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ORIGINAL RESEARCH

Geometric representation of the weighted harmonic mean of n positive values and potential applications.

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Abstract This paper is dedicated to the analysis and detailed study of a procedure to generate both the weighted arithmetic and harmonic means of n positive real numbers. Together with this interpretation, we prove some relevant properties that will allow us to define numerical approximation methods in several dimensions adapted to discontinuities.

Keywords Arithmetic mean · Harmonic mean · Weighted arithmetic mean · Weighted harmonic mean · Reconstruction operators · Adaptation · Singularities

Mathematics Subject Classification 41A05 · 41A10 · 65D17

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1 Introduction

Both the arithmetic and the harmonic means of positive numbers appear in different scientific scenarios varying from subdivision schemes and image processing to signal filtering, solution of partial differential equations, statistics, etcetera. The harmonic mean penalizes large values in the given data, being appropriate, because of this characteristic, for several real world applications. Besides, when the input variables are similar, both means keep that similarity, which is also convenient in applications as it will be seen later in the paper.

Among the applications of these two means we can mention for instance the following: in the field of numerical solution of hyperbolic conservation laws [13,14], for applications involving signal processing [1,2,4,15], in more specific research areas such as image compression and image denoising [5,8], in the fast generation of curves and surfaces by means of subdivision schemes [3,9,10].

In [11], a nonlinear reconstruction operator called PPH (Piecewise Polynomial Harmonic) was extended to nonuniform grids by using a specific weighted harmonic mean instead of the standard harmonic mean. In this paper our aim is to introduce some necessary ingredients to extend in turn this last reconstruction operator to several dimensions. More specifically speaking, we need to dispose of an appropriate mean in several dimensions which satisfies the required basic properties, the two mentioned above, as the harmonic mean does. We carry out this study accompanied by a geometric representation of the weighted harmonic mean of several values, which helps to quickly and intuitively understand the theoretical results.

Nonlinear means appear as good candidates to define adapted reconstruction methods which minimize the undesirable effects generated by the presence of a discontinuity in the data. In fact, we will give a list of already existing methods sharing these properties, and in turn we will summarize how to define families of new methods based on the theory and ideas developed through this paper.

The paper is organized as follows: In Sect. 2, we deal with the general case of considering the weighted arithmetic and harmonic mean of n positive numbers for whatever integer value $n \geq 2$. In Sect. 3, we outline some applications of these results in order to define adapted reconstructions in several dimensions. Finally, in Sect. 4, we give some conclusions.

2 Results on the weighted harmonic mean of *n* values

First, we introduce the definition of weighted harmonic mean H_w that we are going to be using.

Definition 1 Given $a_i > 0, i = 1, \ldots, n$ n positive real numbers and the weights $w_i > 0, i = 1, \ldots, n$ with $\sum_{i=1}^{n} w_i = 1$, the weighted harmonic mean is defined by

$$H_w(a_1, \ldots, a_n) = \frac{1}{\sum\limits_{i=1}^{n} \frac{w_i}{a_i}} = \frac{\prod\limits_{k=1}^{n} a_k}{\sum\limits_{i=1}^{n} w_i \prod\limits_{\substack{k=1 \\ k \neq i}}^{n} a_k},$$

and the weighted arithmetic mean is defined by

$$M_w(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i.$$

We now give the main two results which are crucial in applications in numerical analysis, such as we will show in the section devoted to practical cases. The first lemma has to do with the property of boundedness of the mean by the minimum of its arguments and it is used to define adaptative methods.

Lemma 1 *Let $a_i > 0, i = 1, \ldots, n$ be n positive real numbers and $w_i > 0, i = 1, \ldots, n$ the corresponding weights with $\sum_{i=1}^{n} w_i = 1$. Then, the weighted harmonic mean H_w is bounded as follows*

$$H_w < \frac{a_{i_0}}{w_{i_0}} \leq \frac{a_i}{w_i}, \quad i = 1, \ldots, n,$$

where $\frac{a_{i_0}}{w_{i_0}} = \min\{\frac{a_1}{w_1}, \cdots, \frac{a_n}{w_n}\}$.

Proof

$$H_w = \frac{\prod_{k=1}^{n} a_k}{\sum_{j=1}^{n} w_j \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k} = \frac{a_{i_0}}{w_{i_0}} \frac{\prod_{\substack{k=1 \\ k\neq i_0}}^{n} a_k}{\sum_{j=1}^{n} \frac{w_j}{w_{i_0}} \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k} < \frac{a_{i_0}}{w_{i_0}} \leq \frac{a_i}{w_i}, \quad i = 1, \ldots, n.$$

□

The second lemma deals with how close remains the weighted harmonic mean to the weighted arithmetic mean when the arguments are also close among them. This property is essential to define nonlinear methods which preserve the order of approximation of their linear counterparts from which they are derived. We will also show this relation in the section dedicated to the practical examples.

Lemma 2 *Let $a_i > 0, i = 1, \ldots, n$ be n positive real numbers and $w_i > 0, i = 1, \ldots, n$ the corresponding weights with $\sum_{i=1}^{n} w_i = 1$. If $a_i = O(1), \quad \forall i = 1, \cdots, n, \quad$ and $\quad |a_1 - a_i| = O(h), \quad \forall i = 2, \cdots, n,$ then, the weighted harmonic mean H_w and the weighted arithmetic mean $M_w := \sum_{i=1}^{n} w_i a_i$ satisfy*

$$|M_w - H_w| = O(h^2).$$

Proof Using the expressions of H_w and M_w we have

$$|M_w - H_w| = \left| \sum_{i=1}^{n} w_i a_i - \frac{\prod_{k=1}^{n} a_k}{\sum_{j=1}^{n} w_j \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k} \right| = \left| \frac{\sum_{i=1}^{n} w_i a_i \sum_{j=1}^{n} w_j \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k - \prod_{k=1}^{n} a_k}{\sum_{j=1}^{n} w_j \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k} \right|. \tag{1}$$

Now, paying attention to the fact that given two indices i_0, j_0 such that $1 \leq i_0 < j_0 \leq n$ we have

$$w_{i_0} a_{i_0} w_{j_0} \prod_{\substack{k=1 \\ k\neq j_0}}^{n} a_k = w_{i_0} a_{i_0}^2 w_{j_0} \prod_{\substack{k=1 \\ k\neq i_0, j_0}}^{n} a_k, \tag{2}$$

$$w_{j_0} a_{j_0} w_{i_0} \prod_{\substack{k=1 \\ k\neq i_0}}^{n} a_k = w_{i_0} a_{j_0}^2 w_{j_0} \prod_{\substack{k=1 \\ k\neq i_0, j_0}}^{n} a_k, \tag{3}$$

and just by summing up both terms in (2) and (3) we get

$$w_{i_0} a_{i_0} w_{j_0} \prod_{\substack{k=1 \\ k\neq j_0}}^{n} a_k + w_{j_0} a_{j_0} w_{i_0} \prod_{\substack{k=1 \\ k\neq i_0}}^{n} a_k = w_{i_0} w_{j_0} \prod_{\substack{k=1 \\ k\neq i_0, j_0}}^{n} a_k (a_{i_0}^2 + a_{j_0}^2). \tag{4}$$

For the case $i_0 = j_0$ we get

$$w_{i_0} a_{i_0} w_{j_0} \prod_{\substack{k=1 \\ k\neq j_0}}^{n} a_k = w_{i_0}^2 \prod_{k=1}^{n} a_k. \tag{5}$$

Using the simplifications in (4) and (5) we can rewrite (1) as

$$|M_w - H_w| = \left| \frac{\sum_{i=1}^{n} {w_i}^2 \prod_{k=1}^{n} a_k + \sum_{\substack{i,j=1 \\ i<j}}^{n} w_i w_j (a_i^2 + a_j^2) \prod_{\substack{k=1 \\ k\neq i,j}}^{n} a_k}{\sum_{j=1}^{n} w_j \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k} \right| = \left| \frac{\sum_{\substack{i,j=1 \\ i<j}}^{n} w_i w_j (a_i - a_j)^2 \prod_{\substack{k=1 \\ k\neq i,j}}^{n} a_k}{\sum_{j=1}^{n} w_j \prod_{\substack{k=1 \\ k\neq j}}^{n} a_k} \right| = O(h^2),$$

since by the triangular inequality we have that $|a_i - a_j| \leq |a_i - a_1| + |a_1 - a_j| = O(h)$. □

We introduce the following notation for the vertices of a prism in $\mathbb{R}^n$.

$$\begin{cases} B_i \equiv (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0) \quad i = 1, \cdots, n-1, \\ B_n = (0, \cdots, 0), \\ P_i \equiv (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0, a_i) \quad i = 1, \cdots, n-1, \\ P_n = (0, \cdots, 0, a_n), \end{cases}$$

where the points B_i represent the vertices which lay on the base of the prism and the vertices P_i are nothing more than the points located at the maximum height of the prism at the corresponding points B_i in the base and in the parallel direction to the x_n axis.

We are now ready to give the following two theorems for the weighted harmonic mean, which generalize the geometrical representations using prisms.

Theorem 1 *Let us consider the hyperplane Π which passes through the points $P_i, i = 1, \ldots, n$, $n \geq 2$, given by the equation*

$$\Pi \equiv \; x_n = a_n + \sum_{i=1}^{n} x_i (a_i - a_n). \tag{6}$$

Let us also consider the hyperplane V_n which passes through the points $B_i, i = 1, \ldots, n-1$ and P_n given by the equation

$$V_n \equiv \; \sum_{i=1}^{n-1} x_i + \frac{x_n}{a_n} = 1. \tag{7}$$

and the paraboloids $V_i, i = 1, \ldots, n-1$ given by the equations

$$V_i \equiv x_n = b_i {x_i}^2 + (a_i - b_i) x_i, \tag{8}$$

which pass through $B_1, \ldots, B_{i-1}, P_i, B_{i+1}, \ldots, B_n$ respectively, where the coefficients b_i are given by

$$b_i = \frac{H_w}{\bar{x}_i(\bar{x}_i - 1)}(w_n - w_i), \qquad i = 1, \ldots, n-1. \tag{9}$$

Then, the system of equations formed by (7) and (8) has a unique solution $(\bar{x}_1, \ldots, \bar{x}_n)$ given by

$$\bar{x}_i = w_i \frac{H_w}{a_i}, \quad i = 1, \ldots, n-1, \quad \bar{x}_n = w_n H_w. \tag{10}$$

Moreover, the following two affirmations are true:

a) The height of the prism through the point $(\bar{x}_1, \ldots, \bar{x}_{n-1}, 0)$ coincides with the weighted harmonic mean H_w of $a_i, i = 1, \ldots, n$, that is, the point $(\bar{x}_1, \ldots, \bar{x}_{n-1}, H_w)$ belongs to the hyperplane Π.

b) The height of the prism through the barycenter of the base $GM_w = \sum_{i=1}^{n} w_i B_i$ coincides with the weighted arithmetic mean.

Proof It is immediate to check that the proposed solution satisfies (7) and (8). Let us prove that the solution is unique. By reductio ad absurdum, let us suppose that there exists another solution $x' = (x'_1, \cdots, x'_n)$ with $x' \neq \bar{x}$. Then, denoting $z_i = \bar{x}_i - x'_i, i = 1, \cdots, n$, the system of equations formed by (7) and (8) can be easily transformed into

$$\sum_{i=1}^{n-1} z_i + \frac{z_n}{a_n} = 0, \tag{11a}$$

$$z_n - b_i z_i (\bar{x}_i + x'_i) - (a_i - b_i) z_i = 0, \quad i = 1, \cdots, n-1, \tag{11b}$$

what amounts to a homogeneous linear system of n equations with n unknowns. If we show that this system has only the trivial solution $z = 0$, then we would have proven that $\bar{x} = x'$, what is a contradiction with the

starting supposition. Therefore, $\bar{x}$ would be the unique solution. Let us then prove that system (11) has $z = 0$ as the unique solution. Again by reductio ad absurdum, let us suppose that the system has infinite solutions, that is, $z = \sum_{k=1}^{s} \lambda_k v^k$, where $s = n - r$, being r the rank of the coefficient matrix of the linear system, $\lambda_k \in \mathbb{R}$, and $v^k,\ k = 1, \ldots, s$, represent a base of the kernel of the associated linear map. Let us consider the univariate set of solutions $z = \lambda_1 v^1$, $\lambda_1 \in \mathbb{R}$. By the sake of simplicity, we will drop the superindex and we will write $z = \lambda v$. Thus, we obtain

$$x' = \bar{x} - z = \bar{x} - \lambda v,$$

whose coordinates are given by

$$x'_i = \bar{x}_i - z_i = \bar{x}_i - \lambda v_i. \tag{12}$$

Plugging (12) into (11b) we get

$$\lambda v_n = b_i \lambda v_i (2\bar{x}_i - \lambda v_i) + (a_i - b_i)\lambda v_i, \quad \forall i = 1, \cdots, n-1, \tag{13}$$

and simplifying expression (13) we obtain

$$-\lambda b_i v_i^2 + 2 b_i v_i \bar{x}_i + (a_i - b_i) v_i - v_n = 0, \quad \forall i = 1, \cdots, n-1, \quad \lambda \neq 0. \tag{14}$$

Now, particularizing expression (14) for two different values of λ, $\lambda_1 \neq \lambda_2$, and subtracting both expressions, we reach to

$$(\lambda_1 - \lambda_2) b_i v_i^2 = 0, \quad \forall i = 1, \cdots, n-1. \tag{15}$$

We are going to prove now that there exists $i_0 \in \{1, \cdots, n-1\}$ such that $b_{i_0} \neq 0$ and $v_{i_0} \neq 0$, and therefore, from (15), this would imply that $\lambda_1 = \lambda_2$ what is a contradiction. Thus, $z = 0$ would be the unique solution of the homogeneous linear system and $\bar{x}$ would be the unique solution of the system given by (7) and (8).

Since $v \neq 0$, $\exists v_i \neq 0$ for some $i \in \{1, \cdots, n-1\}$. Otherwise, if $v_i = 0,\ \forall i \in \{1, \cdots, n-1\}$, from (11a) we get $v_n = -a_n \sum_{i=1}^{n-1} v_i = 0$ and $v = 0$, what is not possible. Let us denote I the set of indices for which $v_i \neq 0$. If we suppose that $b_i = 0,\ \forall i \in I$, then from (14) we get $a_i v_i - v_n = 0$. Thus, $v_i = \dfrac{v_n}{a_i},\ \forall i \in I$. Also, from (11a)

$$\sum_{i \in I} \lambda v_i + \lambda \frac{v_n}{a_n} = 0. \tag{16}$$

Now, using in (16) the fact that $v_i = \dfrac{v_n}{a_i},\ \forall i \in I$, we get $v_n = 0$, and in turn, $v = 0$, what gives a contradiction which comes from the supposition $b_i = 0,\ \forall i \in I$. Therefore, $\exists i_0 \in I$, such that $b_{i_0} \neq 0$.

In order to prove now point a) of the theorem, we consider the straight line parallel to the x_n axis passing through $(\bar{x}_1, \ldots, \bar{x}_{n-1}, 0)$, that is

$$r_w \equiv \begin{cases} x_1 = \bar{x}_1, \\ \vdots \\ x_{n-1} = \bar{x}_{n-1}. \end{cases} \tag{17}$$

Cutting this straight line with the hyperplane Π we get the point $(\bar{x}_1, \ldots, \bar{x}_{n-1}, H_w)$, which gives the enunciated result. A similar argument proves point b), just by considering in this case the straight line parallel to the x_n axis passing through the barycenter $GM_w = \sum_{i=1}^{n} w_i B_i = (w_1, \ldots, w_{n-1}, 0)$, and verifying that its intersection point with the hyperplane Π is just the weighted arithmetic mean $M_w = \sum_{i=1}^{n} w_i a_i$. □

Theorem 2 *Let us consider the hyperplane* Π *which passes through the points* $P_i, i = 1, \ldots, n$, $n \geq 2$, *given by the equation*

$$\Pi \equiv \; x_n = a_n + \sum_{i=1}^{n} x_i (a_i - a_n). \tag{18}$$

Let us also consider the paraboloid given by V_n *which passes through the points* $B_i, i = 1, \ldots, n-1$ *and* P_n *given by the equation*

$$V_n \equiv \; x_n = a_n + \sum_{i=1}^{n-1} (c_i x_i^2 - (c_i + a_n) x_i), \tag{19}$$

where the coefficients c_i *are given by*

$$c_i = \frac{\frac{H_w}{n} + (\sum_{j=1}^{n-1} \bar{x}_j - 1) a_n}{(n-1)\bar{x}_i(\bar{x}_i - 1)}, \qquad i = 1, \ldots, n-1, \tag{20}$$

and the paraboloids $V_i, i = 1, \ldots, n-1$ *given by the equations*

$$V_i \equiv x_n = b_i {x_i}^2 + (a_i - b_i) x_i, \tag{21}$$

which pass through $B_1, \ldots, B_{i-1}, P_i, B_{i+1}, \ldots, B_n$ *respectively, where the coefficients* b_i *are given by*

$$b_i = \frac{H_w}{\bar{x}_i(\bar{x}_i - 1)} \left(\frac{1}{n} - w_i\right), \qquad i = 1, \ldots, n-1. \tag{22}$$

Then, the system of equations formed by (19) and (21) has a unique solution $(\bar{x}_1, \ldots, \bar{x}_n)$ *given by*

$$\bar{x}_i = w_i \frac{H_w}{a_i}, \quad i = 1, \ldots, n-1, \quad \bar{x}_n = \frac{H_w}{n}. \tag{23}$$

Moreover, the following two affirmations are true:

a) The height of the prism through the point $(\bar{x}_1, \ldots, \bar{x}_{n-1}, 0)$ *coincides with the weighted harmonic mean* H_w *of* $a_i, i = 1, \ldots, n$, *that is, the point* $(\bar{x}_1, \ldots, \bar{x}_{n-1}, H_w)$ *belongs to the hyperplane* Π.

b) The height of the prism through the barycenter of the base $GM_w = \sum_{i=1}^{n} w_i B_i$ *coincides with the weighted arithmetic mean.*

Proof It is trivial to see that the proposed solution satisfies (19) and (21). Let us prove that the solution is unique. Let us suppose that there exists another solution $x' = (x'_1, \cdots, x'_n)$ with $x' \neq \bar{x}$. Then, denoting $z_i = \bar{x}_i - x'_i, i = 1, \cdots, n$, the system of equations formed by (19) and (21) can be written as

$$\sum_{i=1}^{n-1} \left[c_i z_i (\bar{x}_i + x'_i) - (c_i + a_n) z_i \right] - z_n = 0, \tag{24a}$$

$$z_n - b_i z_i (\bar{x}_i + x'_i) - (a_i - b_i) z_i = 0, \quad i = 1, \cdots, n-1, \tag{24b}$$

what amounts to a homogeneous linear system of n equations with n unknowns. If we show that this system has only the trivial solution $z = 0$, then we would have proven that $\bar{x} = x'$, what is a contradiction with the starting supposition. Therefore, $\bar{x}$ would be the unique solution. Let us then prove that system (24) has $z = 0$ as the unique solution. By reductio ad absurdum, let us suppose that the system has infinite solutions, that is, $z = \sum_{k=1}^{s} \lambda_k v^k$, where $s = n - r$, being r the rank of the coefficient matrix of the linear system, $\lambda_k \in \mathbb{R}$, and v^k, $k = 1, \ldots, s$, represent a base of the Kernel of the associated linear map. Let us consider the univariate set of solutions $z = \lambda_1 v^1$, $\lambda_1 \in \mathbb{R}$. By the sake of simplicity, we will drop the superindex and we will write $z = \lambda v$. Thus, we obtain

$$x' = \bar{x} - z = \bar{x} - \lambda v,$$

whose coordinates are given by

$$x_i' = \bar{x}_i - z_i = \bar{x}_i - \lambda v_i. \tag{25}$$

Plugging (25) into (24b) we get

$$\lambda v_n = b_i \lambda v_i (2\bar{x}_i - \lambda v_i) + (a_i - b_i)\lambda v_i, \quad \forall i = 1, \cdots, n-1, \tag{26}$$

and simplifying expression (26) we obtain

$$-\lambda b_i v_i^2 + 2 b_i v_i \bar{x}_i + (a_i - b_i) v_i - v_n = 0, \quad \forall i = 1, \cdots, n-1, \quad \lambda \neq 0. \tag{27}$$

Now, particularizing expression (27) for two different values of λ, $\lambda_1 \neq \lambda_2$, and subtracting both expressions, we reach to

$$(\lambda_1 - \lambda_2) b_i v_i^2 = 0, \quad \forall i = 1, \cdots, n-1. \tag{28}$$

Before continuing with the main proof, we need to prove the following statement

s1) $sign(c_i) = sign(c_j)\ \forall i, j \in \{1, \ldots, n-1\}$,

where $sign(.)$ denotes the sign function

$$sign(x) := \begin{cases} 1 & x > 0, \\ -1 & x < 0, \\ 0 & x = 0. \end{cases}$$

Statement s1) is proven just by isolating the term H_w in equation $c_i = 0$, that is

$$\begin{aligned} c_i = 0 \Leftrightarrow \frac{H_w}{n} &= a_n (1 - \sum_{j=1}^{n-1} \bar{x}_j) = a_n - H_w (a_n \sum_{j=1}^{n-1} \frac{w_j}{a_j}) \\ \Leftrightarrow H_w &= \frac{a_n}{\frac{1}{n} + a_n \sum_{j=1}^{n-1} \frac{w_j}{a_j}} = \frac{\prod_{k=1}^{n} a_k}{\frac{1}{n} a_1 \ldots a_{n-1} + \sum_{j=1}^{n-1} w_j \prod_{\substack{k=1 \\ k \neq j}}^{n} a_k}. \end{aligned} \tag{29}$$

Comparing expression (29) with the expression of H_w in Definition 1, we get that $c_i = 0 \Leftrightarrow w_n = \frac{1}{n}$, $c_i > 0 \Leftrightarrow w_n < \frac{1}{n}$, and $c_i < 0 \Leftrightarrow w_n > \frac{1}{n}$.

We are ready to continue with the main proof. Since $v \neq 0$, the set of indices I such that $v_i \neq 0, i \in I$, is not empty. Let us suppose that $b_i = 0, \forall i \in I$. From (24), we get

$$v_n = a_i v_i, \tag{30}$$

$$v_n = \sum_{i \in I} (-\lambda c_i v_i + 2 c_i \bar{x}_i - (a_n + c_i)) v_i. \tag{31}$$

Plugging (30) into (31) and using that $v_i \neq 0$, and in turn $v_n \neq 0$, we get that

$$\sum_{i \in I} (-\lambda c_i \frac{v_i}{a_i} + 2 c_i \frac{\bar{x}_i}{a_i} - \frac{a_n + c_i}{a_i}) - 1 = 0, \ \forall \lambda \in \mathbb{R}. \tag{32}$$

From equation (32), since it is true for all value of λ, taking two different values $\bar{\lambda}_1, \bar{\lambda}_2$ we get that

$$-\bar{\lambda}_1 v_n (\sum_{i \in I} \frac{c_i}{a_i^2}) + \sum_{i \in I} (2 c_i \frac{\bar{x}_i}{a_i} - \frac{a_n + c_i}{a_i}) - 1 = 0, \tag{33}$$

$$-\bar{\lambda}_2 v_n (\sum_{i \in I} \frac{c_i}{a_i^2}) + \sum_{i \in I} (2 c_i \frac{\bar{x}_i}{a_i} - \frac{a_n + c_i}{a_i}) - 1 = 0, \tag{34}$$

and subtracting (33) and (34) we get

$$\sum_{i\in I}\frac{c_i}{a_i^2}=0. \tag{35}$$

Taking into account statement s1), since all c_i have the same sign, it must be $c_i=0, i\in I$. In turn, by using (31), this fact implies

$$-\sum_{i\in I}\frac{a_n v_n}{a_i}=v_n\Rightarrow v_n(1+\sum_{i\in I}\frac{a_n}{a_i})=0,$$

what is not viable as $v_n\neq 0$, and we get a contradiction. Therefore, $\exists i\in I$, such that $b_i\neq 0$. From (28), this means that $\lambda_1=\lambda_2$ what gives again a contradiction, this time with the initial supposition. Thus, $z=0$ is the unique solution of the homogeneous linear system and $\bar{x}$ is the unique solution of the system given by (19) and (21).

In order to prove now point a) of the theorem, we consider the straight line parallel to the x_n axis passing through $(\bar{x}_1,\ldots,\bar{x}_{n-1},0)$, that is

$$r_w\equiv\begin{cases}x_1=\bar{x}_1,\\ \vdots\\ x_{n-1}=\bar{x}_{n-1}.\end{cases} \tag{36}$$

Cutting this straight line with the hyperplane Π we get the point $(\bar{x}_1,\ldots,\bar{x}_{n-1},H_w)$, which gives the enunciated result. A similar argument proves point b), just by considering in this case the straight line parallel to the x_n axis passing through the barycenter $GM_w=\sum_{i=1}^{n}w_iB_i=(w_1,\ldots,w_{n-1},0)$, and verifying that its intersection point with the hyperplane Π is just the weighted arithmetic mean $M_w=\sum_{i=1}^{n}w_ia_i$. □

Remark 1 In the non-weighted case, that is, when all $w_i=\frac{1}{n}, i=1,\ldots,n$, all the paraboloids degenerate in diagonal hyperplanes.

In Fig. 1 we represent the situation given in Theorem 2 for three variables, being the situation of Theorem 1 similar. In Fig. 1, in the left part, we show the paraboloids built with the values $a_1=3$, $a_2=4$, $a_3=6$, with the weights $w_1=0.2$, $w_2=0.2$, $w_3=0.6$, and in the right part, the planes obtained for the case of dealing with equal weights $w_1=w_2=w_3=\frac{1}{3}$. These plots correspond to the situation considered in Theorem 2. We observe how the paraboloids degenerate in planes generalizing the case of the non-weighted harmonic mean. We show the intersection of the three paraboloids for the mentioned values and weights. It is interesting to compare the representation of the weighted harmonic mean H_w, which coincides with the height of the prism through the point GH_w (orthogonal projection onto the base of the intersection point of the three paraboloids considered in Theorem 2), with the representation of the weighted arithmetic mean M_w, which amounts to the height of the prism through the barycenter GM_w of the vertices of the triangular base affected by the corresponding weights.

A simpler representation using only hyperplanes is also possible for the general case of dealing with the weighted harmonic mean, as it comes out directly from Remark 1 and from the observation

$$H_{\frac{1}{n}}(\frac{a_1}{w_1},\ldots,\frac{a_n}{w_n})=nH_w(a_1,\ldots,a_n), \tag{37}$$

where $H_{\frac{1}{n}}$ stands for the harmonic mean with uniform weights $w_i=\frac{1}{n}, i=1,\ldots,n$.

More precisely, using the previous notations and defining also

$$\begin{cases}a_i^*=\frac{a_i}{w_i},\\ P_i^*\equiv(0,\cdots,0,\underset{i}{1},0,\cdots,0,a_i^*),\quad i=1,\cdots,n-1,\\ P_n^*=(0,\cdots,0,a_n^*),\\ H_w=H_w(a_1,\ldots,a_n),\ w=(w_1,\ldots,w_n),\\ H_{\frac{1}{n}}^*=H_w(a_1^*,\ldots,a_n^*),\ w=(\frac{1}{n},\ldots,\frac{1}{n}),\\ M_w=M_w(a_1,\ldots,a_n),\ w=(w_1,\ldots,w_n),\\ M_{\frac{1}{n}}^*=M_w(a_1^*,\ldots,a_n^*),\ w=(\frac{1}{n},\ldots,\frac{1}{n}),\end{cases}$$

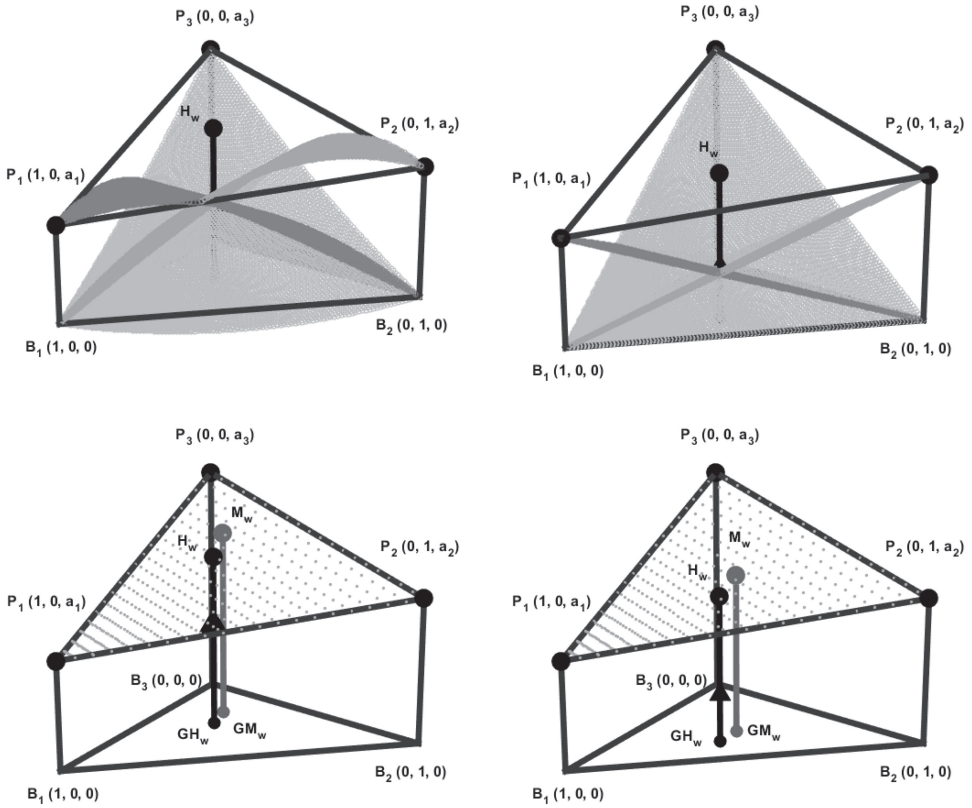

Fig. 1 Representation of weighted harmonic mean of three positive values $a_1 = 3$, $a_2 = 4$, $a_3 = 6$ as the height of the prism through the intersection point of the three paraboloids considered in Theorem 2. Comparison with the weighted arithmetic mean. for the representation of the harmonic mean of the values. Weights $w_1 = 0.2$, $w_2 = 0.2$, $w_3 = 0.6$ to the left and $w_1 = w_2 = w_3 = \frac{1}{3}$ to the right. Upper: Intersection of the three paraboloids. Bottom: Comparison between the weighted harmonic mean and the weighted arithmetic mean

we can give the following corollary.

Corollary 1 *Let us consider the hyperplanes* Π, Π^* *which pass through the points* P_i, *and* P_i^*, $i = 1, \ldots, n$, *respectively,* $n \geq 2$. *They are given by the equations*

$$\Pi \equiv \; x_n = a_n + \sum_{i=1}^{n} x_i (a_i - a_n), \tag{38}$$

$$\Pi^* \equiv \; x_n = \frac{a_n}{w_n} + \sum_{i=1}^{n} x_i \left(\frac{a_i}{w_i} - \frac{a_n}{w_n}\right). \tag{39}$$

Let us also consider the hyperplane V_n^* *which passes through the points* B_i, $i = 1, \ldots, n-1$ *and* P_n^* *given by the equation*

$$V_n^* \equiv \sum_{i=1}^{n-1} x_i + w_n \frac{x_n}{a_n} = 1. \tag{40}$$

and the hyperplanes V_i^*, $i = 1, \ldots, n-1$ *given by the equations*

$$V_i^* \equiv x_n = \frac{a_i}{w_i} x_i, \quad \textit{which pass through } B_1, \ldots, B_{i-1}, P_i^*, B_{i+1}, \ldots, B_n. \tag{41}$$

Then, the system of equations formed by (40) and (41) has a unique solution $(\bar{x}_1, \ldots, \bar{x}_n)$ *given by*

$$\bar{x}_i = w_i \frac{H_w}{a_i}, \quad i = 1, \ldots, n-1, \quad \bar{x}_n = \frac{H^*_{\frac{1}{n}}}{n} = H_w. \tag{42}$$

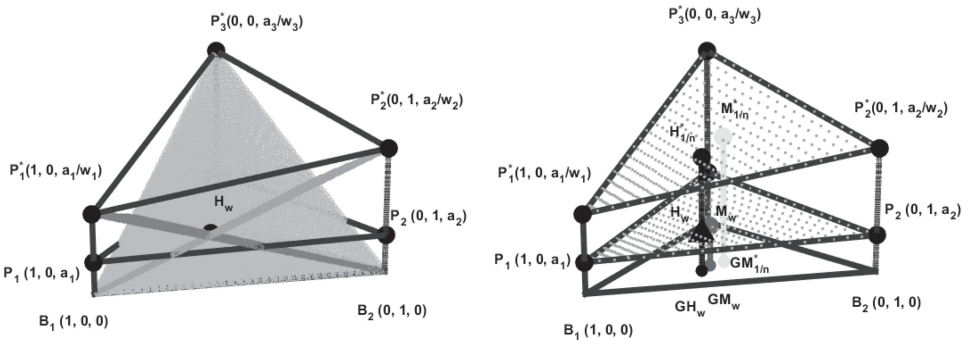

Fig. 2 Representation of weighted harmonic mean according to Corollary 1. Left: Weighted harmonic mean of the three positive values $a_1 = 6$, $a_2 = 7$, $a_3 = 10$ with weights $w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.3$. Right: Comparison among $H^*_{\frac{1}{n}}$, H_w, $M^*_{\frac{1}{n}}$, M_w, in the case of three arguments. In black the harmonic mean, in red the weighted arithmetic mean of the original values, in yellow the arithmetic mean of the modified values

Moreover, the following two affirmations are true:

a) The height of the prism P^* (with vertices P^*_i) through the point $(\bar{x}_1, \ldots, \bar{x}_{n-1}, 0)$ coincides with the harmonic mean of $a^*_i, i = 1, \ldots, n$, that is, $H^*_{\frac{1}{n}}$, and the height of the prism P (with vertices P_i) through the same point is $\frac{1}{n} H^*_{\frac{1}{n}} = H_w$, that is, the point $(\bar{x}_1, \ldots, \bar{x}_{n-1}, H_w)$ belongs to the hyperplane Π.

b) The height of the prism P^* through the barycenter of the triangular base $GM_{\frac{1}{n}} = \sum_{i=1}^{n} \frac{1}{n} B_i$ coincides with the arithmetic mean $M^*_{\frac{1}{n}}$, and the height of the prism P through the weighted barycenter of the triangular base $GM_w = \sum_{i=1}^{n} w_i B_i$ coincides with the weighted arithmetic mean M_w.

Proof It is trivially derived either from Theorem 1 or from Theorem 2 just by applying the relation (37) and observing the Remark 1. □

In Fig. 2 we see the representation of Corollary 1 in $3D$ for a particular choice of the arguments a_i and the weights. We can appreciate the relation (37) between the harmonic mean of the modified arguments a^*_i and the weighted harmonic mean of the original arguments a_i, which is placed at the third part of the height of the prism through the corresponding abscissa. However, no clear relation is observed between the arithmetic mean of the modified values $M^*_{\frac{1}{n}}$ and the weighted arithmetic mean M_w of the original ones.

3 Examples of application

In this section our main purpose is to point out how to use the simple theoretical results presented in previous sections to define a nonlinear reconstruction operator adapted to jump discontinuities. This application is just one possibility of use of the introduced concepts. It can be applied in many other contexts in order to define a nonlinear method from an already existing linear method, just by writing the necessary expressions in terms of a weighted arithmetic mean of some quantities, which act somehow as smoothness indicators. Adaptation after the substitution of the weighted arithmetic mean for the corresponding weighted harmonic mean will take place if only a few of these quantities are affected by the presence of a potential discontinuity, and these affected quantities are not used in the rest of the expressions. These smoothness indicators should also satisfy similar hypothesis to those of Lemma 2 in smooth areas of an hypothetical underlying function in order to maintain the approximation order of the original method. The fact of substituting the arithmetic mean for a corresponding harmonic mean will allow the adaptation thanks to Lemma 1, since the large values, due to the presence of a discontinuity, will be limited.

In real applications, it might be also necessary the application of a translation strategy as it is well explained in [6,7] in order to deal with weighted means of values which do not necessarily have the same sign.

Some examples of already existing methods that use these ideas with the harmonic mean of two values can be found in several applications. Let us mention:

- Point values reconstructions and the related field of subdivision and multiresolution schemes, see [1,11] and the references therein.
- The field of image processing, to define nonlinear compression methods into the cell averages framework inside Harten's multiresolution, see [5].
- Also in the field of image processing for denoising purposes, see [8].
- Generation of curves and surfaces, due to some remarkable properties of the harmonic mean in relation with the definition of convexity preserving reconstruction methods, see for example [10].
- In combination with spline reconstructions, see [9].
- In the solution of hyperbolic conservation laws, see [13,14].

Up to our knowledge, there is only one application using these ideas in $3D$ involving harmonic means of 3 values, [7], and there is no other implementation of nonlinear algorithms based on this methodology in higher dimensions or involving harmonic means of more than 3 values.

4 Conclusions

In this article, we have presented two relevant properties of the harmonic mean that allow for new constructions of numerical methods, such as nonlinear reconstruction operators, subdivision and multiresolution schemes, and solvers of hyperbolic conservation laws. These properties have been presented for any finite number of arguments, with the purpose of generating new algorithms in problems involving N-dimensional spaces. We have given some geometrical representations of both the weighted harmonic mean and the weighted arithmetic mean, where the mentioned properties can be appreciated in an intuitive way. In the last part of the article, we offer a list of examples that illustrate the methodology for the two variables case, and we explain how to use these simple concepts to attain interesting and promising results in defining new methods for higher dimensions. In fact, we give also a reference of a particular new reconstruction method for two dimensional functions which seems to avoid Gibbs effect at the time of retaining some approximation order close to jump discontinuities.

References

1. S. Amat, R. Donat, J. Liandrat, J.C. Trillo, Analysis of a new nonlinear subdivision scheme. Applications in image processing. Found. Comput. Math. **6** (2), (2006), 193-225.
2. S. Amat, K. Dadourian, J. Liandrat, J. C. Trillo, High order nonlinear interpolatory reconstruction operators and associated multiresolution schemes. J. Comput. Appl. Math. **253**, (2013), 163-180.
3. S. Amat, R. Donat, J. C. Trillo, Proving convexity preserving properties of interpolatory subdivision schemes through reconstruction operators. Appl. Math. Comput. **219** (14), (2013), 7413-7421.
4. S.Amat, J.Liandrat, On the stability of PPH nonlinear multiresolution. Appl. Comp. Harm. Anal. **18** (2), (2005), 198-206.
5. S. Amat, J. Liandrat, J. Ruiz, J.C. Trillo, On a nonlinear mean and its application to image compression using multiresolution schemes. Numer. Algorithms. **71** (4) (2016), 729-752.
6. S. Amat, P. Ortiz, J. Ruiz, J.C. Trillo, D. F. Yáñez, The translation operator. Applications to nonlinear reconstruction operators on nonuniform grids. Submitted.
7. S. Amat, P. Ortiz, J. Ruiz, J.C. Trillo, D. F. Yáñez, A nonlinear PPH-type reconstruction based on equilateral triangles. *arXiv* **2022**, arxiv:2202.02293.
8. S. Amat, J. Ruiz, J.C. Trillo, Fast multiresolution algorithms and their related variational problems for image denoising. J. Sci. Comput. **43** (1), (2010), 1-23
9. S.Amat, C.W. Shu, J.Ruiz, J.C. Trillo, On a class of splines free of Gibbs phenomenon. Math. Modell. in Numer. Anal. https://doi.org/10.1051/m2an/2020021,(2020).
10. F. Kuijt, R. van Damme, Convexity preserving interpolatory subdivision schemes. Const. Approx., **14**, (1998), 609-630.
11. P. Ortiz, J.C. Trillo, On the convexity preservation of a quasi C^3 nonlinear interpolatory reconstruction operator on σ quasi-uniform grids. Mathematics. , **9** (4-310), (2021). https://doi.org/10.3390/math9040310.
12. P. Ortiz, J.C. Trillo, PPH nonlinear interpolatory reconstruction operator on non uniform grids: Adaptation around jump discontinuities and elimination of Gibbs phenomenon. *Mathematics.*, **9** (335), (2021). https://doi.org/10.3390/math9040335
13. S. Serna, A class of extended limiters applied to piecewise hyperbolic methods. SIAM J. Sci. Comput. **28** (1), (2006), 123-140.
14. S. Serna, A. Marquina, Power ENO methods: a fifth-order accurate weighted power ENO method. J. Comput. Phys. **194** (2), (2004), 632-658.
15. J.C. Trillo, *Nonlinear multiresolution and applications in image processing*, PhD in the University of Valencia, Spain, (2007).

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