m,r}(q)+\sum_{k=m}^{n-1}[k]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+\sum_{k=m}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)$$
$$+q\sum_{k=m}^{n-1}\frac{q^k}{[k+1]_q^{r-1}}\begin{bmatrix}k\\m\end{bmatrix}_q.$$

Using elementary operations, we get

$$\sum_{k=m+1}^{n-1}[k-m]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+[n-m]_q\begin{bmatrix}n\\m\end{bmatrix}_q\widetilde{H}_{n,r}(q)$$
$$=\sum_{k=m}^{n-1}[k]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+\sum_{k=m}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+q\sum_{k=m}^{n-1}\frac{q^k}{[k+1]_q^{r-1}}\begin{bmatrix}k\\m\end{bmatrix}_q.$$

(3) yields that

$$\sum_{k=m+1}^{n-1}[k]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)-\frac{[m]_q}{q^m}\sum_{k=m+1}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+[n-m]_q\begin{bmatrix}n\\m\end{bmatrix}_q\widetilde{H}_{n,r}(q)$$
$$=\sum_{k=m}^{n-1}[k]_q\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+\sum_{k=m}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+q\sum_{k=m}^{n-1}\frac{q^k}{[k+1]_q^{r-1}}\begin{bmatrix}k\\m\end{bmatrix}_q,$$

and then

$$-\frac{[m]_q}{q^m}\sum_{k=m}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+[m]_q\,\widetilde{H}_{m,r}(q)+[n-m]_q\begin{bmatrix}n\\m\end{bmatrix}_q\widetilde{H}_{n,r}(q)$$
$$=[m]_q\,\widetilde{H}_{m,r}(q)+\sum_{k=m}^{n-1}q^k\begin{bmatrix}k\\m\end{bmatrix}_q\widetilde{H}_{k,r}(q)+q\sum_{k=m}^{n-1}\frac{q^k}{[k+1]_q^{r-1}}\begin{bmatrix}k\\m\end{bmatrix}_q,$$

as claimed. Similarly, the proof of other identity is given. □

Lemma 4 *Let p be any prime number. Then for each $m = 0, 1, 2, .., p-1$,*

$$\sum_{k=m}^{p-1} \frac{q^k}{[k+1]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q \equiv (-1)^m q^{-\binom{m+1}{2}-1} \times \left(\frac{1}{[p]_q} + \frac{p+1}{2}(q-1) + [p]_q \left(\frac{p^2-1}{24}(1-q)^2 - \widetilde{H}_{m,2}(q) \right) \right) \pmod{[p]_q^2},$$

and for each $m = 0, 1, 2, .., p-2$,

$$\sum_{k=m}^{p-1} \frac{q^{2k}}{[k+1]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q \equiv (-1)^m q^{-\binom{m+1}{2}-2} \left(\frac{1}{[p]_q} + \frac{p+1}{2}(q-1) - [p]_q \left(\widetilde{H}_{m,2}(q) + (1-q)\frac{q^{m+1}}{[m+1]_q} - \frac{p^2-1}{24}(1-q)^2 \right) \right) \pmod{[p]_q^2}.$$

Proof By (1), taking $n = p-1$ in Lemma 2, we have

$$\begin{aligned} \sum_{k=m}^{p-1} \frac{q^k}{[k+1]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q &= \frac{q^{-1}}{[m]_q} \begin{bmatrix} p \\ m \end{bmatrix}_q + (-1)^m q^{-\binom{m+1}{2}-1} \widetilde{H}_p(q) \\ &\quad + (-1)^m q^{-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{(-1)^k q^{\binom{k}{2}+2k}}{[k]_q} \begin{bmatrix} p-1 \\ k \end{bmatrix}_q \\ &\quad + (-1)^m q^{-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{(-1)^k q^{\binom{k}{2}+k}}{[k]_q} \begin{bmatrix} p-1 \\ k-1 \end{bmatrix}_q. \end{aligned}$$

(6) yields that

$$\begin{aligned} \sum_{k=m}^{p-1} \frac{q^k}{[k+1]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q &\equiv \frac{q^{-1}}{[m]_q} \begin{bmatrix} p \\ m \end{bmatrix}_q + (-1)^m q^{-\binom{m+1}{2}-1} \widetilde{H}_p(q) \\ &\quad + (-1)^m q^{-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{q^{k(p+1)}}{[k]_q} \left(1 - q^{-p}[p]_q \widetilde{H}_k(q)\right) \\ &\quad + (-1)^{m-1} q^{-p-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{q^{k(p+1)}}{[k]_q} \left(1 - q^{-p}[p]_q \widetilde{H}_{k-1}(q)\right) \\ &= \frac{q^{-1}}{[m]_q} \begin{bmatrix} p \\ m \end{bmatrix}_q + (-1)^m q^{-\binom{m+1}{2}-1} \widetilde{H}_p(q) \\ &\quad + (-1)^m q^{-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{q^{k(p+1)}}{[k]_q} \left(1 - q^{-p}[p]_q \widetilde{H}_k(q)\right) \\ &\quad - (-1)^m q^{-p-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{q^{k(p+1)}}{[k]_q} \left(1 - q^{-p}[p]_q \left(\widetilde{H}_k(q) - \frac{q^k}{[k]_q}\right)\right) \\ &\equiv (-1)^m q^{-\binom{m+1}{2}-1} \widetilde{H}_p(q) - (-1)^m \left(1-q^p\right) q^{-p-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{q^{k(p+1)}}{[k]_q} \\ &\quad \frac{q^{-1}}{[m]_q} \begin{bmatrix} p \\ m \end{bmatrix}_q - (-1)^m [p]_q q^{-2p-\binom{m+1}{2}-1} \sum_{k=1}^{m-1} \frac{q^{k(p+2)}}{[k]_q^2} \pmod{[p]_q^2}. \end{aligned}$$

Applying congruence (7) into above congruence, we have

$$\sum_{k=m}^{p-1}\frac{q^k}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q \equiv \frac{q^{-1}}{[m]_q}\begin{bmatrix}p\\m\end{bmatrix}_q - (-1)^m\,[p]_q\,(1-q)\,q^{-\binom{m+1}{2}-1}\sum_{k=1}^{m-1}\frac{q^k}{[k]_q} + (-1)^m\,q^{-\binom{m+1}{2}-1}\widetilde{H}_p(q) - (-1)^m\,[p]_q\,q^{-\binom{m+1}{2}-1}\sum_{k=1}^{m-1}\frac{q^{2k}}{[k]_q^2} \pmod{[p]_q^2}.$$

Using $\sum_{k=1}^{n}\frac{q^{2k}}{[k]_q^2} = \widetilde{H}_{n,2}(q) - (1-q)\widetilde{H}_n(q)$, we write

$$\begin{aligned}\sum_{k=m}^{p-1}\frac{q^k}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q &\equiv \frac{q^{-1}}{[m]_q}\begin{bmatrix}p\\m\end{bmatrix}_q + (-1)^m\,q^{-\binom{m+1}{2}-1}\widetilde{H}_p(q)\\ &\quad - (-1)^m\,[p]_q\,(1-q)\,q^{-\binom{m+1}{2}-1}\widetilde{H}_{m-1}(q)\\ &\quad - (-1)^m\,[p]_q\,q^{-\binom{m+1}{2}-1}\left(\widetilde{H}_{m-1,2}(q) - (1-q)\widetilde{H}_{m-1}(q)\right) \pmod{[p]_q^2},\end{aligned}$$

and by (4) and (7),

$$\begin{aligned}\sum_{k=m}^{p-1}\frac{q^k}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q &\equiv \frac{q^{-1}}{[m]_q}\begin{bmatrix}p\\m\end{bmatrix}_q\\ &\quad + (-1)^m\,q^{-\binom{m+1}{2}-1}\left(\frac{p-1}{2}(q-1)\ + \frac{p^2-1}{24}(1-q)^2\ [p]_q + \frac{1-(1-q)\,[p]_q}{[p]_q}\right)\\ &\quad - (-1)^m\,[p]_q\,q^{-\binom{m+1}{2}-1}\widetilde{H}_{m-1,2}(q) \pmod{[p]_q^2}.\end{aligned}$$

From the congruence $\begin{bmatrix}p\\m\end{bmatrix}_q \equiv (-1)^{m-1}\,\frac{[p]_q}{[m]_q}q^{-\binom{m}{2}} \pmod{[p]_q^2}$, by (6), we write

$$\begin{aligned}\sum_{k=m}^{p-1}\frac{q^k}{[k+1]_q}\begin{bmatrix}k\\m\end{bmatrix}_q &\equiv (-1)^{m-1}\,q^{-\binom{m}{2}-1}\frac{[p]_q}{[m]_q^2}\\ &\quad + (-1)^m\,q^{-\binom{m+1}{2}-1}\left(\frac{p+1}{2}(q-1)\ + \frac{p^2-1}{24}(1-q)^2\ [p]_q + \frac{1}{[p]_q}\right)\\ &\quad - (-1)^m\,[p]_q\,q^{-\binom{m+1}{2}-1}\widetilde{H}_{m-1,2}(q)\\ &= (-1)^m\,q^{-\binom{m+1}{2}-1}\,[p]_q\left(-\frac{q^m}{[m]_q^2} + \frac{p^2-1}{24}(1-q)^2\ - \widetilde{H}_{m-1,2}(q)\right)\\ &\quad + (-1)^m\,q^{-\binom{m+1}{2}-1}\left(\frac{1}{[p]_q} + \frac{p+1}{2}\,(q-1)\right) \pmod{[p]_q^2},\end{aligned}$$

as claimed. □

Theorem 1 *Let p be any odd prime number. Then for each $m = 0, 1, 2, .., p-2$,*

$$\sum_{k=m}^{p-1} q^k \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_k(q) \equiv (-1)^m \frac{q^{-\binom{m}{2}}}{[m+1]_q} \times \left(1 - [p]_q \left(\widetilde{H}_{m+1}(q) + \left(\frac{p-1}{2} + m + 1\right)(1-q)\right)\right) \pmod{[p]_q^2},$$

$$\sum_{k=m}^{p-1} q^k \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_{k,2}(q) \equiv (-1)^{m-1} \frac{q^{-\binom{m}{2}}}{[m+1]_q} \times \left(\widetilde{H}_m(q) + (1-q)\left(m + 1 - \frac{p+1}{2}\right) + [p]_q \left(-\frac{1}{2}\widetilde{H}_m^2(q) - \frac{1}{2}\widetilde{H}_{m,2}(q) \right.\right.$$
$$\left.\left.(1-q)^2 \left(\frac{p^2-1}{8} - \binom{m+1}{2}\right) - (1-q)\left(m + \frac{1}{2}\right)\widetilde{H}_m(q)\right)\right) \pmod{[p]_q^2},$$

and

$$\sum_{k=m}^{p-1} q^k \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_k^2(q) \equiv (-1)^{m-1} \frac{q^{-\binom{m}{2}}}{[m+1]_q} \times \left(\widetilde{H}_{m+1}(q) + \frac{q^{m+1}}{[m+1]_q} + (1-q)\left(m + \frac{p+1}{2}\right)\right.$$
$$+ [p]_q \left(-\widetilde{H}_m(q)\left((1-q)\left(p + m + \frac{1}{2}\right) + 2\frac{q^{m+1}}{[m+1]_q}\right)\right.$$
$$-\frac{1}{2}\left(\widetilde{H}_m^2(q) + \widetilde{H}_{m,2}(q)\right) + \frac{q^{m+1}(q-1)(2m+p)}{[m+1]_q} - 2\frac{q^{m+1}}{[m+1]_q^2}$$
$$\left.\left.\left.-(1-q)^2\left(\frac{1}{24}(p+1)(7p+5) + mp + \binom{m+1}{2}\right)\right)\right)\right) \pmod{[p]_q^2}.$$

Proof By taking $n = p$ and $r = 2$ in Lemma 3 and using $[p-m]_q = [p]_q - q^{p-m}[m]_q$, we have

$$[m+1]_q \sum_{k=m}^{p-1} q^{k-m} \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_{k,2}(q) = \left([p]_q - q^{p-m}[m]_q\right) \begin{bmatrix} p \\ m \end{bmatrix}_q \widetilde{H}_{p,2}(q) - q \sum_{k=m}^{p-1} \frac{q^k}{[k+1]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q.$$

(7) yields that

$$[m+1]_q \sum_{k=m}^{p-1} q^{k-m} \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_{k,2}(q) \equiv -q \sum_{k=m}^{p-1} \frac{q^k}{[k+1]_q} \begin{bmatrix} k \\ m \end{bmatrix}_q \left([p]_q - q^{-m}[m]_q\left(1 - (1-q)[p]_q\right)\right) \begin{bmatrix} p \\ m \end{bmatrix}_q \widetilde{H}_{p,2}(q) \pmod{[p]_q^2}.$$

By Lemma 4, (5) and (6), we have the desired congruence. Similarly, by Lemmas 2, 3 and 4, (4), (6) and (7), the other congruences are obtained. □

Specially, when $m = 1$ in Theorem 1,

$$\begin{aligned}
&\sum_{k=1}^{p-1} q^k\,[k]_q\,\widetilde{H}_k(q)\\
&\equiv -\frac{1}{[2]_q}\left(1-[p]_q\left(\frac{1}{[2]_q}+1+\frac{p-1}{2}\,(1-q)\right)\right) \pmod{[p]_q^2},\\
&\sum_{k=1}^{p-1} q^k\,[k]_q\,\widetilde{H}_{k,2}(q)\\
&\equiv \frac{1}{2\,[2]_q}\left(p\,(q-1)-q+3+[p]_q\,\frac{\left(p^2-1\right)(q-1)^2-8}{4}\right) \pmod{[p]_q^2},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{p-1} q^k\,[k]_q\,\widetilde{H}_k^2(q)\\
&\equiv \frac{1}{[2]_q}\left(-\frac{1}{2}\frac{-p-2q+pq^2-3q^2-3}{[2]_q}\right.\\
&\quad +[p]_q\left(2p-4q+2pq^2-3pq+q^2-(1-q)^2\,(p+1)\,\frac{1}{24}\,(7p+29)\right.\\
&\quad \left.\left.-\frac{2}{[2]_q^2}+\frac{2}{[2]_q}\,(1-p)\right)\right) \pmod{[p]_q^2}.
\end{aligned}$$

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