Indian J Pure Appl Math (2024) 55:1129–1141
https://doi.org/10.1007/s13226-023-00413-2

ORIGINAL RESEARCH

A study of harmonic Fibonacci polynomials associated With Bernoulli-F and Euler–Fibonacci polynomials

Naim Tuglu · Semra Kuş · Can Kızılateş

Received: 29 November 2021 / Accepted: 12 April 2023 / Published online: 25 April 2023

Abstract In this paper, by the help the F-vacuum operator, we define the harmonic Fibonacci polynomials and harmonic based F-exponential generating function. The harmonic based F-exponential generating function is obtained for the Bernoulli-F polynomials, the Euler-Fibonacci numbers, the Euler-Fibonacci polynomials and the Bernoulli-Fibonacci numbers, and their relations are shown with the harmonic Fibonacci numbers and polynomials.

Keywords Harmonic Fibonacci number · Umbral method · Special polynomial · Exponential generating function

Mathematics Subject Classification 11B37 · 05A40 · 33C52

1 Introduction

Umbral calculus has been used innumerable problems of applied mathematics, theoretical physics, special polynomials and several miscellaneous of mathematics like analysis, combinatorics and topology. Interesting properties of many families of special numbers and polynomials have been studied by the use of methods follow up the umbral calculus [1].

The well-known Fibonacci sequence is defined by the following recurrence relation, for $n \geq 0$

$$F_{n+2} = F_{n+1} + F_n,$$

where $F_0 = 0 \ ve \ F_1 = 1$. These numbers have significant applications in different branches of the mathematical sciences, such as number theory, combinatorics and so on (see, for example [2–6,9]).

Communicated by B. Sury.

N. Tuglu (✉)
Department of Mathematics, University of Gazi, 06500 Ankara, Turkey
E-mail: naimtuglu@gazi.edu.tr

S. Kuş
Mucur Vocational High School, University of Kırşehir Ahi Evran, Kırşehir, Turkey
E-mail: skus@ahievran.edu.tr

C. K. lateş
Department of Mathematics, University of Zonguldak Bulent Ecevit, 67100 Zonguldak, Turkey
E-mail: cankizilates@gmail.com

Fibonacci numbers and their generalizations have been studied by many researchers. For example, in [4] Dil and Mezö defined and studied hyper-Fibonacci numbers, $F_n^{(r)}$, for positive integer r,

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)},$$

with $F_n^{(0)} = F_n$, $F_0^{(r)} = 0$ and $F_1^{(r)} = 1$.

In [6] Tuglu et al. defined the harmonic Fibonacci numbers and hyperharmonic Fibonacci numbers. Harmonic Fibonacci numbers $\mathbb{F}_n$, are defined as follows:

$$\mathbb{F}_n = \sum_{k=1}^{n} \frac{1}{F_k},$$

with $\mathbb{F}_0 = 0$. For more details related to harmonic Fibonacci and hyperharmonic Fibonacci numbers, we refer to [6]. Generating functions are a powerful tool for solving problems in number theory, combinatorics, algebra, probability theory, and other fields of mathematics. Many researchers have given important properties for special type numbers and polynomials using the generating functions. Ozdemir and Simsek [7] defined the two variables of Fibonacci-type polynomials by the following generating function

$$H\left(t; x, y; k, m, n\right) = \sum_{j=0}^{\infty} \mathcal{G}_j\left(x, y; k, m, n\right) t^j = \frac{1}{1 - x^k t - y^m t^{m+n}},$$

where $k, m, n \in \mathbb{N}_0$. They obtained some important properties of these polynomials. Moreover, they gave relationships between the Fibonacci, Jacobsthal, Chebyshev polynomials and the other well known polynomials. For more information related to special numbers and polynomials, we refer to [7,8].

Fibonomial coefficients are known as interesting generalizations of binomial coefficients. For the readers, we provide a summary of the mathematical notations and some basic definitions of the Fibonomial coefficients.

The F-factorial is defined as

$$F_n! = F_n.F_{n-1}.F_{n-2}...F_1, \quad F_0! = 1.$$

The Fibonomial coefficients are defined $0 \le k \le n$ as

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}!F_k!}$$

with $\binom{n}{0}_F = \binom{n}{n}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$. The F-binomial theorem is given by

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

The F-exponential function e_F^x is defined by

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}. \tag{1.1}$$

The Golden derivative or F-derivative operator

$$\partial_F\left(f(x)\right) = \frac{f(\alpha x) - f(\beta x)}{(\alpha - \beta)x},$$

where α and β are roots of the characteristic equation $x^2 - x - 1$. By using the definition of F-derivative operator, we can easily seen that

$$\partial_F\left(x^n\right) = F_n\, x^{n-1}. \tag{1.2}$$

On the other hand, $(p,q)-$integral is defined by [20],

$$\int_0^1 f(x)d_{p,q}(x) = (p-q)\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}f\left(\frac{q^i}{p^{i+1}}\right), \left|\frac{p}{q}\right| > 1. \tag{1.3}$$

If we take $p \to \alpha = \frac{1+\sqrt{5}}{2}$ and $q \to \beta = \frac{1-\sqrt{5}}{2}$ in Equation (1.3), we get F-integral as follows:

$$\int_0^1 f(x)d_{\alpha,\beta}(x) = \int_0^1 f(x)d_F(x),$$

and thus,

$$\int_0^1 f(x)d_F(x) = (\alpha-\beta)\sum_{i=0}^{\infty}\frac{\beta^i}{\alpha^{i+1}}f\left(\frac{\beta^i}{\alpha^{i+1}}\right). \tag{1.4}$$

In particular, if we take $f(x) = x^n$ in (1.4), we obtain

$$\begin{aligned}\int_0^1 f(x)d_F(x) =&(\alpha-\beta)\sum_{i=0}^{\infty}\frac{\beta^i}{\alpha^{i+1}}\left(\frac{\beta^i}{\alpha^{i+1}}\right)^n\\ =&(\alpha-\beta)\sum_{i=0}^{\infty}\frac{1}{\alpha^{n+1}}\left(\frac{\beta^{n+1}}{\alpha^{n+1}}\right)^i\\ =&\frac{\alpha-\beta}{\alpha^{n+1}-\beta^{n+1}}\\ =&\frac{1}{F_{n+1}}.\end{aligned}$$

For more details and properties related to the Fibonomial coefficients, we refer to [5,9,11].

In [5] Krot defined the Bernoulli-F polynomials of order 1, as follows:

$$B_{n,F}(x) = \sum_{k\geq 0}\frac{1}{F_{k+1}}\binom{n}{k}_F x^{n-k}.$$

In [9,10] Bernoulli-Fibonacci polynomials are defined as follows:

$$\sum_{n=0}^{\infty}B_n^F(x)\frac{t^n}{F_n!} = \frac{te_F^{tx}}{e_F^t-1},$$

which, for $x=0$, yields the Bernoulli-Fibonacci numbers, given by $B_n^F(0) = B_n^F$, as follows:

$$\sum_{n=0}^{\infty}B_n^F\frac{t^n}{F_n!} = \frac{t}{e_F^t-1}. \tag{1.5}$$

In [11] Kus et al. defined the Euler-Fibonacci numbers and the Euler-Fibonacci polynomials are defined by

$$E_{n,F} = -\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}, \tag{1.6}$$

and

$$E_{n,F}(x) = \sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}\, x^{n-k}, \tag{1.7}$$

where $E_{0,F}(x)=1$, respectively. The exponential generating functions of Euler-Fibonacci numbers and Euler-Fibonacci polynomials are defined by

$$\sum_{n=0}^{\infty} E_{n,F} \frac{t^n}{F_n!} = \frac{2}{e_F^t + 1}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} E_{n,F}(x) \frac{t^n}{F_n!} = \frac{2e_F^{xt}}{e_F^t + 1}, \tag{1.9}$$

respectively. Then, they obtained the relationship between the Bernoulli-F polynomials and Euler-Fibonacci polynomials.

In the existing literature on the subject, one can find a fairly large number of papers relating various special numbers and special polynomials to combinatorial analysis, number theory, and other areas (see, for details [5,11–19], see also closely related to earlier works cited therein).

We now turn to a recent investigation Dattoli and other people [12], who defined harmonic number umbral vacuum or vacuum by means of the following integral:

$$\varphi_h(z) := \int_0^1 \frac{1-x^z}{1-x} dx,$$

for $\forall z \in \mathbb{R}^+$. They defined the umbral vacuum shift operator as

$$\hat{h} := e^{\partial_z}$$

via umbral calculus. By using the umbral vacuum shift operator, they obtained the following equality

$$\hat{h}^n \varphi_h(z)\Big|_{z=0} = h_n,$$

where h_n is the n-th harmonic number. Moreover they defined the harmonic-based exponential function and harmonic polynomials respectively.

Based on Dattoli's paper, in this paper firstly we give the n-th harmonic Fibonacci number by means of the F-integral. We define the harmonic Fibonacci polynomials. We also define the harmonic based F-generating function via harmonic F-vacuum operator. Using F-integral, we give interesting equality related to Bernoulli-F polynomials. Then we give harmonic based F-exponential generating functions of the Bernoulli-F polynomials, the harmonic Fibonacci numbers, the harmonic Fibonacci polynomials, the Euler-Fibonacci numbers, the Euler-Fibonacci polynomials and the Bernoulli-Fibonacci numbers.

2 Harmonic Fibonacci polynomials, Bernoulli-F polynomials and harmonic based F-exponential generating functions

In this part of our paper, firstly, we obtain the harmonic Fibonacci numbers via F-integral. We define the harmonic based F-exponential generating function and obtain harmonic based F-exponential generating function of Bernoulli-F polynomials. In addition, with the help of the F-integral, we obtain Bernoulli-F polynomials with a different method. Finally, we give the relationship between harmonic Fibonacci numbers and harmonic Fibonacci polynomials of Bernoulli-F polynomials.

Theorem 1 *The following equality holds:*

$$\mathbb{F}_n = \int_0^1 \frac{1-x^n}{1-x} d_F(x).$$

Proof From the definition of F-integral, we find that

$$\begin{aligned}
\int_0^1 \frac{1-x^n}{1-x} d_F(x) &= \int_0^1 \left(x^{n-1}+x^{n-2}+\cdots+1\right) d_F(x) \\
&= \int_0^1 \sum_{k=1}^{n} x^{k-1} d_F(x) \\
&= \sum_{k=1}^{n} \int_0^1 x^{k-1} d_F(x) \\
&= \sum_{k=1}^{n} \frac{1}{F_k} \\
&= \mathbb{F}_n.
\end{aligned}$$

□

Definition 1 For $n \geq 1$, the n-th harmonic Fibonacci polynomial, $\mathbb{F}_n(x)$ is defined by

$$\mathbb{F}_n(x) = x^n + \sum_{k=1}^{n} \mathbb{F}_k \binom{n}{k}_F x^{n-k},$$

where $\mathbb{F}_0(x) = 1$.

The first three harmonic Fibonacci polynomials as follows:

$$\begin{aligned}
\mathbb{F}_1(x) &= x + \sum_{k=1}^{1} \mathbb{F}_k \binom{1}{k}_F x^{1-k} \\
&= x + \mathbb{F}_1 \binom{1}{1}_F \\
&= x + 1,
\end{aligned}$$

$$\begin{aligned}
\mathbb{F}_2(x) &= x^2 + \sum_{k=1}^{2} \mathbb{F}_k \binom{2}{k}_F x^{2-k} \\
&= x^2 + \mathbb{F}_1 \binom{2}{1}_F x + \mathbb{F}_2 \binom{2}{2}_F \\
&= x^2 + x + 2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{F}_3(x) &= x^3 + \sum_{k=1}^{3} \mathbb{F}_k \binom{3}{k}_F x^{3-k} \\
&= x^3 + \mathbb{F}_1 \binom{3}{1}_F x^2 + \mathbb{F}_2 \binom{3}{2}_F x + \mathbb{F}_3 \binom{3}{3}_F \\
&= x^3 + 2x^2 + 4x + \frac{5}{2}.
\end{aligned}$$

The harmonic Fibonacci polynomials and harmonic Fibonacci numbers are given as in Table 1.

Table 1 Harmonic Fibonacci polynomials and harmonic Fibonacci numbers

n	$\mathbb{F}_n(x)$	$\mathbb{F}_n$
0	1	0
1	$x+1$	1
2	x^2+x+2	2
3	$x^3+2x^2+4x+\frac{5}{2}$	$\frac{5}{2}$
4	$x^4+3x^3+12x^2+\frac{15}{2}x+\frac{17}{6}$	$\frac{17}{6}$
5	$x^5+5x^4+30x^3+\frac{75}{2}x^2+\frac{85}{6}x+\frac{91}{30}$	$\frac{91}{30}$
6	$x^6+8x^5+80x^4+150x^3+\frac{340}{3}x^2+\frac{364}{15}x+\frac{379}{120}$	$\frac{379}{120}$

Now we apply the harmonic vacuum operator and vacuum shift operator for harmonic numbers defined by Dattoli et al. [12], for the harmonic Fibonacci numbers with a similar approach. For the harmonic Fibonacci numbers, harmonic F-vacuum operator is defined as

$$\Phi_{\mathbb{F}}(z)=\int_0^1 \frac{1-x^z}{1-x}\, d_F(x), \tag{2.1}$$

and F-vacuum shift operator

$$f=e^{\partial z}. \tag{2.2}$$

Also, we get

$$f_{\mathbb{F}}^n=\mathbb{F}_n,$$

where F-umbral operator $f_{\mathbb{F}}^n$, defines the harmonic Fibonacci numbers, as the action of the F-vacuum shift operator (2.2) on the harmonic F-vacuum operator (2.1).

Definition 2 The harmonic based F-exponential generating function defined by

$$e^{f_{\mathbb{F}}t}=1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!}. \tag{2.3}$$

By virtue of (1.2), the relevant derivatives can be expressed as follows:

$$\begin{aligned}
\partial_{F,t}\left(e^{f_{\mathbb{F}}t}\right)&=\partial_{F,t}\left(1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!}\right)\\
&=\sum_{n=1}^{\infty}\mathbb{F}_n\frac{F_n\,t^{n-1}}{F_n!}\\
&=\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^{n-1}}{F_{n-1}!}\\
&=\sum_{n=0}^{\infty}\mathbb{F}_{n+1}\frac{t^n}{F_n!}\\
&=1+\sum_{n=1}^{\infty}\mathbb{F}_{n+1}\frac{t^n}{F_n!}.
\end{aligned} \tag{2.4}$$

Theorem 2 *The harmonic based F-exponential generating function of harmonic Fibonacci polynomial as follows:*

$$\sum_{n=0}^{\infty}\mathbb{F}_n(x)\frac{t^n}{F_n!}=e_F^{xt}e^{f_{\mathbb{F}}t}. \tag{2.5}$$

Proof By virtue of (1.1) and (2.3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{F}_n(x)\frac{t^n}{F_n!} &= \sum_{n=0}^{\infty}\left(x^n+\sum_{k=1}^{n}\binom{n}{k}_F\mathbb{F}_k x^{n-k}\right)\frac{t^n}{F_n!}\\ &= \sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\mathbb{F}_k}{F_k!}\frac{x^{n-k}}{F_{n-k}!}t^n\\ &= \left(\sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}\right)\left(1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!}\right)\\ &= e_F^{xt}e^{f_{\mathbb{F}}t}.\end{aligned}$$

So the proof is completed. □

In the following theorem, we will give the interesting identity between Bernoulli-F polynomials and F-integral.

Theorem 3 *The following equation holds:*

$$\int_0^1 (x+_F y)^n\, d_F(y) = B_{n,F}(x).$$

Proof Using the definition of Bernoulli-F polynomials

$$B_{n,F}(x)=\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F x^{n-k},$$

and F-integral

$$\int_0^1 x^n d_F(x)=\frac{1}{F_{n+1}},$$

we have

$$\begin{aligned}\int_0^1 (x+_F y)^n\, d_F(y) &= \int_0^1\left(\sum_{k=0}^{n}\binom{n}{k}_F x^k y^{n-k}\right)d_F(y)\\ &=\sum_{k=0}^{n}\binom{n}{k}_F x^k\left(\int_0^1 y^{n-k}d_F(y)\right)\\ &=\sum_{k=0}^{n}\binom{n}{k}_F x^k\frac{1}{F_{n-k+1}}\\ &=\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F x^{n-k}\\ &=B_{n,F}(x).\end{aligned}$$

□

Theorem 4 *For Bernoulli-F polynomials $B_{n,F}(x)$; harmonic based F-exponential function as follows:*

$$\sum_{n=0}^{\infty}B_{n,F}(x)\frac{t^n}{F_n!}=\left(\partial_{F,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1\right)e_F^{xt}.$$

Proof By virtue of (2.3) and (2.4), we find that

$$
\begin{aligned}
\partial_F e^{f_{\mathbb{F}}t} - e^{f_{\mathbb{F}}t} &= \sum_{n=1}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!} - \sum_{n=1}^{\infty} \mathbb{F}_n \frac{t^n}{F_n!} \\
&= \sum_{n=1}^{\infty} (\mathbb{F}_{n+1} - \mathbb{F}_n) \frac{t^n}{F_n!} \\
&= \sum_{n=1}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \right) - 1. \qquad (2.6)
\end{aligned}
$$

Multiplying both sides of the (2.6) by e_F^{xt}, we have

$$
\begin{aligned}
\left(\partial_F e^{f_{\mathbb{F}}t} - e^{f_{\mathbb{F}}t} + 1 \right) e_F^{xt} &= \sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{F_{k+1}!} . \frac{x^{n-k}}{F_{n-k}!} \right) t^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F x^{n-k} \right) \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!}.
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 5 *For $B_{n,F}(x)$ Bernoulli-F polynomials, $\mathbb{F}_n$ harmonic Fibonacci numbers and $\mathbb{F}_n(x)$ harmonic Fibonacci polynomials, the following equation holds:*

$$
B_{n,F}(x) = x^n + \sum_{k=0}^{n} \mathbb{F}_{k+1} \binom{n}{k}_F x^{n-k} - \mathbb{F}_n(x). \qquad (2.7)
$$

Proof We begin the right-hand side of the (2.7), then we get

$$
\begin{aligned}
x^n + \sum_{k=0}^{n} \mathbb{F}_{k+1} \binom{n}{k}_F x^{n-k} - \mathbb{F}_n(x) &= x^n + \sum_{k=0}^{n} \mathbb{F}_{k+1} \binom{n}{k}_F x^{n-k} - \left(x^n + \sum_{k=0}^{n} \mathbb{F}_k \binom{n}{k}_F x^{n-k} \right) \\
&= \sum_{k=0}^{n} (\mathbb{F}_{k+1} - \mathbb{F}_k) \binom{n}{k}_F x^{n-k} \\
&= \sum_{k=0}^{n} \frac{1}{F_{k+1}} \binom{n}{k}_F x^{n-k} \\
&= B_{n,F}(x).
\end{aligned}
$$

So the proof is completed. □

3 Harmonic Fibonacci polynomials, Euler-Fibonacci polynomials, Euler-Fibonacci numbers, harmonic based F-exponential generating functions

In this section, we obtain the harmonic based F-exponential generating functions of the Euler-Fibonacci numbers, the Euler-Fibonacci polynomials and the Bernoulli-Fibonacci numbers. Then we show the relationships between harmonic Fibonacci numbers and harmonic Fibonacci polynomials of the Euler-Fibonacci polynomials.

Theorem 6 *For Euler-Fibonacci numbers $E_{n,F}$, harmonic based F-exponential function as follows:*

$$\sum_{n=0}^{\infty} E_{n,F} \frac{t^n}{F_n!} = \frac{2}{2 + t\left(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1\right)}. \tag{3.1}$$

Proof From (1.8), (2.3) and (2.4), we find that

$$\begin{aligned}
e_F^t + 1 &= 2 + \sum_{n=1}^{\infty} \frac{t^n}{F_n!} \\
&= 2 + \sum_{n=0}^{\infty} \frac{t^{n+1}}{F_{n+1}!} \\
&= 2 + t \sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \\
&= 2 + t \sum_{n=0}^{\infty} \left(\mathbb{F}_{n+1} - \mathbb{F}_n\right) \frac{t^n}{F_n!} \\
&= 2 + t \left(1 + \sum_{n=1}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!} - \sum_{n=1}^{\infty} \mathbb{F}_n \frac{t^n}{F_n!}\right) \\
&= 2 + t\left(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,F} \frac{t^n}{F_n!} &= \frac{2}{e_F^t + 1} \\
&= \frac{2}{2 + t\left(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1\right)}.
\end{aligned}$$

□

Theorem 7 *For Euler-Fibonacci polynomials $E_{n,F}(x)$, harmonic based F -exponential function as follows:*

$$\sum_{n=0}^{\infty} E_{n,F}(x) \frac{t^n}{F_n!} = \frac{2e_F^{xt}}{2 + t\left(\partial_{F,t} e^{f_{\mathbb{F}} t} - e^{f_{\mathbb{F}} t} + 1\right)}.$$

Proof By virtue of (1.7), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,F}(x) \frac{t^n}{F_n!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_F E_{k,F}\, x^{n-k}\right) \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{F_n!}{F_{n-k}! F_k!} E_{k,F}\, x^{n-k}\right) \frac{t^n}{F_n!}
\end{aligned}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{F_{n-k}!F_k!}E_{k,F}\,x^{n-k}\right)t^n$$
$$=\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}. \tag{3.2}$$

Thus, from (3.1), we get

$$\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}=\frac{2e_F^{xt}}{2+t\left(\partial_{F,t}e^{f_{\mathbb{F}}t}-e^{f_{\mathbb{F}}t}+1\right)}.$$

□

Theorem 8 *Let $E_{n,F}\left(x\right)$, $\mathbb{F}_n\left(x\right)$ and $\mathbb{F}_n$ be the Euler-Fibonacci polynomials, the harmonic Fibonacci polynomials and the harmonic Fibonacci numbers, respectively. Then the following equality holds for:*

$$E_{n,F}\left(x\right)=\sum_{k=0}^{n}\binom{n}{k}_F\left(E_{k,F}\mathbb{F}_{n-k}\left(x\right)-E_{k,F}\left(x\right)\mathbb{F}_{n-k}\right).$$

Proof By virtue of (1.7) and (3.2), we find that

$$\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}=\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}. \tag{3.3}$$

Multiplying both sides of the (3.3) by $e^{f_{\mathbb{F}}t}$, we obtain

$$\left(\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}\right)e^{f_{\mathbb{F}}t}=\left(\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}\right)\left(1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!}\right)$$
$$=\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}+\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!}$$
$$=\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}\left(x\right)\,\mathbb{F}_{n-k}\right)\frac{t^n}{F_n!},$$

and

$$\left(\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}\right)e^{f_{\mathbb{F}}t}=\left(\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{F_n!}\right)\left(1+\sum_{n=1}^{\infty}\mathbb{F}_n\frac{t^n}{F_n!}\right)$$
$$=\left(\sum_{n=0}^{\infty}E_{n,F}\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}\mathbb{F}_n(x)\frac{t^n}{F_n!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}\,\mathbb{F}_{n-k}(x)\right)\frac{t^n}{F_n!}.$$

Thus,

$$\sum_{n=0}^{\infty}E_{n,F}\left(x\right)\frac{t^n}{F_n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}\,\mathbb{F}_{n-k}(x)\right)\frac{t^n}{F_n!}-\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F E_{k,F}\left(x\right)\,\mathbb{F}_{n-k}\right)\frac{t^n}{F_n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F\left(E_{k,F}\,\mathbb{F}_{n-k}(x)-E_{k,F}\left(x\right)\,\mathbb{F}_{n-k}\right)\frac{t^n}{F_n!}.$$

Comparing the coefficients of $t^n/F_n!$, on both sides of the above equations we arrive at the desired result. □

Table 2 Harmonic based F-exponential generating functions for some polynomials and numbers

	Polynomial/Number	F-EGF	Harmonic based F-EGF
$B_{n,F}(x)$	Bernoulli-F Polynomial	$\frac{e_F^{xt}\left(e_F^t-1\right)}{t}$	$\left(\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1\right) e_F^{xt}$
$E_{n,F}(x)$	Euler-Fibonacci Polynomial	$\frac{2e_F^{xt}}{e_F^t+1}$	$\frac{2e_F^{xt}}{2+t\left(\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1\right)}$
$E_{n,F}$	Euler-Fibonacci Number	$\frac{2}{e_F^t+1}$	$\frac{2}{2+t\left(\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1\right)}$
$B_n^F(x)$	Bernoulli-Fibonacci Polynomial	$\frac{te_F^{xt}}{e_F^t-1}$	$\frac{e_F^{xt}}{\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1}$
B_n^F	Bernoulli-Fibonacci Number	$\frac{t}{e_F^t-1}$	$\frac{1}{\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1}$
$\mathbb{F}_n(x)$	Harmonic Fibonacci Polynomial	–	$e_F^{xt} e^{f_{\mathbb{F}} t}$
$\mathbb{F}_n$	Harmonic Fibonacci Number	–	$e^{f_{\mathbb{F}} t}-1$

Theorem 9 *For Bernoulli-Fibonacci numbers B_n^F, harmonic based F -exponential function as follows:*

$$\sum_{n=0}^{\infty} B_n^F \frac{t^n}{F_n!}=\frac{1}{\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1}.$$

Proof Using (1.1), we have

$$\begin{aligned}
e_F^t-1 &= \sum_{n=1}^{\infty} \frac{t^n}{F_n!} \\
&= \sum_{n=0}^{\infty} \frac{t^{n+1}}{F_{n+1}!} \\
&= t \sum_{n=0}^{\infty} \frac{1}{F_{n+1}} \frac{t^n}{F_n!} \\
&= t \sum_{n=0}^{\infty} \left(\mathbb{F}_{n+1}-\mathbb{F}_n\right) \frac{t^n}{F_n!} \\
&= t\left(1+\sum_{n=1}^{\infty} \mathbb{F}_{n+1} \frac{t^n}{F_n!}-\sum_{n=1}^{\infty} \mathbb{F}_n \frac{t^n}{F_n!}\right) \\
&= t\left(\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1\right).
\end{aligned} \tag{3.4}$$

From (1.5), we get

$$\sum_{n=0}^{\infty} B_n^F \frac{t^n}{F_n!}=\frac{1}{\partial_{F,t} e^{f_{\mathbb{F}} t}-e^{f_{\mathbb{F}} t}+1}.$$

□

Remark 1 We give the relationship of the first few Bernoulli-F polynomials $B_{n,F}(x)$, harmonic Fibonacci polynomials $\mathbb{F}_n(x)$ and Euler-Fibonacci polynomials $E_{n,F}(x)$ with the graphics in the following figures.

Remark 2 We give the above table for the convenience of the reader.

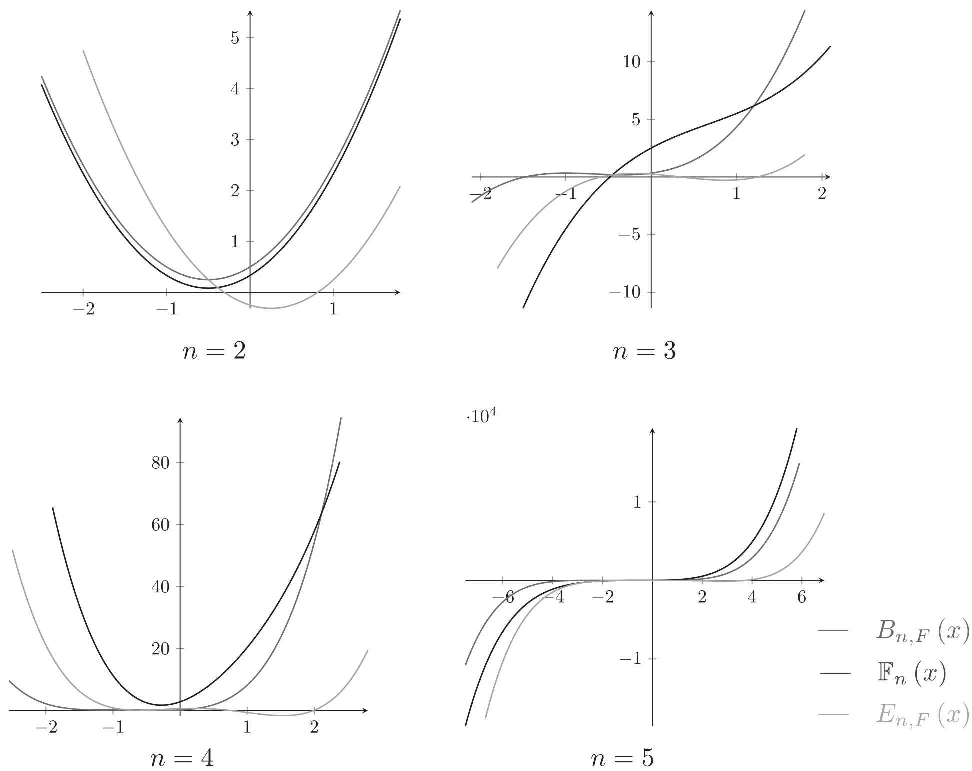

Fig. 1 Graphs of $B_{n,F}(x)$, $\mathbb{F}_n(x)$ and $E_{n,F}(x)$ for $n = 2, 3, 4, 5$

4 Conclusion

In our present research, we have obtained harmonic Fibonacci numbers using F-integral. We have defined the harmonic Fibonacci polynomials. By virtue of harmonic F-vacuum operator, we have obtained harmonic based F- exponential generating functions of the harmonic Fibonacci numbers, the harmonic Fibonacci polynomials, the Euler-Fibonacci numbers, the Euler-Fibonacci polynomials and the Bernoulli-Fibonacci numbers. We have obtained the Bernoulli-F polynomials with a different method and given the relationship between the harmonic Fibonacci numbers and the harmonic Fibonacci polynomials of the Bernoulli-F polynomials. We have shown the relationships between harmonic Fibonacci numbers and harmonic Fibonacci polynomials of the Euler-Fibonacci polynomials.

References

1. S. Roman, The Umbral Calculus; Academic Press: New York, NY, USA, pp. 93-100, (1984).
2. T. Koshy, Fibonacci and Lucas Numbers with Applications, 2nd ed.; Pure and Applied Mathematics; John Wiley & Sons, Inc.: Hoboken, NJ, USA, Volume 1, (2018).
3. S. Vajda, Fibonacci and Lucas Numbers, and the Golden Section, Theory and applications, Ellis Horwood Limited; (1989).
4. A. Dil, I. Mezo, A symmetric algorithm for hyperharmonic and Fibonacci numbers, Appl. Math. Comput. 206, 942–951 (2008)
5. E. Krot, An Introduction to Finite Fibonomial Calculus, Centr. Eur. J. Math., 2, 754-766, (2004).
6. N. Tuglu, C. Kızılates and S. Kesim On the harmonic and hyperharmonic Fibonacci numbers, Adv. Difference Equ., 2015:253, 1-12, (2015).
7. G. Ozdemir , Y. Simsek, Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers, Filomat, 30(4), 969-975, (2016).
8. G. Ozdemir, Y. Simsek, G. V. Milovanović, Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials, Mediterr. J. Math., 14(117), 1-17, (2017).
9. M. Ozvatan, Generalized Golden-Fibonacci Calculus and Applications, Graduate School of Engineering and Sciences of Izmir Institute of Technology, Master Thesis, July (2018).
10. O.K. Pashaev, M. Ozvatan, Bernoulli-Fibonacci Polynomials. arXiv preprint arXiv:2010.15080 (2020).
11. S. Kus, N. Tuglu and T. Kim, Bernoulli F-polynomials and Fibo–Bernoulli matrices, Adv. Difference Equ., 2019:145, (2019).

12. G. Dattoli, B. S. Germano, Licciardi and Martinelli M. R., Umbral Methods and Harmonic Numbers, Axioms, 7, 62,(2018).
13. G. Dattoli, H.M. Srivastava, A note on harmonic numbers, umbral calculus and generating functions, Appl. Math. Lett., 21, 686-693, (2008).
14. G. Dattoli, M. Migliorati, Operational Methods, Harmonic Numbers and Riemann Function, Internal Report Enea Frascati (Rome) RT/2008/29/FIM, Edizioni Scientifiche-ENEA Centro Ricerche Frascati: Rome, Italy, (2008).
15. G. Dattoli, K.Zhukovsky, Umbral Methods, Combinatorial Identities and Harmonic Numbers, Appl. Math., 1, 46–49, (2011).
16. D. Cvijovic, The Dattoli-Srivastava conjectures concerning generating functions involving the harmonic numbers, Appl. Math. Comput., 215, 4040-4043, (2010).
17. N. Kilar, Y. Simsek, A special approach to derive new formulas for some special numbers and polynomials, Turk. J. Math., 44(6):2217-2240, (2020).
18. N. Kilar, Y. Simsek, Identities and relations for Hermite-based Milne–Thomson polynomials associated with Fibonacci and Chebyshev polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM 115, 28 (2021).
19. S. Licciardi, Umbral Calculus a Different Mathematical Language, arXiv:1803.03108v1, 24 Feb (2018).
20. P. N. Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, Results Math., 73, 39, (2018).