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ORIGINAL RESEARCH

Mackey imprimitivity and commuting tuples of homogeneous normal operators

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Abstract In this semi-expository article, we investigate the relationship between the imprimitivity introduced by Mackey several decades ago and commuting d- tuples of homogeneous normal operators. The Hahn–Hellinger theorem gives a canonical decomposition of a $*$- algebra representation ρ of $C_0(\mathbb{S})$ (where $\mathbb{S}$ is a locally compact Hausdorff space) into a direct sum. If there is a group G acting transitively on $\mathbb{S}$ and is adapted to the $*$-representation ρ via a unitary representation U of the group G, in other words, if there is an imprimitivity, then the Hahn–Hellinger decomposition reduces to just one component, and the group representation U becomes an induced representation, which is Mackey's imprimitivity theorem. We consider the case where a compact topological space $S \subset \mathbb{C}^d$ decomposes into finitely many G- orbits. In such cases, the imprimitivity based on S admits a decomposition as a direct sum of imprimitivities based on these orbits. This decomposition leads to a correspondence with homogeneous normal tuples whose joint spectrum is precisely the closure of G- orbits.

Keywords imprimitivity · induced representation · homogeneous operator

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1 Introduction

Let G be a locally compact second countable group and $\mathbb{S}$ be a locally compact Hausdorff transitive G- space. Given a $*$- homomorphism ρ from $C_0(\mathbb{S})$, the algebra of continuous functions vanishing at ∞ on $\mathbb{S}$ to $\mathcal{L}(\mathcal{H})$, the algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$ and a unitary representation U of

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the group G on the same Hilbert space $\mathcal{H}$, the imprimitivity is the relationship

$$U(g)\rho(f)U(g)^* = \rho(g \cdot f), \quad g \in G, \ f \in C_0(\mathbb{S}),$$

where $g \cdot f$ is the function: $(g \cdot f)(s) = f(g^{-1} \cdot s)$, $s \in \mathbb{S}$.

In the transitive case, $\mathbb{S}$ can be taken to be the space of cosets G/H for some closed subgroup H of G. There is a unique quasi-invariant measure (modulo equivalance) μ on G/H, that is, setting $g_*\mu(A) := \mu(g(A))$ for any Borel subset $A \subset \mathbb{S}$, we require that $g_*\mu$ be mutually absolutely continuous with respect to μ for every $g \in G$.

A commuting d- tuple $\boldsymbol{N} = (N_1, \ldots, N_d)$ of normal operators acting on a complex separable Hilbert space $\mathcal{H}$ is said to be homogeneous with respect to a group G if the joint spectrum $\sigma(\boldsymbol{N}) \subset \mathbb{C}^d$ is a G- space and there is a unitary representation U of G on $\mathcal{H}$ such that

$$\begin{aligned} U(g)^*\boldsymbol{N}U(g) &:= (U(g)^*N_1U(g), \ldots, U(g)^*N_dU(g)) \\ &= (g_1(\boldsymbol{N}), \ldots, g_d(\boldsymbol{N})) := g(\boldsymbol{N}), \end{aligned}$$

where g_j, $1 \leqslant j \leqslant d$, are the coordinate functions of the action of G on $\sigma_{\boldsymbol{N}}$, namely,

$$g \cdot s := (g_1(s), \ldots, g_d(s)).$$

We explore the relationship of imprimitivities with commuting d- tuples of homogeneous normal operators. For this, we assume throughout that the G- space $\mathbb{S}$ is a subset of $\mathbb{C}^d$. Somewhat surprisingly, such a relationship was hinted in [2].

The imprimitivity theorem of Mackey has two parts. The first is that any transitive imprimitivity $(\mathbb{S}, U, \rho)$ is equivalent to a canonical imprimitivity, where $\rho(f)$ for $f \in C_0(\mathbb{S})$ is defined to be the operator M_f of multiplication by f on $L^2(\mathbb{S}, \mu, \mathcal{H}_n)$ and U is a multiplier representation on $L^2(\mathbb{S}, \mu, \mathcal{H}_n)$, that is,

$$\big(U(g)h\big)(s) = c(g, s)(g \cdot h)(s), \quad h \in L^2(\mathbb{S}, \mu, \mathcal{H}_n), \ g \in G,$$

where $c : G \times \mathbb{S} \to \mathcal{U}(\mathcal{H}_n)$ is a Borel map taking values in the group of unitary operators $\mathcal{U}(\mathcal{H}_n)$ of the Hilbert space $\mathcal{H}_n$ of dimension n. For U to be a homomorphism, the function c must be a cocycle. The second part of the imprimitivity theorem asserts that such a multiplier representation is induced from a unitary representation of the subgroup H acting on the Hilbert space $\mathcal{H}_n$.

It is evident that the d- tuple of multiplication by coordinate functions $(M_1, \ldots, M_d)$ acting on $L^2(\mathbb{S}, \mu, \mathcal{H}_n)$ is homogeneous. We prove that any d- tuple $\boldsymbol{N}$ of commuting homogeneous normal operators such that $\sigma(\boldsymbol{N})$ is a G- space is actually equivalent to a direct sum of several transitive imprimitivities that are taken to be of the canonical form without loss of generality.

Since in the simplest of examples like that of the closed unit disc, the group of Möbius transformations acts holomorphically but the action is not transitive, we set out to study some class of imprimitivities that do not come from a transitive action. (This example was first mentioned in [3].) We prove that, under mild hypotheses, these are direct sums of transitive imprimitivities. For a precise statement, see Corollary 4.10. We then apply this result to several examples and list all the homogeneous d- tuples of normal operators modulo unitary equivalence.

In the Remark on page 225 of [17], it is stated that "Theorem 6.12 gives a complete analysis of the transitive homogeneous systems of imprimitivity based on X. …In view of this it has not been possible to carry out the analysis of ergodic systems of imprimitivity further than that of Theorem 6.11." In this paper, we identify a class of imprimitivities where neither the G action is transitive nor is the spectral measure of uniform multiplicity and yet the imprimitivities are described completely modulo unitary equivalence, see Theorem 4.9 and Corollary 4.10. We discuss several examples where the theorem applies.

In the concluding section, we state two problems that we think might be of some interest to researchers in operator theory as well as representation theory.

2 Spectral theorem and the Hahn–Hellinger decomposition

We begin by describing $*$- representations of a commutative C^*- algebra. In the case of a non-unital C^*- algebra, a $*$- representation $\rho : C_0(\mathbb{S}) \to \mathcal{L}(\mathcal{H})$ is said to be *non-degenerate* if there is no nonzero $h \in \mathcal{H}$ such that $\rho(f)h = 0$ for all $f \in C_0(\mathbb{S})$.

Definition 2.1 Let $\mathcal{B}$ be the σ- algebra of all Borel sets in $\mathbb{S}$ and $\mathcal{P}(\mathcal{H})$ be the set of projections on a separable complex Hilbert space $\mathcal{H}$. A spectral measure defined on $\mathbb{S}$ is a projection valued map $P : \mathcal{B} \to \mathcal{P}(\mathcal{H})$ such that $P(\mathbb{S}) = I$ and $P(\cup E_k) = \sum_{k=1}^{\infty} P(E_k)$ for any disjoint collection of sets E_k, $k = 1, 2, \ldots$, in $\mathcal{B}$, where the convergence is in the strong operator topology. The spectrum $\sigma(P)$ of a spectral measure is the complement in $\mathbb{S}$ of the union of all those open subsets E of $\mathbb{S}$ such that $P(E) = 0$. A spectral measure is said to be compact if $\sigma(P)$ is compact. If P is a spectral measure for $(\mathbb{S}, \mathcal{B})$ and $x, y \in \mathcal{H}$, then

$$P_{x,y}(E) \equiv \langle P(E)x, y\rangle, \quad x, y \in \mathcal{H}, \quad E \in \mathcal{B},$$

defines a countably additive measure on $\mathbb{S}$. A projection-valued measure P on $(\mathbb{S}, \mathcal{B})$ is called regular if each of the measures $P_{x,y}$ is regular.

Theorem 2.2 (Corollary 1.55, [8]) *Suppose that $\mathbb{S}$ is a locally compact Hausdorff space, and ρ is a nondegenerate $*$- representation of $C_0(\mathbb{S})$ on $\mathcal{H}$. Then there is a unique regular projection-valued measure P on $\mathbb{S}$ such that $\rho(f) = \int f\,dP$ for all $f \in C_0(\mathbb{S})$.*

If S is compact, then the $*$- representation ρ of $C(S)$ extends to the algebra $B(S)$ of bounded Borel functions on S. Here and in what follows, if $\mathbb{S}$ is compact then we denote it by S.

Theorem 2.3 (Chapter IX, Theorem 1.14, [6]) *If S is a compact topological space and $\rho : C(S) \to \mathcal{L}(\mathcal{H})$ is a $*$- representation, there is a unique spectral measure P defined on the Borel subsets of S such that for all x and y in $\mathcal{H}$, $P_{x,y}$ is a regular measure and $\rho(f) = \int f\,dP$ for all $f \in C(S)$.*

We reproduce a key idea of the proof of Theorem 2.3 above from [6]. If $x, y \in \mathcal{H}$, then $\Lambda_{x,y}(f) := \langle \rho(f)x, y\rangle$ is a linear functional on $C(S)$ with $\|\Lambda_{x,y}\| \leqslant \|x\|\|y\|$. Hence there is a unique regular Borel measure $\mu_{x,y}$ such that

$$\langle \rho(f)x, y\rangle = \int f d\mu_{x,y}$$

for all f in $C(S)$. Let ϕ be a bounded Borel function on S and define $[x, y] = \int \phi d\mu_{x,y}$. Evidently, $[\cdot, \cdot]$ is a sesquilinear form and $|[x, y]| \leqslant \|\phi\|\|x\|\|y\|$. Hence there is a unique bounded operator A_ϕ such that $[x, y] = \langle A_\phi x, y\rangle$ and $\|A_\phi\| \leqslant \|\phi\|$. Define $\tilde{\rho} : B(S) \to \mathcal{L}(\mathcal{H})$ by setting $\tilde{\rho}(\phi) = A_\phi$. Thus,

$$\langle \tilde{\rho}(\phi)x, y\rangle = \int \phi d\mu_{x,y}.$$

It is shown in the proof of Theorem 1.14 of [6] that (a) $\tilde{\rho} : B(S) \to \mathcal{L}(\mathcal{H})$ is a $*$- representation and $\tilde{\rho}_{|\,C(S)} = \rho$ and (b) $P(E) := \tilde{\rho}\,(\chi_E)$, $E \in \mathcal{B}$, is a spectral measure, where χ_E is the characteristic function of E.

Remark 2.4 Any $*$- representation ρ of $C(S)$, S compact, not only (uniquely) defines a spectral measure, and it is actually the restriction of a $*$- representation of the algebra of bounded Borel functions on S.

Finally, using the Hahn–Hellinger theorem, one may write down any $*$- representation of $C_0(\mathbb{S})$ in a canonical form, see [16, Theorem 3.7]. Let μ be a σ-finite Borel measure on $\mathbb{S}$, and $\mathcal{H}_n$ denote an n dimensional Hilbert space, $1 \leqslant n \leqslant \aleph_0$. The map π_μ^n defines a $*$- representation of $C(\mathbb{S})$ by setting

$$(\pi_\mu^n(f)h)(x) = f(x)h(x), \quad f \in C(\mathbb{S}),\ h \in L^2\left(\mathbb{S}, \mu, \mathcal{H}_n\right).$$

Theorem 2.5 *Let $\pi : C_0(\mathbb{S}) \to \mathcal{L}(\mathcal{H})$ be a $*$- representation of $C_0(\mathbb{S})$ on a separable Hilbert space $\mathcal{H}$.*

(a) *Then, there exists a Borel measure μ on $\mathbb{S}$, and a sequence $\{E_n : 1 \leq n \leq \aleph_0\}$ of pairwise disjoint Borel sets in S, and Hilbert spaces $\mathcal{H}_n$, $1 \leqslant n \leqslant \aleph_0$, with $\dim \mathcal{H}_n = n$ such that $\mu\left(\mathbb{S} \setminus (\cup_n E_n)\right) = 0$, and*

$$\pi \cong \bigoplus_{1 \leq n \leq \aleph_0} \pi^n_{\mu|_{E_n}}.$$

(b) *Further, if π' is another representation of $C_0(\mathbb{S})$, and if μ' is a measure and if $\left\{E'_n : 1 \leq n \leq \aleph_0\right\}$ is a sequence of pairwise disjoint Borel sets such that*

$$\pi' \cong \bigoplus_{1 \leq n \leq \aleph_0} \pi^n_{\mu'|_{E'_n}},$$

and if $\pi \cong \pi'$, then $\mu \cong \mu'$ and $\mu\left(E_n \triangle E'_n\right) = 0$ for all n, where $\triangle$ denotes "symmetric difference": $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$.

We need to discuss the relationship of the spectrum of the spectral measure P defined on a locally compact space $\mathbb{S}$ with the (joint) spectrum of $\boldsymbol{N}$ (commuting tuple of normal operators) defined by P as in Theorem 2.3. However, there are several different ways in which one may define the spectrum of $\boldsymbol{N}$, and these are discussed below.

Let $\boldsymbol{N} = (N_1, N_2, \cdots, N_d)$ be a commuting tuple of normal operators acting on a Hilbert space $\mathcal{H}$. Some of what we are going to say applies to any commuting tuple of bounded operators but we will restrict ourselves to normal operators. In this case, some of the arguments are much simpler.

First, the *left spectrum* $\sigma_{\text{left}}(\boldsymbol{N})$ is defined to be those $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ such that the left ideal generated by $\{(N_1 - \lambda_1), (N_2 - \lambda_2), \ldots, (N_d - \lambda_d)\}$ in the unital (commutative) C^*- algebra $C^*(\boldsymbol{N})$ generated by $\boldsymbol{N}$ is proper. The right spectrum $\sigma_{\text{right}}(\boldsymbol{N})$ is defined similarly.

From the Gelfand theory, see [15, Theorem B, p. 320], it follows that $C^*(\boldsymbol{N})$ is $*$- isomorphic to the algebra of continuous functions $C(\mathcal{M})$ on the space $\mathcal{M}$ of multiplicative linear functionals of $C^*(\boldsymbol{N})$. Then we have

$$\sigma_{\text{left}}(\boldsymbol{N}) = \sigma_{\text{right}}(\boldsymbol{N}) = \{(\ell(N_1), \ldots, \ell(N_d)) : \ell \in \mathcal{M}\} \subset \mathbb{C}^d, \tag{2.1}$$

see [10, Lemma 2.1] and also the remark following [4, Corollary 4].

Remark 2.6 It may appear that the definition of the spectrum, say, the left spectrum, depends on the algebra $C^*(\boldsymbol{N})$ since $(\lambda_1, \ldots, \lambda_d) \notin \sigma_{\text{left}}(\boldsymbol{N})$, only if we can find $S_1, \ldots, S_d \in C^*(\boldsymbol{N})$ such that $\sum_{j=1}^d S_j(N_j - \lambda_j) = I$. However, if $C^*(\boldsymbol{N}) \subset \mathcal{A}$ such that the identity of $\mathcal{A}$ also serves as the identity in $C^*(\boldsymbol{N})$, then the spectrum $\sigma_{\text{left}}(\boldsymbol{N}) = \sigma_{\mathcal{A}}(\boldsymbol{N})$, see [7, Corollary 8]. A particular case that is important in what follows occurs by choosing $\mathcal{A} = C^*(\boldsymbol{N})'$. Since we have $\sigma_{\text{left}}(\boldsymbol{N}) = \sigma_{\text{right}}(\boldsymbol{N})$, we drop the subscripts "left" and "right" and let $\sigma(\boldsymbol{N})$ denote the spectrum of $\boldsymbol{N}$. This is known as the Harte spectrum.

A second notion of the spectrum is the *joint approximate spectrum* $\sigma_a(\boldsymbol{N})$: The d- tuple $(\lambda_1, \cdots, \lambda_d)$ of complex numbers is said to be in $\sigma_a(\boldsymbol{N})$ if there exists a sequence x_n of unit vectors in $\mathcal{H}$ such that $\left(N_j - \lambda_j\right) x_n \to 0$, $1 \leqslant j \leqslant d$. The proof of the following lemma is in [5]. We provide a proof for the sake of completeness.

Lemma 2.7 *If $\boldsymbol{N}$ is a d- tuple of commuting normal operators, then $\sigma_{\text{left}}(\boldsymbol{N}) = \sigma_a(\boldsymbol{N})$.*

Proof Let $\mathcal{I} = \left\{\sum_{j=1}^d S_j\left(N_j - \lambda_j\right) : S_j \in \mathcal{L}(\mathcal{H})\right\}$.

To prove that $\sigma_a(\boldsymbol{N}) \subseteq \sigma(\boldsymbol{N})$, consider $(\lambda_1, \ldots, \lambda_d) \in \sigma_a(\boldsymbol{N})$, i.e., $\left(N_j - \lambda_j\right) x_n \to 0$, for every j with $\|x_n\| = 1$. If $\mathcal{I}$ is not proper, then $I \in \mathcal{I}$. Hence, there exist $S_j \in \mathcal{L}(\mathcal{H})$, $j = 1, 2, \cdots, d$, such that $\sum_{j=1}^d S_j\left(N_j - \lambda_j\right) = I$, i.e., $\sum_{j=1}^d S_j\left(N_j - \lambda_j\right) x_n = x_n$. This contradicts the fact that $\left(N_j - \lambda_j\right) x_n \to 0$ since $\|x_n\| = 1$. Hence $\mathcal{I}$ is proper and $(\lambda_1, \ldots, \lambda_d) \in \sigma(\boldsymbol{N})$.

To prove that $\sigma(\boldsymbol{N}) \subseteq \sigma_a(\boldsymbol{N})$, consider $(\lambda_1, \ldots, \lambda_d) \in \sigma(\boldsymbol{N})$. Then $\mathcal{I}$ is proper and, therefore, consists of non-invertible elements. Hence $\sum_{j=1}^d \left(N_j - \lambda_j\right)^* \left(N_j - \lambda_j\right) \in \mathcal{I}$ is non-invertible, that is, 0 belongs to the spectrum of the operator $\sum_{j=1}^d \left(N_j - \lambda_j\right)^* \left(N_j - \lambda_j\right)$. Since the operator is self-adjoint, all its spectral values are approximate eigenvalues. Hence, there exists $\{x_n \in \mathcal{H} : \|x_n\| = 1, n \in \mathbb{N}\}$ such that $\sum_{j=1}^d \left(N_j - \lambda_j\right)^* \left(N_j - \lambda_j\right) x_n \to 0$, $n \to \infty$. It follows that $\sum_{j=1}^d \left\|\left(N_j - \lambda_j\right) x_n\right\|^2 \to 0$ and $\left(N_j - \lambda_j\right) x_n \to 0$, $n \to \infty$, for every j. Hence $(\lambda_1, \ldots, \lambda_d)$ is in $\sigma_a(\boldsymbol{N})$. □

Recall that the spectral measure P defines a commuting d- tuple of normal operators as described below. This is the spectral theorem for a d- tuple of commuting operators.

Theorem 2.8 (Theorem 3, [11]) *Let $\boldsymbol{N} = (N_1, \ldots, N_d)$ be a d- tuple of pairwise commuting normal operators acting on a Hilbert space $\mathcal{H}$ and let $W^*(\boldsymbol{N}) := \{N_1, \ldots, N_d\}''$ be the commuting von Neumann algebra consisting of all those operators that doubly commute with $\{N_1, \ldots, N_d\}$. Then there exists a projection-valued spectral measure P with* $\text{supp}(P) = \sigma(\boldsymbol{N})$ *such that*

$$N_j = \int z_j \, dP(z), \quad 1 \leq j \leq d,$$

where $\sigma(\boldsymbol{N})$ is computed relative to $W^*(\boldsymbol{N})$.

From Remark 2.6, we conclude that $\sigma(\boldsymbol{N})$ is unambiguously defined irrespective of whether we define it relative to the algebra $W^*(\boldsymbol{N})$ or the algebra $C^*(\boldsymbol{N})$. The claim that $\text{supp}(P) = \sigma(\boldsymbol{N})$ follows from [5, Theorem 2] as pointed out in [11]. We verify this equality closely following the proof of Theorem 2 from [9, p. 64].

Lemma 2.9 *Assume that P is a compact spectral measure on a locally compact space $\mathbb{S} \subset \mathbb{C}^d$. Then the support of the spectral measure P and the spectrum of the commuting tuple $\boldsymbol{N}$ of operators, where $N_j = \int z_j dP(z)$, $1 \leq j \leq d$, are equal.*

Proof From Lemma 2.7, we have that $\sigma(\boldsymbol{N}) = \sigma_a(\boldsymbol{N})$. Thus, it is enough to prove $\text{supp}(P) = \sigma_a(\boldsymbol{N})$.

Pick $\boldsymbol{\lambda}^0 \notin \text{supp}(P)$. Then there exists an open neighborhood U of $\boldsymbol{\lambda}^0$ such that $P(U) = 0$. Let U' be the complement of U and $\delta = \text{dist}(\boldsymbol{\lambda}^0, U')$. Now, we have

$$\begin{aligned}\sum_{j=1}^{d} \left\|(N_j - \lambda_j^0 I)x\right\|^2 &= \sum_{j=1}^{d} \langle (N_j - \lambda_j^0)^*(N_j - \lambda_j^0)x, x\rangle \\ &= \sum_{j=1}^{d} \int \left(\overline{(\lambda_j - \lambda_j^0)}(\lambda_j - \lambda_j^0)\right) d\langle P(\boldsymbol{\lambda})x, x\rangle\end{aligned}$$

for all $x \in \mathcal{H}$. Since $P(U) = 0$, it follows that

$$\begin{aligned}\Big(\sum_{j=1}^{d} \left\|(N_j - \lambda_j^0 I)x\right\|\Big)^2 &\geqslant \sum_{j=1}^{d} \left\|(N_j - \lambda_j^0 I)x\right\|^2 \\ &= \int_{U'} \sum_{j=1}^{d} |\lambda_j - \lambda_j^0|^2 d\langle P(\boldsymbol{\lambda})x, x\rangle \\ &\geqslant \delta^2 \|x\|^2\end{aligned}$$

for all $x \in \mathcal{H}$. Consequently, $\sum_{j=1}^{d} \left\|\left(N_j - \lambda_j^0\right)x\right\| \geqslant \delta\|x\|$ and therefore $\boldsymbol{\lambda}^0 \notin \sigma_a(\boldsymbol{N})$.

Conversely, if $\boldsymbol{\lambda}^0 \in \text{supp}(P)$, then $P(U) \neq 0$ for every open set containing $\boldsymbol{\lambda}^0$. Hence if δ is any positive number such that $B = \{\boldsymbol{\lambda} : |\boldsymbol{\lambda} - \boldsymbol{\lambda}^0| < \delta\}$, then there is a non-zero vector x in the range of $P(B)$. Now, arguing as in the first half of the proof, we conclude that

$$\Big(\sum_{j=1}^{d} \left\|(N_j - \lambda_j^0 I)x\right\|\Big)^2 = \int_B \sum_{j=1}^{d} |\lambda_j - \lambda_j^0|^2 d\langle P(\boldsymbol{\lambda})x, x\rangle \leqslant \delta^2 \|x\|^2.$$

It follows that $\boldsymbol{\lambda}^0 \in \sigma_a(\boldsymbol{N})$. □

Adapting the terminology from [9], let us say that a spectral measure defined on a subset $\mathbb{S}$ of $\mathbb{C}^d$ is a *complex* spectral measure. It is proved in [9, Theorem 1, p. 63] that a complex spectral measure is regular for $d = 1$. The same proof works with the obvious modifications for any $d \in \mathbb{N}$. We record below this property of a complex spectral measure without proof.

Proposition 2.10 *Every complex spectral measure is regular.*

3 Multiplier representations and Induced representations

This section is devoted to establishing a correspondence between multiplier representations and induced representations. We start with some generalities.

Let G be a locally compact second countable group and $\mathbb{S}$ be a locally compact G- space, that is, there is a map $\alpha : G \times \mathbb{S} \to \mathbb{S}$, such that for a fixed $g \in G$, $s \to \alpha_g(s)$, $\alpha_g(s) := \alpha(g, s)$, is a bijective and continuous map of $\mathbb{S}$, moreover, $g \to \alpha_g$ is a homomorphism. We let $g \cdot s := \alpha(g, s)$, $g \in G$ and $s \in \mathbb{S}$. The action of G on $\mathbb{S}$ is said to be *transitive* if for every pair s_1, s_2 in $\mathbb{S}$, there is a $g \in G$ such that $g \cdot s_1 = s_2$. Let $H \subseteq G$ be a closed subgroup and let $\mathbb{S} := G/H$ be the space of cosets: $\{gH \mid g \in G\}$. Equipped with the action of G by left multiplication: $g'(gH) := (g'g)H$, $g', g \in G$, the coset space $\mathbb{S}$ is a transitive G- space. On the other hand, any transitive G-space must be of this form, see [8, Proposition 2.46]. Following [8], let us say that $\mathbb{S}$ is a *homogeneous* space if

it is homeomorphic to G/H. In this case, we identify $\mathbb{S}$ with G/H and define $(g \cdot f)(s) = f(g^{-1} \cdot s)$ for any function defined on $\mathbb{S}$.

Let $(\mathbb{S}, \mathcal{B})$ be the Borel measurable space, and note that each $g \in G$ defines a continuous map on $\mathbb{S}$ by our assumption. Given a σ-finite measure μ on $\mathbb{S}$, define the *push-forward* $g_*\mu$ of the measure μ by the requirement

$$(g_*\mu)(A) := \mu\big(g \cdot A\big), \quad g \cdot A := \{g^{-1} \cdot s \mid s \in A\}, \quad A \in \mathcal{B}.$$

The measure μ on G is said to be invariant if $g_*\mu = \mu$ and *quasi-invariant* if $g_*\mu$ is equivalent (mutually absolutely continuous) to μ for all $g \in G$.

Remark 3.1 If G is second countable, then we have the following two very useful tools at our disposal.

(1) There is a Borel cross-section $p : G/H \to G$, that is, a Borel subset $B \subset G$ that meets each coset of H in exactly one point, see [1, Corollary, Theorem 3.41], and [14, Lemma 1.1]. Thus, each $g \in G$ can be written uniquely as $g = g_1 g_0$ with $g_0 \in H$ and $g_1 \in B$.
(2) There is a quasi-invariant measure uniquely determined modulo mutual absolute equivalence on $\mathbb{S}$, see [8, Theorem 2.58].

The inducing construction is a way of producing unitary representations of a locally compact group G from unitary representations of a closed subgroup H of G. In this section we describe two pictures of *induced representations*.

(i) The first is that of a multiplier representation, which is more suitable for function theory/operator theory applications.
(ii) The second one is the standard representation theoretic construction of a G- action on a space of vector-valued functions on G that transform according to the given representation of H.

3.1 Multiplier representations

Let $\mathbb{S}$ be a second countable Hausdorff space on which a Lie group G acts transitively. For $x \in \mathbb{S}$ and $g \in G$, $g \cdot x$ denotes the action of g on the point x. If $x_0 \in \mathbb{S}$, the stabilizer of x_0 is the closed subgroup H of G defined by

$$H = \{h \in G : \ h \cdot x_0 = x_0\}.$$

We fix $x_0 \in \mathbb{S}$ in what follows. Since the action is transitive, we can identify the space $\mathbb{S}$ with the homogeneous space G/H. Recall that G/H admits a quasi-invariant measure, which we denote by μ.

Example 3.2 Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : \ |z| < 1\}$ denote the Euclidean unit ball in $\mathbb{C}^n$. The group of bi-holomorphic automorphisms on $\mathbb{B}^n$, denoted by $\text{Möb}(\mathbb{B}^n)$ can be described as follows:

For each $a \in \mathbb{B}^n$, let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a. So, $P_0 = 0$ and

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \quad a \neq 0.$$

Let $Q_a = I - P_a$ be the orthogonal projection onto the orthogonal complement of a. Define

$$\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a\rangle}. \tag{3.1}$$

The bi-holomorphic map φ_a is an involution which interchanges the points 0 and a. Then, the group $\text{Möb}(\mathbb{B}^n)$ is given by

$$\text{Möb}(\mathbb{B}^n) = \{U\varphi_a : a \in \mathbb{B}^n, U \in U(n)\}.$$

The unit ball $\mathbb{B}^n$ can be identified with G/K where $G = \text{Möb}(\mathbb{B}^n)$ and $K = SU(n)$. In this case, it is well known that G/K carries a measure that is G- invariant.

Now we describe the multiplier representations. Let $m : G \times \mathbb{S} \to \mathcal{U}(V)$ be a Borel function, where $\mathcal{U}(V)$ is the space of unitary operators on a complex separable Hilbert space V.

Define

$$T_g f(x) = \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} m(g, x) f(g^{-1} \cdot x),$$

where f comes from $L^2(\mathbb{S}, \mu, V)$. We assume that T_g defines a unitary representation of G. It is easily verified that $g \to T_g$ is a homomorphism if and only if the *multiplier* m satisfies the cocycle identity

$$m(g_1 g_2, x) = m(g_1, x) m(g_2, g_1^{-1} \cdot x), \quad g_1, g_2 \in G, x \in \mathbb{S}. \tag{3.2}$$

Next, set $\sigma(g) = m(g, x_0)$. Notice that,

$$\sigma(hg) = \sigma(h)\sigma(g), \quad h \in H, g \in G. \tag{3.3}$$

In particular, σ restricted to H is a homomorphism of H into $\mathcal{U}(V)$, the group of unitary operators on V and hence σ is a unitary representation of H as m is Borel.

Next, we obtain a general form for the multiplier m using σ (this connects the representation of H and the multiplier m). We start with the definition of a section.

If $s : \mathbb{S} \to G$ is a Borel cross-section as in Remark 3.1, then every $g \in G$ admits a unique decomposition $g = s(x)h$ where $x = g \cdot x_0$ and $h \in H$. We shall need the following (Remark 3.1):

Lemma 3.3 *There exists a Borel cross-section $s : \mathbb{S} \to G$ which is measurable.*

Notice that, for $x \in \mathbb{S}$, $g \in G$,

$$\sigma(s(x)^{-1} g) = m(s(x)^{-1} g, x_0) = m(s(x)^{-1}, x_0) m(g, x) = \sigma(s(x)^{-1}) m(g, x). \tag{3.4}$$

Since,

$$s(g^{-1} \cdot x)^{-1} g^{-1} s(x) \cdot x_0 = s(g^{-1} \cdot x)^{-1} g^{-1} \cdot x = x_0,$$

we have,

$$s(g^{-1} \cdot x)^{-1} g^{-1} s(x) \in H.$$

So, $g^{-1} s(x) = s(g^{-1} \cdot x)h$ for some $h \in H$. Write, $s(x)^{-1} g = h^{-1} s(g^{-1} \cdot x)^{-1}$. By (3.3) we get

$$\sigma(s(x)^{-1} g) = \sigma(h^{-1})\sigma(s(g^{-1} \cdot x)^{-1}), \quad g^{-1} s(x) = s(g^{-1} \cdot x)h.$$

Using the above and (3.4), we get the general form for a multiplier as

$$m(g, x) = \sigma(s(x)^{-1})^{-1} \sigma(h^{-1}) \sigma(s(g^{-1} \cdot x)^{-1}), \quad g^{-1} s(x) = s(g^{-1} \cdot x)h. \tag{3.5}$$

3.2 Induced representations

Next, we describe the representation theoretic construction of an induced representation and show that multiplier representations and induced representations are the same. As above $\mathbb{S} = G/H$ and let quasi μ be the quasi G- invariant measure on $\mathbb{S}$.

Let σ be a unitary representation of H on a Hilbert space $\mathcal{H}_\sigma$. We denote the inner product on $\mathcal{H}_\sigma$ by $\langle \cdot, \cdot \rangle_\sigma$. Let $\mathcal{F}_c$ be the vector space of functions taking values in $\mathcal{H}_\sigma$ defined by

$$\mathcal{F}_c = \{F \in C(G, \mathcal{H}_\sigma) : \ F(gh) = \sigma(h^{-1}) F(g) \text{ and } q(\text{supp}(F)) \text{ is compact } \},$$

where $q : G \to G/H$ is the canonical projection. Notice that, if $F_1, F_2 \in \mathcal{F}_c$, the function $\langle F_1(g), F_2(g) \rangle_\sigma$ depends only on the coset $q(h) = gH$ (as the representation σ is unitary) and so is a function in $C_c(G/H)$.

Define an inner product on $\mathcal{F}_c$ by

$$\langle F_1, F_2 \rangle = \int_{\mathbb{S}} \langle F_1(x), F_2(x) \rangle_\sigma \ d\mu(x).$$

It is easy to verify that the above defines an inner product on $\mathcal{F}_c$. Let $\mathcal{H}$ be the Hilbert space obtained by completing $\mathcal{F}_c$ with respect to this inner product. Notice that the space $\mathcal{F}_c$ is invariant under the operators L_g, $g \in G$, defined by

$$L_g F(g_1) = \left(\frac{d(g_*\mu)}{d\mu}(g_1 \cdot x_0)\right)^{\frac{1}{2}} F(g^{-1}g_1)$$

and the inner product on $\mathcal{F}_c$ is preserved by these maps L_g, $g \in G$ (because μ is quasi G- invariant). It follows that L_g for each $g \in G$ extends to an isometry on $\mathcal{H}$. Since it is easy to see that $L_{g_1 g_2} = L_{g_1} L_{g_2}$ on $\mathcal{F}_c$, each L_g becomes a unitary operator on $\mathcal{H}$. Moreover, the map $g \to L_g F$ from G to $\mathcal{F}_c$ (for a fixed F) is continuous and since the operators L_g are uniformly bounded, they are strongly continuous. Thus we have a unitary representation of G called the induced representation, denoted by $\mathrm{Ind}_H^G(\sigma)$. We may also use L_g to denote this representation when there is no scope for confusion.

Example 3.4 Let σ be the trivial representation of H on $\mathbb{C}$. Then, $\mathcal{F}_c$ is naturally identified with $C_c(G/H) = C_c(\mathbb{S})$ and $\mathcal{F} = L^2(G/H) = L^2(\mathbb{S}, \mu)$. If G/H admits a G- invariant measure μ, the representation $\mathrm{Ind}_H^G(\sigma)$ is the left regular representation of G on $L^2(G/H, \mu)$.

Remark 3.5 The elements in the space $\mathcal{H}$ are in fact functions that take values in $\mathcal{H}_\sigma$, see *Remark 1* in [8, Chapter 6].

For $F \in \mathcal{H}$, define the $\mathcal{H}_\sigma$ valued function $\widetilde{F}$ on $\mathbb{S}$ by $\widetilde{F}(x) = F(s(x))$, where $s : \mathbb{S} \to G$ is the measurable section chosen earlier. Then, it can be verified that the map $F \to \widetilde{F}$ is a unitary transformation from $\mathcal{H}$ onto the Hilbert space $\widetilde{\mathcal{H}}$, where

$$\widetilde{\mathcal{H}} = L^2(\mathbb{S}, \mu, \mathcal{H}_\sigma) = \{\widetilde{F} : \mathbb{S} \to \mathcal{H}_\sigma, \quad \int_{\mathbb{S}} \|\widetilde{F}(x)\|_\sigma^2 d\mu(x) < \infty\}.$$

Transferring the action of G on $\mathcal{H}$ to $\widetilde{\mathcal{H}}$, we obtain the representation $\widetilde{L_g}$ of G defined by

$$\begin{aligned}\widetilde{L_g}\widetilde{F}(x) &= \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} \sigma(h^{-1})\widetilde{F}(g^{-1}s(x) \cdot x_0) \\ &= \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} \sigma(h^{-1})\widetilde{F}(g^{-1} \cdot x), \end{aligned} \tag{3.6}$$

where $h \in H$ is determined by $g^{-1}s(x) = s(g^{-1} \cdot x)h$. The representation $\widetilde{L_g}$ is, of course, unitarily equivalent to $\mathrm{Ind}_H^G(\sigma)$. Recall that the representation $\mathrm{Ind}_H^G(\sigma)$ is realized (as L_g) on the Hilbert space $\mathcal{H}$.

We now show the correspondence between multiplier representations and induced representations. For this, consider the multiplier representation T_g described earlier with the cocycle m and its restriction σ. Note that σ is defined on all of G and, when restricted to H gives a unitary representation of H, see Equation (3.3).

Define the map $\mathcal{A}$ on $\widetilde{\mathcal{H}}$ by $\mathcal{A}f(x) = \sigma(s(x))^{-1} f(x)$. Clearly, $\mathcal{A}$ is an invertible transformation. Let us compute $\mathcal{A}T_g\mathcal{A}^{-1}$. For $f \in \widetilde{\mathcal{H}}$ we have,

$$\begin{aligned}\mathcal{A}T_g\mathcal{A}^{-1}f(x) &= \sigma(s(x))^{-1}\left(T_g\mathcal{A}^{-1}f\right)(x) \\ &= \sigma(s(x))^{-1}\left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} m(g,x)(\mathcal{A}^{-1}f)(g^{-1} \cdot x) \\ &= \sigma(s(x))^{-1}\left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} m(g,x)\sigma(s(g^{-1} \cdot x))f(g^{-1} \cdot x) \\ &= \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} \sigma(h^{-1})f(g^{-1} \cdot x),\end{aligned}$$

where we used (3.5) in the last step. But the above equals $\widetilde{L_g}f(x)$ (see (3.6)) and hence T_g is equivalent to $\mathrm{Ind}_H^G(\sigma)$, with the representation σ from which the induction is done being the representation obtained by restricting the cocycle m as mentioned above.

Conversely, let σ be a unitary representation of H on a Hilbert space $\mathcal{H}_\sigma$. We shall construct a multiplier m such that the corresponding multiplier representation, denoted by T^m_g is unitarily equivalent to $\mathrm{Ind}_H^G(\sigma)$. Notice that we can not define the multiplier m using the form given in (3.5) as $\sigma(s(x))$ does not make sense if $s(x) \notin H$ (which is the case if $x \neq x_0$). However, notice that $s(x)^{-1}gs(g^{-1}\cdot x) \in H$, as

$$s(x)^{-1}gs(g^{-1}\cdot x)\cdot x_0 = s(x)^{-1}g(g^{-1}\cdot x) = s(x)^{-1}\cdot x = x_0.$$

Hence we can define

$$m(g,x) = \sigma(s(x)^{-1}gs(g^{-1}\cdot x)).$$

It is easy to verify that m satisfies the cocycle identity (3.2). Define T^m_g on $\widetilde{\mathcal{H}}$ by

$$T^m_g f(x) = \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} m(g,x) f(g^{-1}\cdot x).$$

Clearly, $g \to T^m_g$ is a unitary representation of G. Now, proceeding as above by defining $\widetilde{\sigma}(g) = m(g, x_0)$ we obtain a unitary representation $\widetilde{\sigma}$ of the subgroup H and that T^m_g is unitarily equivalent to $\mathrm{Ind}_H^G(\widetilde{\sigma})$. Hence, it suffices to prove that σ and $\widetilde{\sigma}$ are unitarily equivalent as *representations of* H. Now, for $h \in H$,

$$\widetilde{\sigma}(h) = m(h,x_0) = \sigma(s(x_0)^{-1}hs(h^{-1}\cdot x_0)) = \sigma(s(x_0))^{-1}\sigma(h)\sigma(s(x_0))$$

as $s(x_0) \in H$ and σ is a representation of H, which proves that σ and $\widetilde{\sigma}$ are equivalent. So, it follows that T^m_g and $\mathrm{Ind}_H^G(\sigma)$ are unitarily equivalent. Thus we have proved the following:

Theorem 3.6 *Let G be a second countable locally compact group and H be a closed subgroup of G.*

(a) Let m be a multiplier defined on $G \times \mathbb{S}$ taking values in the group of unitary operators on a Hilbert space. Then the multiplier representation T^m_g defined by

$$T^m_g f(x) = \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{\frac{1}{2}} m(g,x) f(g^{-1}\cdot x)$$

is unitarily equivalent to $\mathrm{Ind}_H^G(\sigma)$, where the representation σ of H is given by $\sigma(h) = m(h, x_0)$.

(b) Conversely, let σ be a unitary representation of H and let

$$m(g,x) = \sigma(s(x)^{-1}gs(g^{-1}\cdot x)).$$

Then m is a multiplier and $\mathrm{Ind}_H^G(\sigma)$ is equivalent to the multiplier representation T^m_g.

A lot more on the inducing construction and the imprimitivity can be found in sections 1 and 2 of [12]. In particular, see Theorems 4 and 5 of [12].

4 Imprimitivity and homogeneous normal operators

Let $\boldsymbol{N}$ be a commuting tuple of normal operators acting on a complex separable Hilbert space $\mathcal{H}$. Let $\mathcal{C}^*(\boldsymbol{N})$ be the unital C^*- algebra generated by $(N_1, \ldots, N_d)$. This C^*- algebra is $*$- isomorphic to the algebra of continuous functions $C(S)$ for some compact Hausdorff space $S \subset \mathbb{C}^d$ via the Gelfand representation theorem. The joint spectrum $\sigma(\boldsymbol{N})$ of $(N_1, \ldots, N_d)$ is the set S. The continuous functional calculus for the commuting d- tuple $\boldsymbol{N}$ defined using the Gelfand transform provides a $*$- homomorphism $\rho_{\boldsymbol{N}} : C(\sigma(\boldsymbol{N})) \to \mathcal{L}(\mathcal{H})$ by setting $\rho_{\boldsymbol{N}}(f) = f(\boldsymbol{N})$. On the other hand, if $\rho : C(\sigma(\boldsymbol{N})) \to \mathcal{L}(\mathcal{H})$ is any $*$- homomorphism, then setting $N_j = \rho(z_j)$, where z_j are the coordinate functions, we see that $(N_1, \ldots, N_d)$ is a commuting tuple of normal operators. The $*$- homomorphism $\rho_{\boldsymbol{N}}$ induced by $\boldsymbol{N}$ coincides with ρ.

Let G be a second countable locally compact group and $\mathbb{S}$ be a locally compact topological space. We say that $\mathbb{S}$ is a G- space if there is a continuous map $G \times \mathbb{S} \to \mathbb{S}$ such that $(e, s) = s$, where e is the identity in G and for every pair $g, \hat{g}$ in G, $g \cdot (\hat{g} \cdot s) = (g\hat{g}) \cdot s$, $s \in S$. It then follows that for every fixed $g \in G$, the map $g : \mathbb{S} \to \mathbb{S}$, $g(s) := g \cdot s$, $s \in \mathbb{S}$, is a continuous bijection of $\mathbb{S}$. Thus, $g = (g_1, \ldots, g_d)$ with $g_j : \mathbb{S} \to \mathbb{C}$, $1 \leqslant j \leqslant d$. These are the *coordinate functions* of the action $g : \mathbb{S} \to \mathbb{C}^d$ induced by $g \in G$.

In what follows, we assume that $\sigma(\boldsymbol{N})$ is a G- space and that the action of $g \in G$ is holomorphic in some open neighbourhood $\mathcal{O}_g$ of $\sigma(\boldsymbol{N})$. This action of G on $\sigma(\boldsymbol{N})$ lifts to the commuting tuple $\boldsymbol{N}$ by setting

$$g \cdot \boldsymbol{N} := (g_1 \cdot \boldsymbol{N}, \ldots, g_d \cdot \boldsymbol{N}). \tag{4.1}$$

Definition 4.1 Let $\boldsymbol{N} = (N_1, \ldots, N_d)$ be a commuting tuple of normal operators such that the spectrum $\sigma(\boldsymbol{N})$ is a G- invariant compact set. The commuting d- tuple $\boldsymbol{N}$ is said to be *homogeneous* if there is a unitary representation U of G on $\mathcal{H}$ such that

$$U_g^* \boldsymbol{N} U_g := (U_g^* N_1 U_g, \ldots, U_g^* N_d U_g) = g \cdot \boldsymbol{N}, \quad g \in G. \tag{4.2}$$

To discuss induced representations for a locally compact second countable group G, Mackey introduced the notion of an imprimitivity which we recall below. For any locally compact Hausdorff space $\mathbb{S}$, we let $C_0(\mathbb{S})$ denote the C^*- algebra of continuous functions vanishing at ∞, that is, if $f \in C_0(\mathbb{S})$, and ϵ is any positive number, then there is a compact subset K of $\mathbb{S}$ such that $|f(s)| < \epsilon$ for all $s \notin K$. The action of the group G lifts to an action on $C_0(\mathbb{S})$: $(g, f) \to g \cdot f$, where $(g \cdot f)(s) = (f \circ g^{-1})(s)$, $g \in G$, $s \in \mathbb{S}$.

Definition 4.2 Suppose that U is a unitary representation of the group G on some Hilbert space $\mathcal{H}$ and $\rho : C_0(\mathbb{S}) \to \mathcal{L}(\mathcal{H})$ is a $*$- homomorphism of the C^*- algebra of continuous functions on $\mathbb{S}$ vanishing at infinity. The *imprimitivity* introduced by Mackey is the requirement

$$U_g \rho(f) U_g^* = \rho(g \cdot f), \quad f \in C_0(\mathbb{S}), \tag{$\dagger$}$$

where $(g \cdot f)(s) = f(g^{-1} \cdot s)$, $s \in \mathbb{S}$.

If (S, U, ρ) is an imprimitivity for some compact set S, then the d- tuple $(\rho(z_1), \ldots, \rho(z_d))$ of commuting normal operators is *homogeneous* by definition, see ($\dagger$) above, with $\sigma(\rho(z_1), \ldots, \rho(z_d)) = S$. The other way round, the theorem below shows that if $\boldsymbol{N}$ is a d- tuple of *homogeneous* normal operators with associated representation U, then $(\sigma(\boldsymbol{N}), U, \rho_{\boldsymbol{N}})$ is an imprimitivity.

Theorem 4.3 *Let $\boldsymbol{N} := (N_1, \ldots, N_d)$ be a d- tuple of commuting normal operators defined on a complex separable Hilbert space $\mathcal{H}$. Assume that $\boldsymbol{N}$ is homogeneous under the action of a group G with associated representation U. Then $(\sigma(\boldsymbol{N}), U, \rho_{\boldsymbol{N}})$ is an imprimitivity.*

Proof. By hypothesis, we have $U_g^* \boldsymbol{N} U_g = g \cdot \boldsymbol{N}$, $g \in G$, for some unitary representation U of G. Taking the adjoint of both sides, we get $U_g^* \boldsymbol{N}^* U_g = (g \cdot \boldsymbol{N})^*$, where $\boldsymbol{N}^* = (N_1^*, \ldots, N_d^*)$ and similarly, $(g \cdot \boldsymbol{N})^* = ((g_1 \cdot \boldsymbol{N})^*, \ldots, (g_d \cdot \boldsymbol{N})^*)$. Or, equivalently,

$$(g \cdot \boldsymbol{N})^* = (\overline{g_1} \cdot \boldsymbol{N}^*, \ldots, \overline{g_d} \cdot \boldsymbol{N}^*),$$

where $\overline{g_j}$ denotes the complex conjugate of the coordinate functions g_j, $1 \leq j \leq d$. We adopt the convention: $(g \cdot \boldsymbol{N})^* = \bar{g} \cdot \boldsymbol{N}^*$. Thus, if $\boldsymbol{N}$ is a commuting d- tuple of homogeneous normal operators, then we also have $U_g^* \boldsymbol{N}^* U_g = \bar{g} \cdot (\boldsymbol{N}^*)$, $g \in G$.

The homogeneous d- tuple $\boldsymbol{N}$ defines a $*$- homomorphism $\rho_{\boldsymbol{N}} : C(\sigma(\boldsymbol{N})) \to \mathcal{L}(\mathcal{H})$ and the homogeneity ensures that

$$\begin{aligned} U_g^*(\rho(z_1), \ldots, \rho(z_d)) U_g &:= U_g^* \rho(\boldsymbol{z}) U_g \\ &= (g_1 \cdot \boldsymbol{z}, \ldots, g_d \cdot \boldsymbol{z}) := \rho(g \cdot \boldsymbol{z}), \ \boldsymbol{z} \in \sigma(\boldsymbol{N}), \quad g \in G. \end{aligned} \tag{4.3}$$

Similarly, the equality $U_g^* \boldsymbol{N}^* U_g = \bar{g} \cdot (\boldsymbol{N}^*)$ becomes

$$U_g^* \rho(\bar{\boldsymbol{z}}) U_g = \rho(\bar{g} \cdot \bar{\boldsymbol{z}}), \quad g \in G. \tag{4.4}$$

Setting $f_m(\boldsymbol{z}) = z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$, and $f_{m,n}(\boldsymbol{z}) = f_m \overline{f_n}$, we have that

$$\begin{aligned} U_g^* \rho(f_m) U_g &= U_g^* N_1^m \cdots N_d^{m_d} U_g \\ &= \left((U_g^* N_1 U_g)^{m_1} (U_g^* N_2 U_g)^{m_2} \cdots (U_g^* N_d U_g)^{m_d} \right) \\ &= (U_g^* \rho(z_1) U_g)^{m_1} \cdots (U_g^* \rho(z_d) U_g)^{m_d} \\ &= \rho(g_1 \cdot \boldsymbol{z})^{m_1} \cdots \rho(g_d \cdot \boldsymbol{z})^{m_d} \\ &= \rho(g \cdot f_m). \end{aligned}$$

Similarly, as before,

$$\begin{aligned} U_g^*\rho(\overline{f_n})U_g &= U_g^* N_1^{*n_1}\dots N_1^{*n_d}U_g \\ &= \big((U_g^*N_1^*U_g)^{n_1}(U_g^*N_2^*U_g)^{n_2}\cdots(U_g^*N_d^*U_g)^{n_d}\big) \\ &= (U_g^*\rho(\overline{z_1})U_g)^{n_1}\cdots(U_g^*\rho(\overline{z_d})U_g)^{n_d} \\ &= \rho(\overline{g_1}\cdot\bar{z})^{n_1}\cdots\rho(\overline{g_d}\cdot\bar{z})^{n_d}) \\ &= \rho((\overline{g_1}\cdot\bar{z})^{n_1}\cdots(\overline{g_d}\cdot\bar{z})^{n_d}) \\ &= \rho(g\cdot\overline{f_n}) \end{aligned}$$

Finally, the computation below shows that $U_g^* f_{m,n}U_g = \rho(g\cdot f_{m,n})$:

$$\begin{aligned} U_g^*\rho(f_{m,n})U_g &= U_g^*\rho(f_m)U_gU_g^*\rho(\overline{f_n})U_g = \rho(g\cdot f_m)\rho(g\cdot\overline{f_n}) \\ &= \rho\big((g\cdot f_m)(g\cdot\overline{f_n})\big) = \rho(g\cdot f_{m,n}). \end{aligned}$$

Now if p is a polynomial of the form $p = \sum_{m,n} a_{m,n}f_{m,n}$, we clearly have $U_g^*\rho(p)U_g = \rho(g\cdot p)$. By the Stone–Weierstrass theorem, these polynomials are dense in $C(\sigma(\boldsymbol{N}))$. Thus,

$$U_g^*\rho(f)U_g = \rho(g\cdot f), \quad f\in C(\sigma(\boldsymbol{N})). \quad \square$$

Next, we show that every imprimitivity $(\mathbb{S}, U, \rho)$ based on a locally compact G- space $\mathbb{S}$ defines a homogeneous commuting normal tuple $\boldsymbol{N}$ with spectrum $\bar{\mathbb{S}}$, where $\bar{\mathbb{S}}$ is the closure of $\mathbb{S}$. For this, it would be useful to describe the imprimitivity in an equivalent form involving a *spectral measure* on $\mathbb{S}$.

Definition 4.4 Let G be a second countable locally compact group and $\mathbb{S}$ be a locally compact G- space. Suppose that U is a unitary representation of G on a Hilbert space $\mathcal{H}$ and P is a regular $\mathcal{H}$-projection-valued measure on $\mathbb{S}$. Then $(\mathbb{S}, U, P)$ is said to be a system of imprimitivity if

$$U(g)P(E)U(g)^{-1} = P(g\cdot E) \tag{4.5}$$

for all $g\in G$ and every Borel subset E of $\mathbb{S}$.

It is proved in [8, p. 180] that the Definitions 4.2 and 4.4 are equivalent. For the proof of Proposition 4.6 given below, we will find it convenient to use the formulation of imprimitivity in Definition 4.4.

The *imprimitivity theorem* due to Mackey says, among other things, that if $(\mathbb{S}, U, \rho)$ is a transitive system of imprimitivity, then U is unitarily equivalent to the induced representation $\mathrm{Ind}_H^G(\sigma)$ for some unitary representation σ of H, where the homogeneous space $\mathbb{S}$ is assumed to be of the form G/H, G locally compact, $H\subset G$ closed.

Applying the Hahn–Hellinger theorem to the spectral measure P, writing it in the canonical form Theorem 2.5, and imposing the imprimitivity condition of Equation (4.5) from Definition 4.4 gives a canonical form of the imprimitivity $(\mathbb{S}, U, P)$ as described below, see [16, Proposition 4.1 and Theorem 4.3(b)] (and also [17, Theorem 6.12], [12, Theorem 4]).

Theorem 4.5 (Imprimitivity theorem) *Let $\mathbb{S} = G/H$ be a homogeneous G- space and μ be a quasi-invariant measure on $\mathbb{S}$ (there is always one such uniquely determined modulo mutual absolute equivalence). Assume that $(\mathbb{S}, U, \rho)$ is an imprimitivity acting on some separable complex Hilbert space $\mathcal{H}$. Then there is a Hilbert space V such that $\mathcal{H}$ is isometrically isomorphic to $\mathcal{H} = L^2(\mathbb{S}, \mu, V)$, where*

(i) *μ is a quasi-invariant measure on $\mathbb{S}$ determined uniquely modulo equivalence,*
(ii) *the representation ρ is of the form $\rho(f) = M_f$, $f\in C_0(\mathbb{S})$, and*
(iii) *the representation U is of the form*

$$(U(g)f)(s) = \sqrt{\frac{d\mu(g\cdot s)}{d\mu(s)}}\sigma(h)f(g\cdot s).$$

Here $h\in H$ is determined from the relation $g\,p(g^{-1}\cdot s) = p(s)h$, $s\in\mathbb{S}$, where $p: G/H\to G$ is a Borel cross-section.

We let $\mathcal{B}$ denote the σ- algebra of Borel subsets of $\mathbb{C}^d$. The Borel σ- algebra of $X \subset \mathbb{C}^d$ then consists of all subsets of X of the form $X \cap B$, $B \in \mathcal{B}$. If E is a subset of X, then we say that a measure μ *lives* on E if $\mu(X \setminus E) = 0$. Two measures μ and ν are said to be mutually singular if they live on two disjoint sets.

We have gathered all the tools from spectral theory of commuting tuples of normal operators to prove that every imprimitivity $(\mathbb{S}, U, \rho)$ based on a locally compact transitive G- space $\mathbb{S}$ is the restriction of an imprimitivity based on $\bar{\mathbb{S}}$.

Theorem 4.6 *Suppose that $\mathbb{S}$ is a locally compact transitive G- space and the action of G extends to $\bar{\mathbb{S}}$, the closure of $\mathbb{S}$ with $g \cdot \partial\mathbb{S} \subseteq \partial\mathbb{S}$.*

(1) *If $(\mathbb{S}, U, P)$ is an imprimitivity, then there exists a unique spectral measure $\hat{P}$ defined on the Borel σ- algebra $\mathcal{B}$ of $\bar{\mathbb{S}}$ satisfying the imprimitivity condition (4.5) with $\hat{P}(E) = P(E)$ for every Borel subset E of $\mathbb{S}$. Moreover,* $\mathrm{supp}(P) = \bar{\mathbb{S}}$.
(2) *If $(\mathbb{S}, U, P)$ is an imprimitivity, then it defines uniquely a homogeneous commuting tuple of normal operators $\boldsymbol{N}$ such that $\sigma(\boldsymbol{N}) = \mathrm{supp}(\hat{P}) = \bar{\mathbb{S}}$, where $\hat{P}$ is the spectral measure of $\boldsymbol{N}$.*

Proof The imprimitivity on $\mathbb{S}$ comes equipped with a spectral measure P defined on the Borel subsets of $\mathbb{S}$. Define $\hat{P}$ on any Borel subset E of $\bar{\mathbb{S}}$ by setting

$$\hat{P}(E) = \begin{cases} P(E \cap \mathbb{S}) & \text{if } E \cap \mathbb{S} \neq \varnothing, \\ 0 & \text{if } E \subseteq \partial\mathbb{S}. \end{cases}$$

Thus, $\hat{P}$ is an extension of the spectral measure P to the Borel σ- algebra on $\bar{\mathbb{S}}$ with $\hat{P}(\partial\mathbb{S}) = 0$. Clearly, these properties determine $\hat{P}$ uniquely.

Pick $s \in E$ and use Urysohn's Lemma to find a continuous function f defined on $\bar{\mathbb{S}}$ with $f(s) = 1$, $f = 0$ on E^c. If we had $P(E) = 0$, then $\int_{\mathbb{S}} f dP = 0$. This is not possible since the $*$- homomorphism ρ of $C(\mathbb{S})$ defined by $\rho(f) = \int_{\mathbb{S}} f dP$ cannot be the zero operator. Hence, $P(E) \neq 0$ whenever E is open in $\bar{\mathbb{S}}$. Consequently, $\mathrm{supp}(P) = \bar{\mathbb{S}}$.

Starting with the imprimitivity, $(\mathbb{S}, U, P)$, we define the spectral measure $\hat{P}$ on $\bar{\mathbb{S}}$ as in the Theorem. Since the action of G leaves $\mathbb{S}$ and $\partial\mathbb{S}$ invariant, it follows that $(\bar{\mathbb{S}}, U, \hat{P})$ is an imprimitivity. Or, equivalently, $(\bar{\mathbb{S}}, U, \rho)$ is an imprimitivity, where $\rho(f) = \int_{\bar{\mathbb{S}}} f d\hat{P}$ is a $*$- homomorphsim of $C(\bar{\mathbb{S}})$. By the imprimitivity theorem, the $*$- homomorphsim ρ can be realized on the Hilbert space $L^2(\mathbb{S}, \mu, V)$ in the form $\rho(f) = M_f$, $f \in C(\bar{\mathbb{S}})$. The d- tuple $(\rho(z_1), \ldots, \rho(z_d))$ of multiplication by the coordinate functions is homogeneous. To verify this claim, we have to find a unitary representation Γ of G and show that

$$\Gamma(g)^* \rho(z_k) \Gamma(g) = g \cdot z_k, \quad k = 1, \ldots, d,$$

where $g \cdot (\boldsymbol{z} \mapsto z_k) = z_k(g \cdot \boldsymbol{z}) = g_k(\boldsymbol{z})$. (Recall that $g(\boldsymbol{z}) = (g_1(\boldsymbol{z}), \ldots, g_d(\boldsymbol{z}))$.) Evidently, this is the imprimitivity condition for the particular choice of the function $z_k := \boldsymbol{z} \mapsto z_k$. Hence choosing $\Gamma = U$, where U is the unitary from the Imprimitivity theorem, we are done. Moreover, the spectrum of this d- tuple is $\bar{\mathbb{S}}$ as shown in the first half of the proof. □

Definition 4.7 Let G be a second countable locally compact group and $\mathbb{S}$ be a locally compact G- space. We say that a representation U of G acting on a Hilbert space $\mathcal{H}$ consisting of functions (equivalence classes modulo zero sets) is a *multiplier representation* if there is a concrete realisation of U as follows:

$$(U_g f)(z) = (m(g^{-1}, z))^{-1} f(g^{-1} z), \quad z \in \mathbb{S},\ f \in \mathcal{H},\ g \in G.$$

Let $S \subset \mathbb{C}^d$ be a compact G- space. Assume that S decomposes into finitely many G orbits: $S = S_0 \cup S_1 \cup \cdots \cup S_r$. Thus, G acts transitively on each S_j, and therefore $S_j \cong G/H_j$ for some closed subgroup H_j of G, $0 \leqslant j \leqslant r$. There exists a unique quasi-invariant measure on each of these orbits modulo mutual absolute equivalence. We choose and fix one such quasi-invariant measure on S_j, say, μ_j.

Lemma 4.8 *Let μ be a Borel measure on S that is quasi-invariant with respect to the G- action on S. Then $\mu \cong \sum_{j=0}^{r} \mu_j$.*

Proof The proof follows from the uniqueness of the quasi-invariant measure μ_j on S_j, $0 \leqslant j \leqslant r$. □

Theorem 4.9 *Let $\boldsymbol{N}$ be a homogeneous d- tuple of commuting normal operators acting on some Hilbert space $\mathcal{H}$ and let $S := \sigma(\boldsymbol{N})$ be the spectrum of $\boldsymbol{N}$. Assume that S is a G- space and that $S = \cup_{j=0}^{r} S_j$, where each S_j is a G- orbit. Then the imprimitivity $(S, U, \rho_{\boldsymbol{N}})$ induced by $\boldsymbol{N}$ is equivalent to the imprimitivity $(S, \pi_\mu, \hat{U})$, i.e., there is a unitary*

$$\Gamma : \mathcal{H} \to \oplus L^2(E_n, \mu, \mathcal{H}_n)$$

such that $\Gamma\rho(f)\Gamma^* = \pi_\mu(f)$, $f \in C(S)$ and $\Gamma U\Gamma^* = \hat{U}$ is a multiplier representation.

Proof Applying the Hahn–Hellinger theorem to the $*$- representation ρ of $C(S)$, we find a disjoint sequence E_n, $n \geqslant 1$, of Borel subsets of S such that (i) $S = \bigcup_{n=1}^{\infty} E_n$, (ii) $\mathcal{H} \cong \oplus L^2(E_n, \mu, \mathcal{H}_n)$, and (iii) $\rho(f) \cong \oplus_{n\geqslant 1}\pi^{(n)}(f)$, $f \in C(S)$, where $\pi^{(n)} = \pi^n_{\mu|_{E_n}}$ is the canonical $*$- representation of $C(S)$.

Since ρ and $g \cdot \rho$ are unitarily equivalent, it follows from the uniqueness assertion in the Hahn–Hellinger theorem that $\mu \cong \mu \circ g^{-1}$ and $\mu(E_n \triangle g \cdot E_n) = 0$, $g \in G$. Since μ is quasi-invariant it lives on a union of G-orbits, namely, $S_0 \cup S_1 \cup \cdots \cup S_r$.

Let $\hat{n}$ be the multiplicity function defined on S, i.e., $\hat{n} : S \to \mathbb{N}$, $\hat{n}_{|E_n} = n$. By the Hahn–Hellinger theorem, $\hat{n} \circ g = \hat{n}$ for all $g \in G$. It follows that $\hat{n}$ is a constant function on each orbit of G. Consequently, $S_j \subseteq E_{n_j}$ for some n_j. Let $\boldsymbol{S}_j = \cup\{S_k : S_k \subseteq E_{n_j}\}$. Hence $S = \cup\{\boldsymbol{S}_j : j \in \boldsymbol{F} \subseteq \{0, 1, \ldots, r\}\}$. We also see by the uniqueness portion of the Hahn–Hellinger theorem, that $\mu_{n_j}(E_{n_j} \triangle \boldsymbol{S}_j) = 0$. Thus, $\mu_{|\boldsymbol{S}_j} = \mu_{j_1} + \cdots + \mu_{j_k}$, where μ_{j_k} is the quasi-invariant measure that lives on S_{j_k} with $S_{j_k} \subseteq E_{n_j}$. It follows that modulo unitary equivalence via an unitary $\Gamma : \mathcal{H} \to \oplus_{n=1}^{\infty} L^2(E_n, \mu, \mathcal{H}_n)$ we have

$$\mathcal{H} \cong \bigoplus_{j\in\boldsymbol{F}} L^2(E_{n_j} \cap \boldsymbol{S}_j, \mu_{|\boldsymbol{S}_j}, \mathcal{H}_{n_j}) = \bigoplus_{j=0}^{r} L^2(S_j, \mu_{|S_j}, \mathcal{H}_{n_j}).$$

Let $\hat{\rho} = \Gamma\rho\Gamma^*$ and $\hat{U} = \Gamma U\Gamma^*$ and note that the pair $(\hat{\rho}, \hat{U})$ satisfy the imprimitivity condition. For each $j = 0, 1, \ldots, r$, define the unitary representation V^j_g on $L^2(S_j, \mu_j, \mathcal{H}_{n_j})$ by setting

$$V^j_g\psi(x) = \big(\tfrac{d(\mu_j\circ g^{-1})}{d\mu_j}(x)\big)^{1/2}\big(\psi \circ g^{-1}\big)(x),\ x \in S_j, \quad \psi \in L^2(S_j, \mu_j, \mathcal{H}_{n_j}).$$

Let $V_g = \oplus_{j=0}^{r} V^j_g$. By construction, $(S, \hat{\rho}, V_g)$ is an imprimitivity. Hence

$$\hat{U}^*_g\hat{\rho}\hat{U}_g = \hat{\rho}(g \cdot f) = V^*_g\hat{\rho}V_g, \quad g \in G.$$

Hence the unitary operators $\hat{U}_gV^*_g$, $g \in G$, commute with the set of normal operators $\{\hat{\rho}(f) : f \in C(S)\}$. It follows that the reducing subspaces $L^2(S_j, \mu_j, \mathcal{H}_{n_j})$ of the operators $\hat{\rho}(f)$ ($f \in C(S)$) are also invariant under the unitary operators $\hat{U}_gV^*_g$ ($g \in G$). Consequently, these reducing subspaces are also invariant under $\hat{U}_g$. Therefore, $\hat{U}_g$ has the form $\oplus_{j=0}^{r}\hat{U}^j_g$, where $\hat{U}^j_g$ is the restriction of $\hat{U}_g$ to the subspace $L^2(S_j, \mu_j, \mathcal{H}_{n_j})$.

Recall that $\hat{U}_gV^*_g$ must be a multiplication by a Borel function $c_g(\cdot)$ defined on S taking values in the unitary operators acting on $\oplus_{j=0}^{r}\mathcal{H}_{n_j}$. Furthermore, since $L^2(S_j, \mu_j, \mathcal{H}_{n_j})$ are reducing subspaces for $\hat{U}_gV^*_g$, we conclude that $c_g = \oplus_{j=0}^{r}c^j_g$. Hence $\hat{U}^j_gV^{j*}_g = \oplus_{j=0}^{r}c^j_g$. The representation V^j_g is a multiplier representation by construction with multiplier $\big(\frac{d(g\cdot\mu_j)}{d\mu_j}\big)^{1/2}$, we have that

$$\begin{aligned}\hat{U}^j_g f(x) &= c^j_g(x)(V_g f)(x)\\ &= \big(\tfrac{d(\mu_j\circ g^{-1})}{d\mu_j}(x)\big)^{1/2}c^j_g(x)(g \cdot f)(x),\end{aligned}$$

for all $x \in S_j$ and $f \in L^2(S_j, \mu_j, \mathcal{H}_{n_j})$. Since $\hat{U}^j_g$ is a homomorphism, it follows that $(\frac{d(\mu_j\circ g^{-1})}{d\mu})^{1/2}c^j_g$ is a cocycle. Moreover, $\big(\frac{d(g\cdot\mu_j)}{d\mu_j}\big)^{1/2}$ is evidently a cocycle and therefore c^j_g is a cocyle taking values in the group of unitary operators acting on $\mathcal{H}_{n_j}$. □

We can assert more than what is claimed in the theorem. This is the corollary below, and its proof is apparent from the proof of the theorem.

Corollary 4.10 *Let $\boldsymbol{N}$ be a homogeneous d- tuple of commuting normal operators acting on some Hilbert space $\mathcal{H}$ and let $S := \sigma(\boldsymbol{N})$ be the spectrum of $\boldsymbol{N}$. Assume that $S = \cup_{j=0}^{r} S_j$, where each S_j is a G- orbit and is not necessarily compact. Then there exist quasi-invariant measures μ_j living on S_j such that $\boldsymbol{N}$ is unitarily equivalent to the direct sum of $\boldsymbol{M}^{(j)}$ of the multiplication by the coordinate functions acting on the Hilbert space $L^2(S_j, \mu_j, \mathcal{H}_{n_j})$, $\dim(\mathcal{H}_{n_j}) = n_j$, $0 \leqslant j \leqslant r$.*

We point out that $\mathcal{H}_{n_j}$ may be isomorphic to $\mathcal{H}_{n_k}$ even if $j \neq k$.

Remark 4.11 Let $\boldsymbol{N}$ be a homogeneous d- tuple. Theorem 4.9 ensures that we may take, modulo unitary equivalence and without loss of generality, the unitary representation U intertwining the commuting d- tuple $(N_1, \ldots, N_d)$ with $(g_1(\boldsymbol{N}), \ldots, g_d(\boldsymbol{N}))$ to be a multiplier representation.

Conversely, assume that the d- tuple $\boldsymbol{N}$ of normal operators is realized as multiplication M_k, $k = 1, \ldots, d$, by coordinate functions z_k on some $L^2(S, \mu, V)$. If there is a multiplier representation U on $L^2(S, \mu, V)$ of the form $f \mapsto c(g, z) f(g \cdot z)$, then

$$\begin{aligned}\big(M_k U(g) f\big)(z) &= M_k c(g, z) f(g \cdot z) \\ &= g_k(g \cdot z) c(g, z) f(g \cdot z) \\ &= \big(U(g) M_{g_k} f\big)(z),\ 1 \leqslant k \leqslant d.\end{aligned}$$

Thus, U intertwines the commuting d- tuple $\boldsymbol{N}$ with $g \cdot \boldsymbol{N}$. Hence $\boldsymbol{N}$ is homogeneous.

5 Examples

The Möbius group acts transitively on the open unit disc $\mathbb{D}$ and the unit circle $\mathbb{T}$. However although the Möbius group acts on the closed disc $\overline{\mathbb{D}}$, this action is no longer transitive. Indeed, $\overline{\mathbb{D}}$ is the disjoint union of two orbits, namely, $\mathbb{D}$ and $\mathbb{T}$. When the action is transitive, the canonical form of the imprimitivity is described by Mackey. Hence if N is a homogeneous normal operator with $\sigma(N) = \mathbb{T}$, then it must be a n- fold direct sum of the operator of multiplication by α, $\alpha \in \mathbb{T}$ on $L^2(\mathbb{T}, d\theta)$, where $d\theta$ is the arc length measure. However, it is not obvious what are the homogeneous normal operators N with $\sigma(N) = \overline{\mathbb{D}}$. These are described explicitly in a forthcoming paper of the first named author with A. Korányi, see also [3, Theorem 6.6].

5.1 The case of a product domain

In this subsection, we describe commuting pairs $\boldsymbol{N}$ of homogeneous normal operators with $\sigma(\boldsymbol{N}) = \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. The subset $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$ of $\mathbb{C}^2$ is a G- space, where G is the subgroup

$$\{\phi := (\phi_1, \phi_2) \mid \phi_k(z_k) = \beta_k \tfrac{z_k - \alpha_k}{1 - \bar{\alpha}_k z_k},\ \beta_k \in \mathbb{T}, \alpha_k \in \mathbb{D}, k = 1, 2\}$$

of the group of bi-holomorphic automorphisms of $\mathbb{D} \times \mathbb{D}$. The automorphism ϕ extends to an automorphism of $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ with $\phi(\partial\mathbb{D}^2) \subseteq \partial\mathbb{D}^2$. To identify homogeneous (under the G- action) pairs of commuting normal operators, we first note that the spectrum of such a pair must be a G- invariant compact subset of $\mathbb{C}^2$. To find these, note that the orbit through a point $(z_1, z_2) \in \mathbb{T} \times \mathbb{D}$ is $\mathbb{T} \times \mathbb{D}$; similarly $\mathbb{D} \times \mathbb{T}$ is also a G- orbit. If $(z_1, z_2) \in \mathbb{T} \times \mathbb{T}$, the G- orbit is $\mathbb{T} \times \mathbb{T}$. These are all the G- orbits in the boundary of $\mathbb{D} \times \mathbb{D}$. Closures of these orbits give us compact sets that are G- invariant. Moreover, if (z_1, z_2) is in $\mathbb{D}^2$, then the G- orbit through this point is $\mathbb{D}^2$. Thus, all the compact G- invariant subsets of $\mathbb{C}^2$ are

$$\overline{\mathbb{D}} \times \overline{\mathbb{D}}, \quad \mathbb{T} \times \overline{\mathbb{D}}, \quad \overline{\mathbb{D}} \times \mathbb{T}, \quad \mathbb{T} \times \mathbb{T}.$$

Among these, the group G acts transitively only on $\mathbb{T} \times \mathbb{T}$. Consequently, imprimitivities based on $\mathbb{T} \times \mathbb{T}$, or equivalently, associated pairs $\boldsymbol{N}$ of homogeneous normal operators, are described by Mackey's theorem. The remaining three cases can be dealt with using Corollary 4.10; however, we provide an elementary analysis below.

If we consider a commuting pair of homogeneous normal operators $\boldsymbol{N}$ with $\sigma_{\boldsymbol{N}} = \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, then it must be unitarily equivalent to the pair of multiplication operators $\boldsymbol{M} = (M_1, M_2)$ acting on $L^2(\overline{\mathbb{D}} \times \overline{\mathbb{D}}, \mu, \mathcal{H}_n)$, where μ

is quasi-invariant with respect to the group G and $\dim \mathcal{H}_n = n$. The restriction of the measure μ to the transitive G- space $\mathbb{D}\times\mathbb{D}$, $\mathbb{D}\times\mathbb{T}$, $\mathbb{T}\times\mathbb{D}$ and $\mathbb{T}\times\mathbb{T}$ is uniquely determined since the group acts on these transitively. These are the measures: $\mu_1 := dA\times dA$, $\mu_2 := dA\times d\theta$, $\mu_3 := d\theta\times dA$ and $\mu_4 := d\theta\times d\theta$, respectively. (Here, dA and $d\theta$ denote the area and the arc length measures, respectively.) Evidently, $\mu = \mu_1+\mu_2+\mu_3+\mu_4$. Moreover, μ_j, $1\leqslant j\leqslant 4$, are mutually singular. Consequently, $L^2(\overline{\mathbb{D}}\times\overline{\mathbb{D}},\mu,\mathcal{H}_n)$ must be a direct sum of the form

$$L^2(\mathbb{D}\times\mathbb{D},\mu_1,\mathcal{H}_{n_1})\oplus L^2(\mathbb{D}\times\mathbb{T},\mu_2,\mathcal{H}_{n_2})\oplus L^2(\mathbb{T}\times\mathbb{D},\mu_3,\mathcal{H}_{n_3})\oplus L^2(\mathbb{T}\times\mathbb{T},\mu_4,\mathcal{H}_{n_4}),$$

where $n = n_1+n_2+n_3+n_4$. These are reducing subspaces of the homogeneous pair of operators (M_1, M_2) acting on $L^2(\overline{\mathbb{D}}\times\overline{\mathbb{D}},\mu,\mathcal{H}_n)$. We have therefore proved that any pair $\boldsymbol{N}$ of homogeneous normal operators with $\sigma(\boldsymbol{N}) = \overline{\mathbb{D}}\times\overline{\mathbb{D}}$ must be unitarily equivalent to the direct sum of the pair (M_1, M_2) of multiplication operators acting on the Hilbert space $\bigoplus_{j=1}^4 L^2(X_j,\mu_j,n_j)$, where $X_1 = \mathbb{D}\times\mathbb{D}$, $X_2 = \mathbb{D}\times\mathbb{T}$, $X_3 = \mathbb{T}\times\mathbb{D}$, $X_4 = \mathbb{T}\times\mathbb{T}$.

The two cases where the spectrum of $\boldsymbol{N}$ is either $\mathbb{T}\times\overline{\mathbb{D}}$ or $\overline{\mathbb{D}}\times\mathbb{T}$ are described exactly in the same manner.

5.2 Cartan domains

Let $\mathcal{D}\subset\mathbb{C}^d$ be an irreducible bounded symmetric domain (in its standard realization as a Cartan domain) and G be the connected component containing the identity of the bi-holomorphic automorphism group of $\mathcal{D}$. We note that the G- action on $\mathcal{D}$ extends to some open neighborhood of $\mathcal{D}\cup\partial\mathcal{D}$. The G- action is transitive on $\mathcal{D}$. It is known that, under the G action, the topological boundary $\partial\mathcal{D}$ splits into the disjoint union

$$S_0\cup S_1\cup\cdots\cup S_{r-1},$$

of G- orbits so that $\overline{S_j} = S_0\cup S_1\cup\cdots\cup S_j$, $0\leqslant j\leqslant r-1$. Here, S_0 is the Shilov boundary and is necessarily compact.

The only possibilities for the spectrum of a d- tuple of commuting homogeneous normal operators are compact G- invariant subsets of $\mathbb{C}^d$. These are the sets: $\overline{\mathcal{D}}$ and $\overline{S}_j$, $0\leqslant j\leqslant r-1$. Fixing the spectrum of a d- tuple of commuting homogeneous normal operators to be one of these sets, we see that the hypotheses of Corollary 4.10 are met. Therefore, we have a complete description of the commuting d- tuples of homogeneous normal operators, or equivalently, all the imprimitivities based on the G- spaces $\overline{S}_j$, $0\leqslant j\leqslant r-1$ and $\overline{\mathcal{D}}$.

6 Open problems

Let $\mathbb{S}$ be a bounded open connected subset of $\mathbb{C}^d$. Following the well-established tradition in Operator theory, we propose to study homomorphisms of the "disc" algebra $\mathcal{A}(\mathbb{S})\subset C(\overline{\mathbb{S}})$, where $\mathcal{A}(\mathbb{S})$ consists of all those complex-valued functions f such that there is an open set U containing $\overline{\mathbb{S}}$ and f is holomorphic on U. Now, pick any algebra homomorphism $\varrho:\mathcal{A}(\mathbb{S})\to\mathcal{L}(\mathcal{H})$ and a unitary representation of a locally compact second countable group G on $\mathcal{H}$, and say that $(\mathbb{S},\varrho,U)$ is a "holomorphic imprimitivity" if

$$U(g)^*\varrho(f)U(g) = \varrho(g\cdot f),\quad g\in G,\ \ f\in\mathcal{A}(\mathbb{S}).$$

A homomorphism ϱ as above clearly defines a d- tuple of commuting bounded linear operators $(T_1 := \varrho(z_1),\ldots,T_d := \varrho(z_d))$ that is *homogeneous*, namely,

$$U(g)^*(T_1,\ldots,T_d)U(g) = (g_1(T_1),\ldots,g_d(T_d)),\quad g\in G.$$

Here we must assume that the joint spectrum (we take it to be the one defined in the sense of Taylor) of $\boldsymbol{T}$ is contained in $\overline{\mathbb{S}}$.

Conversely, starting with a d- tuple $\boldsymbol{T}$ of homogeneous commuting bounded linear operators with $\sigma(\boldsymbol{T})\subseteq\overline{\mathbb{S}}$, a homomorphism ϱ is defined by setting $\varrho_{\boldsymbol{T}}(f) = f(\boldsymbol{T})$ via the holomorphic functional calculus.

A detailed study of the class of algebra homomorphisms $\varrho:\mathcal{A}(\mathbb{S})\to\mathcal{L}(\mathcal{H})$ that are restrictions of $*$-homomorphisms $\hat{\rho}:B(\mathbb{S})\to\mathcal{L}(\mathcal{K})$, where $\mathcal{K}\supset\mathcal{H}$ and $\hat{\rho}(f)(\mathcal{H})\subseteq\mathcal{H}$ has been very fruitful. In this case the commuting d- tuple of operators $(T_1,\ldots,T_d)$ is called subnormal.

When $\mathbb{S}$ is only locally compact, as pointed out earlier, the $*$- representation $\hat{\rho}$ extends to a $*$- representation of the algebra $B(\mathbb{S})$ of bounded Borel functions on $\mathbb{S}$ and $\mathcal{A}(\mathbb{S})\subset B(\mathbb{S})$. Thus, we can speak of $\hat{\rho}(f)$ for $f\in\mathcal{A}(\mathbb{S})$.

The first question in the context of this paper is the following. Suppose that $(\mathbb{S}, \varrho, U)$ is a holomorphic imprimitivity based on $\mathcal{H}$, and there exists a $*$- homomorphism $\hat{\rho} : C_0(\mathbb{S}) \to \mathcal{L}(\mathcal{K})$, $\mathcal{K} \supset \mathcal{H}$, such that (a) $\mathcal{H}$ is invariant for $\hat{\rho}(f)$, $f \in \mathcal{A}(\mathbb{S})$, and (b) $\hat{\rho}_{|\mathcal{H}}(f) = \rho(f)$, $f \in \mathcal{A}(\mathbb{S})$. Then does it follow that there is a unitary representation $\hat{U}$ of G on $\mathcal{K}$ such that $(\mathbb{S}, \hat{\rho}, \hat{U})$ is an imprimitivity?

Let $(\mathbb{S}, \hat{\rho}, U)$ be an imprimitivity. The second question asks for a description of all the simultaneous invariant subspaces of $\hat{\rho}$ and $\hat{U}$.

The first question has an affirmative answer when $\mathbb{S}$ is either $\overline{\mathbb{D}}$ or $\mathbb{T}$. This is in a forthcoming paper of the first named author jointly with A. Kóranyi. In the same paper, although there are some partial results describing the simultaneous invariant subspaces in the case of $\overline{\mathbb{D}}$ and $\mathbb{T}$, the answer to the question of simultaneous invariant subspaces is far from complete even in this very simple case.

When the answer to the second question is affirmative, restricting to the simultaneous invariant subspace we obtain homogeneous subnormal operators. If the answer to the first question is also affirmative, which is very likely, and if we have a description of all the simultaneous invariant subspaces, then all the homogeneous subnormal operators would be the restriction of homogeneous normal operators (or imprimitivities) to these subspaces.

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