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ORIGINAL RESEARCH

On omitted value for transcendental semigroups

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Abstract In this paper, we define omitted value for transcendental semigroups and show that in a transcendental semigroup there can be at most one omitted value. Several examples of transcendental semigroups having omitted value are given. We define Baker wandering domain for transcendental semigroups. For a transcendental semigroup having a Baker wandering domain W as well as an omitted value a, we prove that a lies in the Julia set. We show that if two transcendental semigroups G and G' are conjugate under a map ψ and G has omitted value z_0 then G' has omitted value $\psi(z_0)$. We show that if $G = < f, g >$ such that g is conjugate of f under the map ψ and f has omitted value z_0 then $O(G) = \{z_0\}$ if and only if $\psi(z_0) = z_0$. Finally, we conclude this paper by giving some problems for future research.

Keywords Entire function · Omitted value · Transcendental semigroup · Conjugate semigroup · Baker wandering domain

Mathematics Subject Classification 37F44 · 37F10 · 30D05

1 Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental function. By complex dynamics, we mean the study of iterative behaviour of complex valued functions. By iteration of an entire or meromorphic function f we mean the behaviour of the sequence $\{f^n(z_0)\}$ where $z_0 \in \mathbb{C}$ and $f^n = f^{n-1} \circ f$ for all $n \in \mathbb{Z}_{\geq 1}$. By convention $f^0(z)$ is defined as $f^0(z) = z$. The theory of complex dynamics mostly began with the works of Fatou [8] and Julia [10]. Since then a lot of developments have happened in complex dynamics, specially in rational functions, but in case of transcendental functions there are still many problems which remain unsolved. A.Hinkannen and G.J.Martin introduced the concept of rational semigroups in [9]. K.K.Poon extended this concept to transcendental semigroups in [15]. In this paper definitions of the Fatou and Julia set of a transcendental semigroup were given as well as many results from rational semigroups were extended to transcendental semigroups. D.Kumar and S.Kumar worked on the escaping sets of various transcendental semigroups in [12]. In this paper our main focus is to extend the concept of omitted value to transcendental semigroups.

We say that a point $z_0 \in \mathbb{C}$ is an omitted value of the function f if z_0 does not belong to the range of f.

A semigroup is a set S together with a binary operation $\circ$ such that S is closed and associative under $\circ$. From [15], a transcendental semigroup is defined to be a semigroup G generated by a family of transcendental entire

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functions together with the binary operation composition of functions $\circ$. In mathematical terms let $F=\{f_1,f_2,\ldots\}$ be a family of transcendental entire functions. Let $G=\{h: h = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_k}$ for some $k \in \mathbb{N}$ and each $f_{i_r} \in F$ for all $r = 1, 2, ..., k\}$. We say that G is transcendental semigroup generated by the family F and is denoted by $G =< f_1, f_2, \ldots >$. We say that G is finitely generated if the family of generators is finite.

For an entire or meromorphic function f we define Fatou set to be the set $F(f) = \{z \in \mathbb{C} : \{f^n : n \in \mathbb{N}\}$ is defined and normal in some neighbourhood of $z\}$. We define the Julia set to be complement of the Fatou set. Various properties of the Fatou and Julia sets are discussed in [3]. A maximal connected open subset of the Fatou set of a function f is called a Fatou component. For a function f, if U is a Fatou component then $f^n(U)$ is contained in some Fatou component which we denote by U_n. A component U of f is said to be pre-periodic if there exists $n > m \geq 0$ such that U_n is same as U_m. When $m = 0$ then we say that U is periodic. Periodic Fatou components of a meromorphic function can be of 5 types called attractive domain, parabolic domain, Siegel disc, Herman ring and Baker domain [3]. Existence of Herman rings for a transcendental meromorphic function having at least one omitted value is discussed in [6,7]. When a Fatou component is not pre-periodic then it is called a wandering domain. A Baker wandering domain of a function f is a multiply connected wandering component U of $F(f)$ such that for n large enough, U_n is bounded, multiply connected, surrounds 0 and $f^n(z) \longrightarrow \infty$ as $n \longrightarrow \infty$ for $z \in U$.

The Fatou set of a transcendental semigroup G is defined to be the largest open subset of $\mathbb{C}$ for which the family of functions in G is a normal family. The Julia set is defined as the complement of the Fatou set. They are denoted by $F(G)$ and $J(G)$ respectively. For more details one can see [15].

We know from Picard's theorem that a non-constant entire function takes all the values in the complex plane with at most one exception. So Picard's theorem implies that the maximum number of omitted values of a non-constant entire function is 1. From [13], we know that $J(e^z) = \mathbb{C}$. So for $f(z) = e^z$, the omitted value lies in the Julia set. But as mentioned in [4] $f(z) = \lambda e^z$ with $0 < \lambda < e^{-1}$ has the omitted value 0 in the Fatou set.

For a transcendental semigroup G, we say that a subset A of $\mathbb{C}$ is forward invariant if for all $f \in G$, $f(A) \subset A$. Similarly we say A is backward invariant if for all $f \in G$, $f^{-1}(A) \subset A$. Let $O^-(z) = \{w \in \mathbb{C} :$ there exists $g \in G$ such that $g(w) = z\}$. We define the exceptional set of G as $E(G) = \{z \in \mathbb{C} : O^-(z)$ is finite$\}$ as mentioned in [15].

Two entire functions f and g are said to be conjugate if there exists a conformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that we have $\phi \circ f = g \circ \phi$. Conformal map means an analytic and univalent map of the complex plane $\mathbb{C}$ that is exactly of the form $az + b$ for $a, b \in \mathbb{C}$ and $a \neq 0$.

Two finitely generated transcendental semigroups G and G' are said to be conjugate under a conformal map $\phi : \mathbb{C} \to \mathbb{C}$ if they have the same number of generators and corresponding generators are conjugate under ϕ [12].

In Section 2, we define omitted value for transcendental semigroup. Some examples of transcendental semigroups having omitted value are also given there. Behaviour of omitted value under iteration and functional composition is discussed here. We discuss some basic properties of omitted value including an analogue of Picard's theorem. In Section 3, we define Baker wandering domain for transcendental semigroup and establish some relations between Baker wandering domain and omitted value. We discuss a method to construct wandering domains in transcendental semigroups in this section. In Section 4, we discuss connection between conjugacy and omitted value for both transcendental entire functions and transcendental semigroups. Finally we give some direction of works for future researchers.

2 Preliminaries

Let G be a transcendental semigroup. We denote the set of omitted value of transcendental semigroup by $O(G)$ and we define it as $O(G) = \{z \in \mathbb{C} : O^-(z)$ is empty$\}$.

We can show that the set of all omitted values of a transcendental semigroup G is simply the intersection of set of all omitted values of the functions in G.

Lemma 2.1 *Let G be a transcendental semigroup. Let $O(g)$ be the set of omitted values of $g \in G$. Then $O(G) = \cap_{g\in G} O(g)$.*

Proof By definition of $O(G)$, we have $O(G) = \{z \in \mathbb{C}: O^-(z)$ is empty$\}$.

Now suppose $z \in O(G)$. Then $O^-(z)$ is empty. That means for any $g \in G$, z has no preimage. This implies z is an omitted value of g for every $g \in$ G. So $z \in \cap_{g\in G} O(g)$. So $O(G) \subset \cap_{g\in G} O(g)$.

Now let $z \in \cap_{g\in G} O(g)$. So $z \in O(g)$ for all $g \in G$. So there does not exist any complex number w such that $g(w) = z$ for any $g \in G$. So $O^-(z)$ is empty for any $g \in G$. So $z \in O(G)$. So $\cap_{g\in G} O(g) \subset O(G)$. Hence we have $O(G) = \cap_{g\in G} O(g)$.

Remark 1 We will prove later that $O(G)$ is either empty or singleton set for any transcendental semigroup G. This implies $O(G)$ is always closed.

Now we are giving an example of semigroup whose set of omitted values is empty.

Example 2.1 Let $G_1 =< f, g >$ where $f(z)= \sin z$ and $g(z) =e^z$. We know that $\sin z$ takes every complex value in the complex plane whereas e^z has a single omitted value 0 in $\mathbb{C}$. So $O(f)=\emptyset$ and $O(g) = \{0\}$. Since $O(f) = \emptyset$ we have $\cap_{g\in G} O(g) = \emptyset$. Therefore from Lemma 2.1, we can say $O(G)$ is an empty set.

Now we consider another example of semigroup whose set of omitted values is non-empty.

Example 2.2 Let $G_2 =< f, g >$ where $f(z) = e^z$ and $g(z) = e^{-z}$. Now $f^r(z)= e^{f^{r-1}(z)}$ and $g^s(z) = e^{-g^{s-1}(z)}$. So $(f^p \circ g^q)(z) = e^{f^{p-1}(g^q(z))}$. Clearly $f^p \circ g^q$ cannot take the value $\{0\}$. So 0 is an omitted value of $f^p \circ g^q$ for all $p, q \in \mathbb{N}$ and by Picard's theorem it is the only such value.

Now $(g^t \circ f^u)(z) = e^{-g^{t-1}(f^u(z))}$. Again clearly $g^t \circ f^u$ cannot take the value $\{0\}$. So 0 is an omitted value of $g^t \circ f^u$ for all $t, u \in \mathbb{N}$ and by Picard's theorem it is the only such value.

So if $h \in G_2$ then $O(h) = \{0\}$. This implies that $\cap_{h\in G} O(h) = \{0\}$. Then from Lemma 2.1, we can say that $O(G_2) = \{0\}$.

Remark 2 We can construct a transcendental semigroup with nonzero omitted value b simply by defining two generators $r(z) = f(z) + b$ and $s(z) = g(z) + b$ where f, g are defined as in Example 2.2. Proceeding similarly to Example 2.2, we have b as the omitted value of the transcendental semigroup.

Lemma 2.2 *If a is an omitted value of a transcendental entire function f then a is also omitted value of f^k for all $k \in \mathbb{N}$.*

Proof Let a be an omitted value of f. Suppose there exists $z_0 \in \mathbb{C}$ such that for some $k \in \mathbb{N}$, $f^k(z_0) = a$. This implies $f(f^{k-1}(z_0)) = a$. This implies a has a preimage for f which is a contradiction. As z_0 and k are both arbitrarily chosen so $f^k(z) \neq a$ for all $z \in \mathbb{C}$.

Remark 3 Using Lemma 2.1 and Lemma 2.2 we can say that if $G =< f >$ then $O(G) = O(f)$.

Lemma 2.3 *If a is an omitted value of two transcendental entire functions f and g then $f \circ g$ and $g \circ f$ also has omitted value a.*

Proof Suppose $f \circ g$ does not have omitted value a. So there exists $z_0 \in \mathbb{C}$ such that $(f \circ g)(z_0) = a$. That is $f(g(z_0)) = a$ which contradicts a is omitted value of f. So $f \circ g$ omits a. Similarly $g \circ f$ omits a.

Remark 4 From Lemma 2.2 and Lemma 2.3, we can say that if f, g are two transcendental entire functions both having omitted value z_0 then $f^k \circ g^l$ and $g^m \circ f^n$ has omitted value z_0 for any $k, l, m, n \in \mathbb{N}$.

We now give a result which determines cardinality of the set of omitted values in transcendental semigroup.

Lemma 2.4 *Let G be a transcendental semigroup. Then $|O(G)| \leq 1$.*

The proof of this lemma is immediate from Picard's theorem and Lemma 2.1.

We can see in Example 2.1, cardinality of $O(G_1)$ is 0 whereas in Example 2.2, cardinality of $O(G_2)$ is 1.

Now we check the invariance properties of the set $O(G)$.

Lemma 2.5 *Let G be a transcendental semigroup. Then either $O(G)$ is empty or $O(G)$ is not forward invariant.*

Proof Here G is a transcendental semigroup and $O(G) \subset \mathbb{C}$. Assume $O(G) \neq \emptyset$. Then from Lemma 2.4, $O(G)$ is singleton set. Let $O(G)=\{z_0\}$. Let us assume that $O(G)$ is forward invariant. So $g(O(G)) \subset O(G)$ for all $g \in G$. As $O(G)$ is singleton so it has only two subsets $\emptyset$ and $O(G)$ itself. But $g(O(G))$ cannot be $\emptyset$ as domain of g is $\mathbb{C}$. If $g(O(G)) = O(G)$ then we have $g(z_0) = z_0$. This implies that z_0 is not omitted value of g. Hence $z_0 \notin O(G)$ which is a contradiction. So $O(G)$ is not forward invariant.

Remark 5 If G is a transcendental semigroup then it is obvious that $g^{-1}(O(G)) = \emptyset$. So in a sense $O(G)$ is backward invariant.

3 Baker wandering domain for transcendental semigroups

By wandering domain of a transcendental semigroup G we mean a component W of $F(G)$ such that there exists infinitely many distinct components U_k of $F(G)$ and a sequence $\{f_k\}$ in G such that $f_k(U) \subset U_k$ [15]. The functions f_k are called wandering functions. We now give some lemmas and a theorem which will be helpful in constructing some wandering domains for transcendental semigroups.

Lemma 3.1 *Let $f(z)$ be a transcendental entire function of period T. Let $f_n(z) = z + f(z) + nT$, $n \in \mathbb{Z}_{\geq 0}$. Then*

(i) $f_n^k(z) = f_0^k(z) + nkT$, ii)$f_l^m \circ f_p^q(z) = f_0^{(m+q)}(z) + (lm + pq)T$ for all $l, m, p, q \in \mathbb{Z}_{\geq 0}$.

Proof i) $f_0(z) = z + f(z)$. Now let $n \in \mathbb{N}$.

$$\begin{aligned} f_n^2(z) &= f_n(f_n(z)) = f_n(z) + f(f_n(z)) + nT \\ &= z + f(z) + nT + f(z + f(z) + nT) + nT = z + f(z) + f(z + f(z)) + 2nT \\ &= f_0(z) + f(f_0(z)) + 2nT = f_0(f_0(z)) + 2nT = f_0^2(z) + 2nT. \end{aligned}$$

So the statement i) is true for $k = 2$ for any choice of n. Assume the statement is true for $k = m$. So $f_n^m(z) = f_0^m(z) + mnT$. Now $f_n^{(m+1)}(z) = f_n(f_n^m(z)) = f_n(f_0^m(z) + mnT) = f_0^m(z) + mnT + f(f_0^m(z) + mnT) + nT = f_0^m(z) + f(f_0^m(z) + (m+1)nT = f_0(f_0^m(z)) + (m+1)nT = f_0^{m+1}(z) + (m+1)nT$.

So if the statement i) is true for $k = m$ then it is also true for $k = m + 1$. So by principle of mathematical induction, we can say $f_n^k(z) = f_0^k(z) + nkT$ for every $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$.

ii) We can see that $f_0(z + T) = f_0(z) + T$. Consequently $f_0^k(z + P) = f_0^k(z) + P$ for all $k \in \mathbb{Z}_{\geq 0}$. Now $f_l^m \circ f_p^q(z) = f_l^m(f_0^q(z) + pqT) = f_0^m(f_0^q(z) + pqT) + lmT = f_0^m(f_0^q(z + pqT)) + lmT = f_0^{m+q}(z + pqT) + lmT = f_0^{m+q}(z) + pqT + lmT = f_0^{m+q}(z) + (lm + pq)T$.

Lemma 3.2 *Let $G =< f_0, f_1, ... >$ be a transcendental semigroup such that f_i's are defined as in Lemma 3.1. Then $F(G) = F(f_0)$.*

Proof Using Lemma 3.1 and techniques similar to it we can see that any $g \in G$ is of the form $g(z) = f_0^r(z) + sT$ for some $r, s \in \mathbb{Z}$. Also clearly g and f_0^r commutes with each other for any $g \in G$. So by [2], we can say $J(g) = J(f_0)$. So for any $g \in G$, $F(g) = F(f_0)$. Hence we can say that $F(G) = F(f_0)$ as $F(f_0)$ is the largest open subset of $\mathbb{C}$ in which every element of G is normal.

Theorem 3.1 *Let $f(z)$ be a transcendental entire function of period T. Let $z_k = kT$ for every $k \in \mathbb{N}$. If $f(z_k) = 0$ and $f'(z_k) = -1$ for every $k \in \mathbb{N}$ then $G =< f_0, f_1, f_2, ... >$ has wandering domains where $f_n(z) = z + f(z) + nT$ for every $n \in \mathbb{Z}_{\geq 0}$.*

Proof $f_0(z) = z + f(z)$. So $f_0'(z) = 1 + f'(z)$. So every z_k is a superattracting fixed point of $f_0(z)$. Let W_k be the Fatou component containing z_k. Now using Lemma 3.2, $F(G) = F(f_0)$. Also $f_n(z_k) = z_k + nT = (k + n)T = z_{(k+n)}$. So $f_n(W_k) \subset W_{k+n}$ for every $n \in \mathbb{Z}_{\geq 0}$. So for every k, W_k is wandering domain of G with wandering functions $\{f_n\}$.

The following lemma from [14], will be useful for our work.

Lemma 3.3 *Let $G \subset \mathbb{C}$ be a region and the closure of $f(G)$ for an entire function f contains an omitted value. Then G is unbounded.*

Using above lemma we can prove the following lemma.

Lemma 3.4 *Let G be a transcendental semigroup and B be a region in $\mathbb{C}$. If there exists at least one $g \in G$ such that $\overline{g(B)} \cap O(G) \neq \emptyset$ then B is unbounded.*

Proof Suppose $\overline{g(B)} \cap O(G)=\{z_0\}$. Then $z_0 \in \overline{g(B)}$ *and* $z_0 \in O(G)$. This implies $z_0 \in \overline{g(B)}$ and z_0 is omitted value of $g \in G$ as $O(G) = \cap O(g)$. Thus $\overline{g(B)}$ contains an omitted value. Hence B is unbounded by Lemma 3.3.

We know that if a transcendental entire function has an unbounded Fatou component then all the Fatou components are simply connected [1]. We recall by a Baker wandering domain of a function f, we mean a multiply connected wandering component U of $F(f)$ such that for n large enough, U_n is bounded, multiply connected and surrounds 0 and $f^n(z) \longrightarrow \infty$ as $n \longrightarrow \infty$ for $z \in U$. It is known that if Baker wandering domain exists for a function then all the Fatou components are bounded.

Now, we define Baker wandering domain for transcendental semigroups.

Definition 3.1 Let G be a transcendental semigroup and W be component of $F(G)$. We say that W is a Baker wandering domain of G if,

(i) W is a multiply connected wandering Fatou component with wandering functions $\{f_n\}$ such that $f_i(W) \subset W_i$ where W_is are Fatou components,
(ii) every W_n is bounded, multiply connected and surrounds 0 for large enough n,
(iii) $f_k(z) \longrightarrow \infty$ as $k \longrightarrow \infty$ for all $z \in W$.

From the definition, we can see that Baker wandering domain is a particular type of wandering domain. We now give a theorem to discuss about the existence of Baker wandering domain in the presence of omitted value.

Theorem 3.2 *Let G be a transcendental semigroup with an omitted value z_0. If $F(G)$ has a component U such that $z_0 \in \overline{g(U)}$ for some $g \in G$ then G does not have Baker wandering domain.*

Proof Assume $F(G)$ has a component U such that $z_0 \in \overline{g(U)}$ for some $g \in G$. By Lemma 3.4, we can say that U is unbounded. From [11], we know that if $F(G)$ has an unbounded component then all the components of $F(G)$ is simply connected. So G does not contain any multiply-connected Fatou component and consequently does not have any Baker wandering domain.

Next theorem describes the location of omitted value of a transcendental semigroup in presence of Baker wandering domain. The following lemma [16] is required for the proof.

Lemma 3.5 *Suppose G is a transcendental semigroup. If $F(g)$ has no unbounded component for any $g \in G$ then $E(G) \subset J(G)$.*

It is obvious by definition $O(G) \subset E(G)$.

Theorem 3.3 *Let G be a transcendental semigroup with omitted value z_0. If G has a Baker wandering domain then $z_0 \in J(G)$.*

Proof Let G has a Baker wandering domain W and $z_0 \in F(f)$ for all $f \in G$. So f has a Fatou component W_f such that $W \subset W_f$ for all $f \in G$. As W is multiply connected, there exists at least one $g \in G$ such that W_g is multiply connected. This implies every component of $F(g)$ is bounded. But as $z_0 \in F(g)$, proceeding as in the proof of Lemma 3.5, we can say that $F(g)$ has a component which contains an unbounded sequence. This is a contradiction. This implies $z_0 \in J(g) \subset J(G)$.

Corollary 3.1 *If G is a transcendental semigroup with a Baker wandering domain then all Fatou components of G are bounded.*

The above corollary is immediate from the fact that $F(g)$ has no unbounded components for at least one $g \in G$ and $F(G) \subset F(g)$.

4 Conjugate semigroups with omitted value

From [12], we have the definition of conjugate semigroups as mentioned in the introduction. In this section we discuss some properties of the set of omitted values related to transcendental semigroup and their conjugate semigroups. But before that we give examples of conjugate entire functions and conjugate transcendental semigroups.

Example 4.1 Let $f(z) = e^z$. Let $\psi(z) = az + b$ for all $z \in \mathbb{C}, a \neq 0$. So $\psi^{-1}(z) = \frac{z-b}{a}$ for all $z \in \mathbb{C}, a \neq 0$. Now define $g(z) = \psi \circ f \circ \psi^{-1}(z)$. So $g(z) = \psi \circ f(\frac{z-b}{a})$. This implies $g(z) = \psi(e^{(\frac{z-b}{a})})$. Hence $g(z) = ae^{(\frac{z-b}{a})} + b$. So g is conjugate map of f under ψ.

Example 4.2 Let G and G' be two transcendental semigroups such that $G =< f, g >$ with $f(z) = e^z + 1$ and $g(z) = e^{-z} + 1$, $G' =< f', g' >$ with $f'(z) = 3(e^{\frac{z}{3}} + 1)$ and $g'(z) = 3(e^{-\frac{z}{3}} + 1)$. So G and G' have the same number of generators. Now let $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(z) = 3z$ for all $z \in \mathbb{C}$. So we have

$(\phi \circ f)(z) = \phi(e^z + 1) = 3(e^z + 1) = f'(3z) = (f' \circ \phi)(z)$.
$(\phi \circ g)(z) = \phi(e^{-z} + 1) = 3(e^{-z} + 1) = g'(3z) = (g' \circ \phi)(z)$.

So f is conjugate to f' and g is conjugate to g'. So the semigroups G and G' are conjugate under the map $\phi(z) = 3z$ for all $z \in \mathbb{C}$.

The following lemma is necessary to prove some results later.

Lemma 4.1 *If f and g be two entire functions conjugate under the conformal map ψ and f has omitted value z_0 then g has omitted value $\psi(z_0)$.*

Proof Given f has omitted value z_0. If possible let there exist $z_1 \in \mathbb{C}$ such that $g(z_1) = \psi(z_0)$. As g and f are conjugate to each other under ψ so $g = \psi \circ f \circ \psi^{-1}$. So $\psi \circ f \circ \psi^{-1}(z_1) = \psi(z_0)$. Hence $f \circ \psi^{-1}(z_1) = z_0$. Implies $f(\psi^{-1}(z_1)) = z_0$. This contradicts that z_0 is an omitted value for f. So $\psi(z_0)$ is an omitted value of g.

Theorem 4.1 *Let G and G' be two conjugate transcendental semigroups under the map $\psi : \mathbb{C} \to \mathbb{C}$ and $O(G) = \{z_0\}$. Then $O(G') = \{\psi(z_0)\}$.*

Proof Let $G =< f_1, f_2, ..., f_n >$. Then we can write $G' = < \psi \circ f_1 \circ \psi^{-1}, \psi \circ f_2 \circ \psi^{-1}, ..., \psi \circ f_n \circ \psi^{-1} >$. Let $h' \in G'$. That means $h' = \psi \circ f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k} \circ \psi^{-1}$ for some $i_1, i_2, .., i_k \in \{1, 2...n\}$. So there exists a corresponding element $h \in G$ such that $h' = \psi \circ h \circ \psi^{-1}$. Now as z_0 is the omitted value of G, so z_0 is an omitted value of h. By Lemma 4.1, $\psi(z_0)$ is the omitted value of $h' = \psi \circ h \circ \psi^{-1}$. As h' is arbitrary hence $O(h') = \psi(z_0)$ for all $h' \in G'$. Therefore, $O(G') = \{\psi(z_0)\}$.

For the Example 4.2, we have $O(G) = \{1\}$ and $O(G') = \{3\}$. Here G and G' are conjugate under the map $\phi(z) = 3z$. So we have $O(G') = \{\phi(1)\}$. This result verifies the above theorem.

Remark 6 By a semigroup isomorphism we mean a map between two semigroups G and G', which is bijective and homomorphism. Let $(G, \circ)$ and $(G', \circ)$ be conjugate under the conformal map ϕ. It is obvious that $\psi : G \to G'$, $\psi(f) = \phi \circ f \circ \phi^{-1}$ is a bijective map. Also for any $f, g \in G$, $\psi(f \circ g) = \phi \circ (f \circ g) \circ \phi^{-1} = (\phi \circ f \circ \phi^{-1}) \circ (\phi \circ g \circ \phi^{-1}) = \psi(f) \circ \psi(g)$. So ψ is a semigroup isomorphism.

Theorem 4.2 *Let $G =< f, g >$ such that g is conjugate of f under the map ψ. Assume f has omitted value z_0. Then $O(G) = \{z_0\}$ if and only if $\psi(z_0) = z_0$.*

Proof At first let $O(G) = \{z_0\}$. As g is conjugate of f under the map ψ, so from Lemma 4.1, we have that g has omitted value $\psi(z_0)$. This implies $O(g) = \{\psi(z_0)\}$. But $O(G) = \{z_0\}$ and $O(G) = \cap_{g \in G} O(g)$, therefore we must have $O(g) = \{z_0\}$. Hence $\psi(z_0) = z_0$.

Conversely let $\psi(z_0) = z_0$. As $G =< f, g >$, so by Lemma 2.2 and Lemma 2.3, we can say that $O(G) = O(f) \cap O(g)$. Let us assume $O(g) \neq \{z_0\}$. So there exist $w \in \mathbb{C}$ such that $g(w) = z_0$. So $\psi \circ f \circ \psi^{-1}(w) = z_0$. That means $f \circ \psi^{-1}(w) = \psi^{-1}(z_0) = z_0$. This implies $f(\psi^{-1}(w)) = z_0$. This contradicts that z_0 is an omitted value of f. So $O(g) = \{z_0\}$. Hence $O(G) = \{z_0\}$.

Corollary 4.1 *Let $G =< f_1, f_2, ..., f_n >$ such that each f_i, $1 \leq i \leq n$ is conjugate to f_1 by the map ψ_i. Assume f_1 has an omitted value z_0. Then $O(G) = \{z_0\}$ if and only if $\psi_i(z_0) = z_0$ for each $1 \leq i \leq n$.*

Future prospect

This paper will be beneficial to future researchers for extending results on omitted value of functions to transcendental semigroups. The effect of the presence of omitted value on pre-periodic components may be explored in future. From [5] we know about the existence of Baker omitted value. For a transcendental entire or meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$, with an essential singularity at ∞, Baker omitted value is defined as an omitted value $a \in \widehat{\mathbb{C}}$ of f with the additional property that there exists $r_0 > 0$ such that for any r with $0 < r < r_0$, each component of the boundary of $f^{-1}(D_r(a))$ is bounded, where $D_r(a)$ is the ball around a with radius r with respect to the spherical metric. It is known that an entire function can have Baker omitted value at ∞ only. The idea of Baker omitted value can be explored in the context of transcendental semigroups in future. Specifically, we will study the dynamics of transcendental semigroups around the Baker omitted value in our future course of works.

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