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ORIGINAL PAPER

From central limit theorems to continuous tensor products

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Dedicated to the memory of Kalyanapuram Rangachari Parthasarathy.

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Introduction

After reading K.R. Parthasarathy's (KRP's) monograph *Probability measures on metric spaces* [8] in 1968 I decided to apply for a post-doc position at Manchester University to work with him. When I eventually arrived there in 1969 KRP's interests had begun to focus on some intriguing connections between central limit theorems in classical probability theory and analogous limit theorems for unitary representations of locally compact groups. The fact that he was starting on a new mathematical venture, combined with his extraordinary willingness to share problems and ideas, allowed me to begin working with him on these problems almost immediately after my arrival.

One of the important problems in classical probability theory is to determine which probability distributions on the real line arise as weak limits of sums of *uniformly infinitesimal* independent random variables. By using the correspondence between probability measures and positive definite functions on $\mathbb{R}$ this problem becomes equivalent to finding all continuous positive definite functions which occur as limits of *uniformly infinitesimal arrays of positive definite functions* (explained below). All such limits can be described by the *Lévy-Khinchine formula* (cf. e.g., [5]). In 1963, KRP, Ranga Rao and Varadhan proved a version of the Lévy-Khinchine formula for limits of uniformly infinitesimal arrays of positive definite functions on locally compact second countable abelian groups (cf. [11, Theorem 7.1] or [8, Corollary IV.7.1]).

A few years later R. Streater [19], motivated by papers by Araki and Woods ([1], [2]), showed that analogous limits of products of uniformly infinitesimal arrays of positive definite functions on more general groups can be used to construct so-called *factorisable representations* of certain current groups, leading to interesting mathematical models of quantum fields. KRP was immediately captivated by the problem of finding an analogue of the Lévy-Khinchine formula for such limits on general locally compact second countable groups.

On the following pages I will attempt to give a brief account of what progress we made on this problem in [13]. I hasten to emphasise that the lion's share of this progress should be credited to my former friend and collaborator KRP.

Limit theorems for positive definite functions on groups

Let G be a Polish (i.e. complete, separable, metric) group with identity element 1_G, $\mathcal{H}$ a complex, separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let $V\colon g \mapsto V_g$ be a (weakly) continuous, unitary,

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cyclic representation of G on $\mathcal{H}$ with a cyclic unit vector v. If we set

$$\phi(g) = \langle V_g v, v\rangle \tag{1}$$

for every $g \in G$, then the function $\phi\colon G \to \mathbb{C}$ is continuous and *positive definite*, i.e.

$$\sum\nolimits_{i,j=1}^{m} c_i \bar{c}_j \phi(g_j^{-1} g_i) = \Big\| \sum\nolimits_{i=1}^{m} c_i V_g v \Big\|^2 \geq 0$$

for every $m \geq 1$ and every choice of $g_1, \ldots, g_m$ in G and $c_1, \ldots, c_m$ in $\mathbb{C}$. Furthermore, ϕ is *normalized* in the sense that $\phi(1_G) = 1$.

Conversely, if $\phi\colon G \to \mathbb{C}$ is a continuous normalized positive definite function, then there exists a continuous unitary representation V of G on a complex separable Hilbert space $\mathcal{H}$ with a cyclic unit vector v satisfying (1) for every $g \in G$ (cf. e.g. [3, Construction 2.B.4] or [13, Theorem 1.2]).

If (V, v) and (V', v') are two continuous, unitary, cyclic representations of G on Hilbert spaces $\mathcal{H}, \mathcal{H}'$, respectively, and if

$$\phi(g) = \langle V_g v, v\rangle, \qquad \phi'(g) = \langle V'_g v', v'\rangle,$$

for every $g \in G$, then the tensor product representation $V \otimes V'$ on $\mathcal{H} \otimes \mathcal{H}'$, restricted to the cyclic subspace of $\mathcal{H} \otimes \mathcal{H}'$ generated by $v \otimes v'$, satisfies that

$$\langle (V_g \otimes V'_g)\,(v \otimes v'), (v \otimes v')\rangle = \phi(g)\phi'(g), \tag{2}$$

for every $g \in G$. This shows that the pointwise product $\phi\phi'$ of two continuous, positive definite functions is again positive definite and arises from the tensor product of the representations arising from ϕ and ϕ'.

Assume now that the group G is locally compact, second countable, and abelian. We denote by $\widehat{G} = \mathrm{Hom}(G, \mathbb{S})$ the *dual* or *character group* of G, consisting of all continuous homomorphisms from G into the group $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$. Under the topology of uniform convergence on compact sets $\widehat{G}$ is again a locally compact and second countable abelian group (cf. [17] or [8, Chapter IV]).

For every $\gamma \in \widehat{G}$ and $g \in G$, we write $\langle g, \gamma\rangle \in \mathbb{S}$ for the value at $g \in G$ of the character $\gamma \in \widehat{G}$. *Bochner's Theorem* allows one to find, for every continuous positive definite function $\phi\colon G \to \mathbb{C}$, a unique Borel probability measure μ_ϕ on $\widehat{G}$ such that

$$\phi(g) = \int_{\widehat{G}} \langle g, \gamma\rangle d\mu_\phi(\gamma) \tag{3}$$

for every $g \in G$. Conversely, if μ is a Borel probability measure on $\widehat{G}$, then the function $\hat{\mu}\colon G \to \mathbb{C}$ defined by

$$\hat{\mu}(g) = \int_{\widehat{G}} \langle g, \gamma\rangle d\mu(\gamma) \tag{4}$$

is continuous, positive definite, and normalized. One can reconcile the equations (1) and (3) by setting $\mathcal{H} = L^2(\widehat{G}, \mu_\phi)$, and by considering the unitary representation V of G on $\mathcal{H}$ given by

$$(V_g f)(\chi) = \langle g, \gamma\rangle f(\gamma)$$

for every $f \in \mathcal{H}$, $g \in G$ and $\gamma \in \widehat{G}$. The constant function $v = 1 \in \mathcal{H}$ is cyclic for V, and

$$\langle V_g v, v\rangle = \int_{\widehat{G}} \langle g, \gamma\rangle \, d\mu_\phi(\chi) = \phi(g)$$

for every $g \in G$.

If μ, μ' are two Borel probability measures on $\widehat{G}$, the product of the positive definite functions $\hat{\mu}, \hat{\mu}'$ in (4) satisfies that

$$\hat{\mu}\hat{\mu}' = \widehat{\mu * \mu'}, \tag{5}$$

where $\mu * \mu'$ is the *convolution* of μ and μ', i.e. the probability measure on $\widehat{G}$ with

$$\int_{\widehat{G}} f d(\mu * \mu') = \int_{\widehat{G}} \int_{\widehat{G}} f(\chi\chi')\, d\mu(\chi)\, d\mu'(\chi')$$

for every bounded Borel function $f\colon \widehat{G} \to \mathbb{C}$.

Infinitely divisible probability measures

A probability measure μ on $\widehat{G}$ is *infinitely divisible* if there exists, for every $n \geq 1$, a probability measure ν_n on $\widehat{G}$ with $\mu = \nu_n^{*n} = \nu_n * \cdots * \nu_n$. If μ on $\widehat{G}$ is infinitely divisible, then (4) – (5) imply that $\hat{\mu}$ is a continuous *infinitely divisible positive definite function*, i.e. that there exists, for every $n \geq 1$, a continuous normalized positive definite function ϕ_n with $\hat{\mu} = \phi_n^n$. The infinitely divisible continuous normalized positive definite functions on G are thus in one-to-one correspondence with the infinitely divisible Borel probability measures on $\widehat{G}$.

For $G = \mathbb{R}$, the celebrated *Lévy-Khinchine formula* states that a continuous, positive definite function $\phi : \mathbb{R} \to \mathbb{C}$ is infinitely divisible if and only if

$$\log \phi(t) = it\beta - \tfrac{t^2\sigma^2}{2} + \int_{\mathbb{R}} \left(\exp itx - 1 - \tfrac{itx}{1+x^2}\right) \tfrac{1+x^2}{x^2}\, d\nu(x) \tag{6}$$

for some $\beta \in \mathbb{R}$, $\sigma^2 \geq 0$, and some finite Borel measure ν on $\mathbb{R}$ with $\nu(\{0\}) = 0$ (cf. [5, §18, Theorem 1]).

The formula on the right hand side of (6) has three parts: after exponentiation, the first part ($t \mapsto it\beta$) determines a character of $\mathbb{R}$ and amounts to a translation of the probability measure μ_ϕ in question. The second part comes from a homomorphism ($t \mapsto t\sigma/2$) from $\mathbb{R}$ to $\mathbb{R}$ and corresponds to the *Gaussian part* of our measure μ_ϕ. The third, and most intimidating looking, part refers to the *Poisson part* of μ_ϕ: it is—in essence—a fancy linear combination of terms of the form $t \mapsto e^{itx} - 1$ which will reappear much later in this account (cf. (9))

In the derivation of analogous Lévy-Khinchine formulas for infinitely divisible positive definite functions on arbitrary locally compact, second countable abelian groups G in [11] or [8], one obtains the same three parts as in (6) with the possible addition of a fourth term coming from potential nontrivial open subgroups of the group G.

Ever since [5], the discussion of infinitely divisible positive definite functions has been broadened to include limits of uniformly infinitesimal arrays of probability measures on $\mathbb{R}$ and, more generally, on locally compact second countable abelian groups.

Definition 1 Let $\mathcal{M} = \{\mu_{n,j} \,|\, n \geq 1, 1 \leq j \leq k_n\}$ be a triangular array of probability measures on a locally compact, second countable abelian group G. The array $\mathcal{M}$ is *uniformly infinitesimal* if

$$\lim_{n\to\infty} \sup_{1\leq j\leq k_n} \mu_{n,j}(G \setminus \mathcal{N}) = 0 \tag{7}$$

for every neighbourhood $\mathcal{N}$ of the identity in G. Equation (7) is equivalent to the condition that

$$\lim_{n\to\infty} \sup_{1\leq j\leq k_n} \sup_{g\in K} |\hat{\mu}_{n,j}(g) - 1| = 0$$

for every compact set $K \subset G$.

A uniformly infinitesimal array $\mathcal{M}$ *converges* to a probability measure μ on G if the sequence of probability measures $(\mu_n := \prod_{j=1}^{k_n} \mu_{k,j},\ n \geq 1)$ converges to μ in the weak topology (cf. [8, Section II.6]).

The following result follows from [8, Theorem V.5.2] and [13, Theorem 12.5]:

Theorem 2 *Let G be a locally compact, second countable abelian group. If a probability measure μ on G is the limit of a uniformly infinitesimal triangular array of probability measures on G, then μ is infinitely divisible.*

Conversely, if G is connected and locally connected, and if μ is an infinitely divisible probability measure on G, then μ is the limit of a uniformly infinitesimal triangular array of probability measures on G.

For refinements and extensions of the convergence results for uniformly infinitesimal triangular arrays of positive definite functions in [13] the reader may wish to consult the follow-up paper [10] by KRP.

From infinitely divisible positive definite functions to cocycles

If G is an arbitrary Polish group G, then (2) implies that every infinitely divisible positive definite function ϕ on G defines a continuous, unitary, cyclic representation V of G on a complex separable Hilbert space $\mathcal{H}$ which can, for every $n \geq 1$, be written as the tensor product of n copies of some other cyclic representation V_n of G on some Hilbert space $\mathcal{H}_n$. According to R. Streater [19], such "infinitely divisible representations" of G lead to the construction of factorisable representations of current groups and to interesting mathematical models of quantum

fields. KRP's earlier interest in a Lévy-Khinchine formula for infinitely divisible probability measures on locally compact abelian groups led him to be immediately captivated by the problem of finding an analogous formula for infinitely divisible positive definite functions and, more generally, for limits of uniformly infinitesimal arrays of positive definite functions on general Polish groups.

We start with two definitions.

Definition 3 A *triangular array* of continuous normalized positive definite functions $\Phi = \{\phi_{n,j} | n \geq 1, 1 \leq j \leq k_n\}$ on a Polish group G is *uniformly infinitesimal* if

$$\lim_{n\to\infty} \sup_{1\leq j\leq k_n} \sup_{g\in K} |\phi_{n,j}(g) - 1| = 0$$

for every compact set $K \subset G$.

Such a uniformly infinitesimal triangular array Φ *converges* to a continuous positive definite function ϕ if ϕ is the limit of the sequence $(\phi_n := \prod_{j=1}^{k_n} \phi_{n,j}, n \geq 1)$ in the topology of uniform convergence on compact subsets of G.

As in Theorem 2 one can show that limits of uniformly infinitesimal arrays of continuous positive definite functions on a group G are infinitely divisible, and that the reverse implication holds under certain hypotheses on the group G.

When I arrived in Manchester in 1969 KRP had just proved the following striking result:

Theorem 4 *([13, Theorem 11.2]) Let G be a Polish group, and let $\phi\colon G \to \mathbb{C}$ be a limit of a uniformly infinitesimal array of continuous normalized positive definite functions. Then there exist an open subgroup $G_0 \subset G$, a continuous unitary representation U of G_0 on a complex separable Hilbert space $\mathcal{K}$ with inner product $\langle\cdot,\cdot\rangle$, and a continuous map $\delta\colon G_0 \to \mathcal{K}$ such that*

$$U_g\delta(h) = \delta(gh) - \delta(g), \tag{8}$$

$$\phi(gh)\phi(g)^{-1}\phi(h)^{-1} = \exp\langle\delta(h), \delta(g^{-1})\rangle \tag{9}$$

for all $g, h \in G_0$.

Any continuous function $\delta\colon G_0 \to \mathcal{K}$ satisfying (8) is called a (1-)*cocycle* for the representation U. Such a cocycle is a *coboundary* (or *trivial*) if there exists a $z \in V$ such that

$$\delta(g) = V_g z - z \tag{10}$$

for every $g \in G_0$. If two cocycles differ by a coboundary they are called *cohomologous*.

From cocycles to infinitely divisible projective representations

After having proved Theorem 4, KRP tried to obtain a converse to this result: given a continuous unitary representation V of a Polish group G on a Hilbert space $\mathcal{K}$ and a continuous cocycle $\delta\colon G \to \mathcal{K}$ satisfying (8), can one find a continuous infinitely divisible positive definite function $\phi\colon G \to \mathbb{C}$ satisfying (9)?

At that time I had just read KRP's notes [9] and noticed that the map $L\colon G \times G \to \mathbb{C}$ defined by

$$L(g, g') = \langle\delta(g'), \delta(g^{-1})\rangle \tag{11}$$

is a *2-cocycle*:

$$L(g_1, g_2) + L(g_1g_2, g_3) = L(g_1, g_2g_3) + L(g_2, g_3) \tag{12}$$

for all $g_1, g_2, g_3 \in G$. Since the real part $\Re L$ of L satisfies the equation

$$\Re L(g, g') = \tfrac{1}{2}\big(\|\delta(g)\|^2 + \|\delta(g')\|^2 - \|\delta(gg')\|^2\big)$$

for all g, g', $\Re L$ is a *2-coboundary*, i.e., there exists a continuous map $b\colon G \to \mathbb{C}$ such that

$$\Re L(g_1, g_2) = b(g_1g_2) - b(g_1) - b(g_2).$$

The imaginary part $\Im L$ of L may or may not be a 2-coboundary, but it certainly satisfies the cocycle equation (12). If we put

$$\sigma(g, g') = \exp(i \cdot \Im L(g, g')), \tag{13}$$

the resulting map $\sigma : G \times G \to \mathbb{S}$ is again a 2-cocycle: it satisfies the multiplicative analogue

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3) \tag{14}$$

of (12) for all $g_1, g_2, g_3 \in G$.

In order to construct the putative positive definite function ϕ in (9) and its associated unitary representation V on a Hilbert space $\mathcal{H}$ in (1) we introduce the *symmetric Fock space* $\operatorname{Exp}\mathcal{K}$ over $\mathcal{K}$, given by

$$\operatorname{Exp}\mathcal{K} = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K} \otimes_s \mathcal{K} \oplus \mathcal{K} \otimes_s \mathcal{K} \otimes_s \mathcal{K} \oplus \cdots, \tag{15}$$

where $\otimes_s$ denotes the *symmetric* tensor product of Hilbert spaces (cf. [13, pp. 29-30]). For every $v \in \mathcal{K}$, put

$$\operatorname{Exp} v = 1 \oplus v \oplus \tfrac{1}{\sqrt{2}}(v \otimes v) \oplus \tfrac{1}{\sqrt{3}}(v \otimes v \otimes v) \cdots \in \operatorname{Exp}\mathcal{K}.$$

If $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the obvious inner product on $\operatorname{Exp}\mathcal{K}$, then

$$\langle\!\langle \operatorname{Exp} v, \operatorname{Exp} v' \rangle\!\rangle = \exp\langle v, v' \rangle$$

for all $v, v' \in \mathcal{K}$. We denote by $\mathcal{H} \subset \operatorname{Exp}\mathcal{K}$ the closed linear span of the set $\{\operatorname{Exp}\delta(g) \mid g \in G\}$. For every $g, h \in G$, let

$$V_g \operatorname{Exp}\delta(h) = \exp(L(g, h) - \|\delta(g)\|^2/2) \operatorname{Exp}\delta(gh), \tag{16}$$

where L is defined in (11). Since

$$\langle\!\langle V_g \operatorname{Exp}\delta(h_1), V_g \operatorname{Exp}\delta(h_2) \rangle\!\rangle = \langle\!\langle \operatorname{Exp}\delta(h_1), \operatorname{Exp}\delta(h_2) \rangle\!\rangle$$

for every $h_1, h_2 \in G$, V_g extends by linearity to a unitary operator on $\mathcal{H}$. The map $g \mapsto V_g$ from G into the unitary group $\mathcal{U}(\mathcal{H})$ is continuous, but will in general not be a unitary representation of G: it is a *projective representation* satisfying

$$\sigma(g, g')V_{gg'} = V_g V_{g'} \tag{17}$$

for all $g, g' \in G$, where the *multiplier* σ of V is the 2-cocycle appearing in (13). By definition, the vector $v = \operatorname{Exp}\delta(1_G) = \operatorname{Exp} 0 \in \mathcal{H}$ is *cyclic* for this projective representation, and

$$\Phi(g) := \langle\!\langle V_g v, v \rangle\!\rangle = \exp(-\|\delta(g)\|^2/2) \tag{18}$$

for every $g \in G$. Hence

$$\Phi(gg')\Phi(g)^{-1}\Phi(g')^{-1} = \exp(\Re\langle\delta(g'), \delta(g^{-1})\rangle).$$

This isn't quite what we wanted in (9). However, if σ is a coboundary, i.e., if

$$\sigma(g, gg') = \beta(gg')b(g)^{-1}b(g')^{-1} \tag{19}$$

for some Borel (and, in fact, automatically continuous) map $\beta : G \to \mathbb{S}$, then the map

$$\tilde{V} : g \mapsto \overline{\beta(g)} V_g$$

is a continuous unitary representation of G, and $\tilde{\Phi}(g) = \langle\!\langle \tilde{V}_g v, v \rangle\!\rangle$ satisfies that

$$\tilde{\Phi}(gg')\tilde{\Phi}(g)^{-1}\tilde{\Phi}(g')^{-1} = \langle\!\langle \tilde{V}_g v, v \rangle\!\rangle$$

for every $g \in G$. It is easy to check that the map $\tilde{\Phi} : G \to \mathbb{C}$ has all the good properties one could hope for: it is continuous, normalized, positive definite, infinitely divisible, and it is a limit of a uniformly infinitesimal array of continuous positive definite functions. Since this is obviously a very nice situation, let me introduce some ad-hoc (and completely non-standard) terminology:

Definition 5 Let U be a continuous unitary representation of a Polish group G on a complex separable Hilbert space $\mathcal{K}$. A continuous 1-cocycle $\delta\colon G \to \mathcal{K}$ is *rectifiable* if the 2-cocycle $\sigma\colon G \times G \to \mathbb{S}$ in (13) is a coboundary in the sense of (19).

Which conditions on the pair (V, δ) will guarantee that δ is rectifiable?

Proposition 6 (1) *A cocycle $\delta\colon G \to \mathcal{K}$ is a coboundary if and only if it is norm-bounded.* (2) *If δ is a coboundary, then it is rectifiable.*

Proof (1) Suppose that δ is bounded in the sense that $K = \sup_{g\in G} \|\delta(g)\|_\infty$. For every $g \in G$ and $v \in \mathcal{K}$, let

$$A_g x = V_g x - \delta(g).$$

Since δ is bounded, the set $S = \{A_g 0 \mid g \in G\} \subset \mathcal{K}$ is bounded in norm, and so is the closed convex hull $C(S)$ of S in $\mathcal{K}$. By the Ryll-Nardzewski fixed point theorem [7, p. 444], there exists a point $z \in C(S)$ such that $A_g z = z$ for every $g \in G$, so that δ is indeed trivial. Conversely, if δ is a coboundary, then it is obviously bounded.

(2) If $\delta(g) = V_g z - z$ for all g, then

$$\begin{aligned} L(g,h) &= \langle \delta(h), \delta(g^{-1})\rangle = \langle U_h z - z, U_{g^{-1}} z - z\rangle - \|z\|^2 \\ &= (\langle U_{gh} z - z, z\rangle - \|z\|^2) - (\langle U_g z, z\rangle - \|z\|^2) - (\langle U_h z, z\rangle - \|z\|^2). \end{aligned}$$

Hence the cocycle σ in (13) satisfies (19) with $\beta(g) = \exp(i \cdot \Im(\langle U_g z, z\rangle)$, and is thus a 2-coboundary. □

If the 2-cocycle σ in (13) is nontrivial (i.e., not a 2-coboundary), the projective representation V in (16) – (17) is again infinitely divisible in the sense that there exists, for every $n \geq 1$, a cyclic projective representation $V^{(n)}$ of G on a Hilbert space $\mathcal{H}^{(n)}$ with multiplier $\sigma^{(n)} = \exp(i \cdot \Im L/n)$ and cyclic vector $v^{(n)}$ such that V is unitarily equivalent to the projective representation $V^{(n)} \otimes \cdots \otimes V^{(n)}$ of G on the n-fold tensor product $\mathcal{H}^{(n)} \otimes \cdots \otimes \mathcal{H}^{(n)}$, restricted to the cyclic subspace generated by $v^{(n)} \otimes \cdots \otimes v^{(n)}$.

Since there is such a close correspondence between continuous 1-cocycles of continuous unitary representations of G and continuous infinitely divisible projective representations of G, any attempt to classify all infinitely divisible projective representations of G (and, in particular, all infinitely divisible positive definite functions on G) would require a classification of all 1-cocycles of continuous unitary representations of our Polish group G.

If G is locally compact and second countable, the usual decomposition techniques allow one to restrict one's attention to *irreducible* representations of G: if we can write a unitary representation U of G as a direct integral $U = \int_X^\oplus U^\omega \, d\mu(\omega)$ of irreducible unitary representations over some finite measure space (Ω, μ), then [13, Theorem 13.2] shows that every continuous cocycle δ for U is of the form $\delta(g) = \int_X^\oplus \delta^\omega(g) \, d\mu(\omega)$, where each δ^ω is a continuous cocycle of the representation U^ω.

For example, if G is abelian, every irreducible unitary representation U of G is one-dimensional and thus a continuous homomorphism from G into $\mathbb{S}$. If $U_h \neq 1$ for some $h \in G$ then $U_h - 1$ is invertible, so that every $v \in \mathcal{H} = \mathbb{C}$ is of the form $v = U_h b - b$ for some $b \in \mathbb{C}$. In particular, any cocycle δ for U satisfies that

$$\delta(h) = U_h b - b$$

for some $b \in \mathcal{H}$. If $h' \in G$ is a second element with $U_{h'} \neq 1$, then

$$\delta(h') = U_{h'} b' - b'$$

for some $b' \in \mathbb{C}$, and

$$\begin{aligned} \delta(hh') &= U_h \delta(h') + \delta(h) = U_{hh'} b' - U_h b' + U_h b - b \\ &= U_{h'} \delta(h) + \delta(h') = U_{hh'} b - U_h b + U_h b' - b', \end{aligned}$$

so that

$$(U_h - 1)(U_{h'} - 1)(b - b') = 0$$

and $b = b'$. It follows that $\delta(g) = U_g b - b$ for every $g \in G$, i.e. that δ is trivial.

Proposition 7 *Let G be an abelian Polish group, U a continuous irreducible representation of G on $\mathcal{H} = \mathbb{C}$, and $\delta\colon G \to \mathbb{C}$ a cocycle of V. If U is nontrivial, then δ is a coboundary* (*cf.* (10)). *If U is trivial* (*i.e., if $U_h = 1$ for all h*), *then δ is a homomorphism from G into $\mathbb{C}$.*

Proposition 7 might give the impression that, for abelian groups, only trivial representations could have non-trivial cocycles (i.e. cocycles which are not coboundaries). However, the Lévy-Khinchine formula (6) is evidence that non-compact locally compact abelian groups can have unitary representations with unbounded cocycles, but without nonzero invariant vectors. For obvious reasons such cocycles are sometimes called *generalised coboundaries*. These generalised coboundaries give rise to the "Poisson parts" of the Lévy-Khinchine formulas (cf. [13, Theorem 16.3]).

Furthermore, if $\delta\colon G \to \mathcal{K}$ is a nontrivial homomorphism arising from the trivial representation of G on a Hilbert space $\mathcal{K}$, then δ is not only nontrivial (since it is unbounded), but it can also give rise to a nontrivial 2-cocycle $\sigma\colon G \times G \to \mathbb{S}$ in (13) – (14). A simple example of this is described in [13, § 8]: if $G = \mathcal{K} = \mathbb{C}$ and U is the trivial representation of G on $\mathcal{K}$, then the identity map $\delta\colon z \mapsto z$ from G to $\mathcal{K}$ defines a cocycle for U on $\mathcal{K}$ which gives rise to the projective representation V in (16) with multiplier $\sigma(z_1, z_2) = \exp(-i \cdot \Im(z_2 \bar{z_1}))$ in (17).

In [13, Part III] there are a few further results about cocycles for irreducible representations of other classes of groups (e.g., connected nilpotent or semi-simple Lie groups), but a more complete picture arises from the papers [4] by P. Delorme and [22] by A.M. Veršhik and S.I. Karpushev. The latter paper explains in particular the apparent scarcity of nontrivial 1-cocycles for irreducible unitary representations of locally compact groups (for terminology we refer to [22]).

Theorem 8 *([22, Theorem 2]) Let G be a locally compact second countable group, and let V be an irreducible unitary representation of G on a complex separable Hilbert space $\mathcal{K}$ which has a nontrivial continuous 1-cocycle. Then V is infinitely small* (*i.e., Hausdorff unseparable from the trivial representation*).

For nontrivial irreducible unitary representations, non-rectifiability of 1-cocycles is, of course, even "scarcer" than non-triviality. The following proposition is a corollary of [9, p. 40, Corollary 1].

Proposition 9 *If G is a connected and simply connected semi-simple Lie group, U a continuous irreducible unitary representation of G on a complex Hilbert space $\mathcal{K}$, and $\delta\colon G \to \mathcal{K}$ a continuous cocycle for V, then δ is rectifiable in the sense of Definition 5.*

Factorisable projective representations

Before moving on to representations of current groups in the spirit of [1] and [19], let me point out an important property of the projective representation V of G on the Hilbert space $\mathcal{H} \subset \operatorname{Exp}\mathcal{K}$ arising from a unitary representation U of G on $\mathcal{K}$ and a cocycle $\delta\colon G \to \mathcal{K}$ in (16) – (17): if $\mathcal{K}$ can be written as a direct sum $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ of two closed U-invariant subspaces we can decompose the cocycle $\delta\colon G \to \mathcal{K}$ as a direct sum $\delta_1 \oplus \delta_2$ of two cocycles $\delta_i\colon G \to \mathcal{K}_i$ and construct the corresponding infinitely divisible projective representations V_i of G on $\mathcal{H}_i \subset \operatorname{Exp}\mathcal{K}_i$ as above. Then the projective representation V of G on $\mathcal{H}$ is unitarily equivalent to the tensor product representation $V_1 \otimes V_2$ on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the spaces $\mathcal{H}_i$.

This elementary observation can be developed to construct *factorisable representations* of so-called *current groups*. The following exposition is based on [12] – [13].

Assume for convenience that X is a compact metrizable space with a nonatomic Borel probability measure μ, G a Polish group, U a continuous unitary representation of G on a complex Hilbert space $\mathcal{K}$ with inner product $\langle \cdot, \cdot \rangle$, and $\delta\colon G \to \mathcal{K}$ a continuous 1-cocycle of U. We denote by $\boldsymbol{\mathcal{K}} = \int_X^{\oplus} \mathcal{K}\, d\mu = L^2_\mu(X, \mathcal{K})$ the Hilbert space of square-integrable maps $f\colon X \to \mathcal{K}$ with inner product $[f, f'] = \int_X \langle f(x), f'(x) \rangle\, d\mu(x)$.

Let $\Gamma = C(X, G)$ be the group of continuous maps $\gamma\colon X \to G$, furnished with pointwise multiplication and the topology of uniform convergence, and define a continuous unitary representation $\mathbf{U}$ of Γ on $\boldsymbol{\mathcal{K}}$ setting

$$(\mathbf{U}_\gamma f)(x) = U_{\gamma(x)} f(x)$$

for every $\gamma \in \Gamma$, $x \in X$, and $f \in \boldsymbol{\mathcal{K}}$. The cocycle δ gives rise to a continuous 1-cocycle $\boldsymbol{\delta}\colon \Gamma \to \boldsymbol{\mathcal{K}}$ for $\mathbf{U}$ with

$$\boldsymbol{\delta}(\gamma)(x) = \delta(\gamma(x)),$$

and

$$\mathbf{U}_{\gamma_1}\boldsymbol{\delta}(\gamma_2) = \boldsymbol{\delta}(\gamma_1\gamma_2) - \boldsymbol{\delta}(\gamma_1)$$

for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $x \in X$.

As in (15) we define the symmetric Fock space $\operatorname{Exp}\mathcal{K}$ with inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and set, for every $\gamma, \gamma' \in \Gamma$,

$$\mathbf{V}_\gamma \operatorname{Exp}\boldsymbol{\delta}(\gamma') = \exp\left(\mathbf{L}(\gamma, \gamma') - \tfrac{1}{2}\|\boldsymbol{\delta}(\gamma)\|^2\right) \operatorname{Exp}\boldsymbol{\delta}(\gamma'), \tag{20}$$

where

$$\mathbf{L}(\gamma, \gamma') = [\boldsymbol{\delta}(\gamma'), \boldsymbol{\delta}(\gamma^{-1})].$$

We denote by $\mathcal{H} \subset \operatorname{Exp}\mathcal{K}$ the closed linear span of the set $\{\operatorname{Exp}\boldsymbol{\delta}(\gamma) \mid \gamma \in \Gamma\}$. As in (12), (14) and (16) we observe that the maps $(\gamma_1, \gamma_2) \mapsto \mathbf{L}(\gamma_1, \gamma_2) \in \mathbb{C}$ and $(\gamma_1, \gamma_2) \mapsto \boldsymbol{\sigma}(\gamma_1, \gamma_2) \in \mathbb{S}$ are 2-cocycles. The maps $\mathbf{V}_\gamma$ in (20) extend by linearity to unitary operators on $\mathcal{H}$, and

$$\boldsymbol{\sigma}(\gamma, \gamma')\mathbf{V}_{\gamma\gamma'} = \mathbf{V}_\gamma \mathbf{V}_{\gamma'},$$

for all $\gamma, \gamma' \in \Gamma$, where

$$\boldsymbol{\sigma}(\gamma, \gamma') = \exp(i \cdot \Im(\mathbf{L}(\gamma, \gamma')).$$

As in (18) we note that the vector $\operatorname{Exp}(1_\Gamma)$ is cyclic for the projective representation $\mathbf{V}$ on $\mathcal{H}$.

In order to explain the factorisability property of the cyclic projective representation $\mathbf{V}$ of Γ on $\mathcal{H}$ we set, for any nonempty, open subset $\mathcal{O} \subset X$,

$$\Gamma_{\mathcal{O}} = \{\gamma \in \Gamma \mid \gamma(x) = 1_G \text{ for every } x \in X \setminus \mathcal{O}\}$$

and denote by $\mathcal{H}_{\mathcal{O}}$ the closed linear span of $\{\operatorname{Exp}\boldsymbol{\delta}(\gamma) \mid \gamma \in \Gamma_{\mathcal{O}}\} \subset \mathcal{H}$. If $\mathcal{O}_1, \mathcal{O}_2$ are two disjoint nonempty open subsets of X, then

$$\Gamma_{\mathcal{O}_1 \cup \mathcal{O}_2} \cong \Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2},$$
$$\mathcal{H}_{\mathcal{O}_1 \cup \mathcal{O}_2} \cong \mathcal{H}_{\mathcal{O}_1} \otimes \mathcal{H}_{\mathcal{O}_2},$$

and the restriction to $\Gamma_{\mathcal{O}_1 \cup \mathcal{O}_2}$ of $\mathbf{V}$ is unitarily equivalent to the representation $(\gamma_1, \gamma_2) \mapsto \mathbf{V}_{\gamma_1} \otimes \mathbf{V}_{\gamma_2}$ of $\Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2}$. For further results and details concerning factorisable representations of current groups arising from cocycles of unitary representations we refer to the papers [20], [21] and [6].

The factorisability of $\mathbf{V}$ means that the operator algebras generated by the unitary operators $\{\mathbf{V}_\gamma \mid \gamma \in \Gamma_{\mathcal{O}_i}\}$, $i = 1, 2$, are *independent*. For physical considerations this local independence condition should really be weakened to hold only if the closures of the sets $\mathcal{O}_i$, $i = 1, 2$, are disjoint. In [18] and [15], KRP and I made attempts in this direction, where we considered the group $\Gamma = C_C^\infty(\mathbb{R}, G)$ of all C^∞-maps with compact supports from $\mathbb{R}$ to a connected and simply connected Lie group G. By viewing the elements of Γ as smooth maps from $\mathbb{R}$ into a semidirect product $G^{(n)}$ of G with the exponential group of a Lie algebra involving the first n derivatives of the maps $\gamma : \mathbb{R} \to G$ (the n-th *Leibnitz extension* of the Lie algebra of G) and constructing factorisable representations of this latter group along the lines of [13] we obtained representations of current groups with slightly more restricted factorisability properties.

After 1975 KRP and I still collaborated on a couple of occasions (cf. [14], [16]), but by that time our research interests had begun to diverge, and KRP's focus had moved on to *quantum stochastic calculus*, *quantum probability* and related problems. However, my memories of KRP's enthusiasm for mathematics, his intellectual generosity, and his – and Shyama's – personal kindness, remain undimmed. Some years ago I was mistakenly listed as one of KRP's former PhD students in the *Mathematics Genealogy Project*; although this is factually incorrect, there may nevertheless be a grain of truth in this mis-attribution.

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