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ORIGINAL RESEARCH

Cuspidal crosscaps and folded singularities on a maxface and a minface

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Abstract For a given zero mean curvature surface X (in the Lorentz Minkowski space) having folded singularity, we construct a family of maxface and minface, having increasing cuspidal crosscaps, 'converging' to X. We include a general discussion of this.

Keywords Mix-type zmc surface · Zero mean curvature · Maxface · Minface

Mathematics Subject Classification 53A35

1 Introduction

Space-like (time-like) zero mean curvature surfaces in the Lorentz-Minkowski space $\mathbb{E}^3_1$ are locally area maximizing (minimizing respectively) surfaces. These are similar to the minimal surface in $\mathbb{R}^3$. But in contrast to the minimal surface in $\mathbb{R}^3$, these surfaces appear with non isolated singularities [1,4,6,10]. The generalized maximal (time-like minimal) immersion having only non-isolated singularities are called maxface (minface respectively).

These non-isolated singularities are of various types. As front or frontal, on these surfaces, we see singularities as cuspidal edge, cuspidal crosscaps, swallowtails, etc. There are other space-like (time-like) zero mean curvature surfaces that are neither front nor frontal but appear with other types of singularities, such as folded singularities, cone-like, etc.

In [7], the authors gave an example of a family of maxface X_n for each natural number n, that has swallowtails in an increasing order. Further, in [5], the authors talked about Trinoids (a maxface) with swallowtails whose computer graphics look cone-like singularities. Moreover, they remarked that the computer graph of many swallowtails together looks cone-like. In [3], the authors justified this phenomenon by producing a sequence of maxface with an increasing number of swallowtails, converging to the maxface with cone-like singularity.

We see similar phenomena (at least graphically) in cuspidal crosscaps. The graph of cuspidal crosscaps looks like the folded singularity. Similar to swallowtails and cone-like discussed in [3], we can ask a question: given a maxface (minface) having folded singularity, does there exist a family of maxface (minface respectively) having an increasing number of cuspidal crosscaps and 'converging' to the former one?

In this article, in section 4, we give such a family and prove the following:

Theorem *(Theorem 4.1) Let X be a maxface (minface) having folded singularity. Then there is a sequence X_n of maxfaces (sequence Y_n of minfaces) having an increasing number of cuspidal crosscaps, and the sequence 'converges' (in the norm $\|.\|_\Omega$) to X.*

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To construct such a family, we will find the condition on the singular Björing data to have a cuspidal crosscap. For the maxface, such conditions are already derived in [13]. In this article, in section 3, we find similar conditions on the singular Björling data for the minface, such that they have a cuspidal edge, swallowtails, and cuspidal crosscaps.

Besides finding the sequence above, such conditions may be useful in various situations. For example, these are required initial conditions to find interpolating surfaces containing given two curves with the prescribed nature of singularities.

2 Preliminaries

The Lorentz Minkowski space is a vector space $\mathbb{R}^3$ with the metric $ds^2 := dx^2 + dy^2 - dz^2$; we denote it by $\mathbb{E}^3_1$. Space-like zero mean curvature immersion is called maximal immersion. If we allow singularities, then we call it a generalized maximal immersion. Moreover, if singularities are due to those points where the limiting tangent plane contains a light-like vector, then we call such a generalized maximal immersion as maxface [14]. In the following, we will discuss two methods to construct the generalized maximal immersion [2,4,8,14].

2.1 Weierstrass-Enneper representation for the generalized maximal immersion

Let M^2 be a Riemann surface. Let g be a meromorphic function and ω a holomorphic 1-form on M^2 such that $Re\int_\gamma \left(\frac{1}{2}(1+g^2), \frac{i}{2}(1-g^2), -g\right)\omega = 0$ for all loops γ on M^2. Then the map

$$f(z) := Re\int_{z_0}^{z}\left(\frac{1}{2}(1+g^2), \frac{i}{2}(1-g^2), -g\right)\omega$$

is well-defined and gives a generalized maximal immersion [4,14]. Moreover, any generalized maximal immersion can be written in this manner.

2.2 Singular Björling representation for the generalized maximal immersion

Let $\gamma : I \to \mathbb{E}^3_1$ be a real analytic null curve and $L : I \to \mathbb{E}^3_1$ be a real analytic null vector field such that $\langle\gamma'(u), L(u)\rangle = 0$ and $\gamma' \not\equiv 0$ or $L \not\equiv 0$. Let $g(z)$ be the analytic extension of

$$g(t) := \begin{cases} \frac{\gamma_1'(t)+i\gamma_2'(t)}{\gamma_3'(t)} & \text{if } \gamma_3'(t) \not\equiv 0 \\ \frac{L_1(t)+iL_2(t)}{L_3(t)} & \text{if } L_3(t) \not\equiv 0. \end{cases}$$

defined on $U \subset \mathbb{C}$, $I \subset U$. If $|g(z)| \not\equiv 1$, then the map

$$X_{max,\gamma,L}(u+iv) := \frac{1}{2}(\gamma(u+iv) + \gamma(u-iv)) - \frac{i}{2}\int_{u-iv}^{u+iv} L(w)dw \tag{2.1}$$

is the generalized maximal surface with $u-v$ as a conformal parameter and $X_{max,\gamma,L}(u,0) = \gamma(u)$, $\frac{\partial}{\partial v}|_{(u,0)}X_{max,\gamma,L} = L(u)$ [2,8].

Similar to the maximal immersion, we have time-like zero mean curvature immersion and minface. Below we will recall the Weierstrass Enneper representation and the Björling formula for a minface [1,9].

2.3 Weierstrass-Enneper representation for minface [1]

Let M^2 be a smooth manifold. A smooth map $X : M^2 \to \mathbb{E}^3_1$ is a minface if and only if at each point of M^2, there exists a local coordinate system (u, v) in a domain U, the functions $g_1(u)$, $g_2(v)$, and the 1-forms $\omega_1 = \hat{\omega}_1 du$, $\omega_2 = \hat{\omega}_2 dv$ such that

1. $g_1 g_2 \not\equiv 1$ on an open dense subset of U
2. $\hat{\omega}_1 \neq 0$, $\hat{\omega}_2 \neq 0$ at each point on U.

Table 1 Criterion on the Singular Björling data for maxface

Nature of singularities at p	γ'	L	γ''	γ'''	L'	L''	$\gamma_1'\gamma_2''-\gamma_1''\gamma_2'$	$L_1L_2'-L_1'L_2$
Function's value at p								
Cuspidal-edge	$\neq 0$	$\neq 0$	–	–	–	–	$\neq 0$	$\neq 0$
Swallowtails	$=0$	$\neq 0$	$\neq 0$	–	–	–	–	$\neq 0$
Cuspidal-Crosscaps	$\neq 0$	$=0$	–	–	$\neq 0$	–	$\neq 0$	–

Moreover, X can be decomposed into two regular null curves.

$$X(u,v)=\frac{1}{2}\left(\int_{u_0}^{u}(2g_1, 1-{g_1}^2, -1-{g_1}^2)\omega_1+\int_{v_0}^{v}(-2g_2, 1-{g_2}^2, 1+{g_2}^2)\omega_2\right)+X(u_0,v_0). \quad (2.2)$$

The quadruple $(g_1, g_2, \omega_1, \omega_2)$ is called the real Weierstrass data. The set of singular points on U of a minface X correspond to the set $\{(u,v)\in U : g_1(u)g_2(v)=1\}$.

2.4 Singular Björling representation for the generalized time-like minimal surface

Let $\gamma : (a,b)\to \mathbb{E}_1^3$ be a smooth null curve, and $L : (a,b)\to \mathbb{E}_1^3$ be a smooth null vector field such that $\langle \gamma'(t), L(t)\rangle \equiv 0$ and $\gamma'\not\equiv 0$ or $L\not\equiv 0$ and $\gamma'\not\equiv \pm L$. Then the map

$$X_{min,\gamma,L}(u,v) := \frac{1}{2}(\gamma(u+v)+\gamma(u-v))+\frac{1}{2}\int_{u-v}^{u+v}L(s)ds, \quad (2.3)$$

defined on $\lozenge_{a,b} := \{(u,v)\in\mathbb{R}^2 : a<u+v<b, \text{ and } a<u-v<b\}$ is a generalized time-like minimal surface. Furthermore, $X_{min,\gamma,L}(u,0)=\gamma(u)$, $\frac{\partial}{\partial v}X_{min,\gamma,L}(u,0)=L(u)$, $\forall u\in(a,b)$.

Both maxface and minface admit various singularities. In the table (1), we mention the criterion given in [13] on the singular Björling data for maxface having various singularities. But for the minface we do not have a criterion to check the type of singularity in terms of the Björling data till now.

For the minface, the criterion to check the type of singularity in terms of the Weierstrass data is given in Fact 4.1 of [1]. We recall it here. The minface with Weierstrass data $(g_1, \hat{\omega_1}, g_2, \hat{\omega_2})$ has

(1) Cuspidal edge at p if and only if at p,

$$\frac{g_1'}{g_1^2\hat{\omega_1}}-\frac{g_2'}{g_2^2\hat{\omega_2}}\neq 0,\ \frac{g_1'}{g_1^2\hat{\omega_1}}+\frac{g_2'}{g_2^2\hat{\omega_2}}\neq 0. \quad (2.4)$$

(2) Cuspidal crosscap at p if and only if at p,

$$\frac{g_1'}{g_1^2\hat{\omega_1}}-\frac{g_2'}{g_2^2\hat{\omega_2}}=0,\ \frac{g_1'}{g_1^2\hat{\omega_1}}+\frac{g_2'}{g_2^2\hat{\omega_2}}\neq 0,\ \text{ and } \left(\frac{g_1'}{g_1^2\hat{\omega_1}}\right)'\frac{g_2'}{g_2}+\left(\frac{g_2'}{g_2^2\hat{\omega_2}}\right)'\frac{g_1'}{g_1}\neq 0. \quad (2.5)$$

(3) Swallowtails at p if and only if at p,

$$\frac{g_1'}{g_1^2\hat{\omega_1}}-\frac{g_2'}{g_2^2\hat{\omega_2}}\neq 0,\ \frac{g_1'}{g_1^2\hat{\omega_1}}+\frac{g_2'}{g_2^2\hat{\omega_2}}=0,\ \text{ and } \left(\frac{g_1'}{g_1^2\hat{\omega_1}}\right)'\frac{g_2'}{g_2}+\left(\frac{g_2'}{g_2^2\hat{\omega_2}}\right)'\frac{g_1'}{g_1}\neq 0. \quad (2.6)$$

The above singularities of a maxface and minface appear when we consider these as front or frontal. Now we discuss another type of singularity called folded singularity that appears on maxface and minface but not as frontal or front.

Definition 2.1 (Folded singularity) [8]. The $X_{max,\gamma,L}$ and $X_{min,\gamma,L}$ have folded singularities along γ if $L=0$ and γ is non degenerate curve.

There are many other singularities, but we recalled only selected one, as per the required discussion of this article. For more details on the topic, we refer [12] and [9].

3 Conditions on the singular Björling data for minface

We start with $\{\gamma, L\}$ the singular Björling data such that

- At those points where $\gamma' \neq 0$, we have $\gamma_2' \neq \pm\gamma_3'$, and
- A those points where $L \neq 0$, we have $L_2 \neq \pm L_3$.

Let $X_{min,\gamma,L}$ be the generalized time-like minimal surface as in (2.3). Below we will find the Weierstrass quadruple for $X_{min,\gamma L}$ in terms of γ and L.

3.1 Relation between the singular Björling data and Weierstrass quadruple for the minface

After changing the coordinate (u, v) to (t, s) by $s = u + v$ and $t = u - v$ and for some $x_0 \in (a, b)$, $X_{min,\gamma,L}$ as in (2.3), changes to

$$\tilde{X}(t, s) = \frac{1}{2}\left(\gamma(t) + \int_t^{x_0} L(\eta)d\eta\right) + \frac{1}{2}\left(\gamma(s) + \int_{x_0}^{s} L(\eta)d\eta\right). \tag{3.1}$$

We take $\alpha(t) := \frac{1}{2}\left(\gamma(t) + \int_t^{x_0} L(\eta)d\eta\right)$ and $\beta(s) := \frac{1}{2}\left(\gamma(s) + \int_{x_0}^{s} L(\eta)d\eta\right)$. It turns out that both α and β are null curves. $\alpha'(t) = \frac{1}{2}(\gamma'(t) - L(t))$, and $\beta'(s) = \frac{1}{2}(\gamma'(s) + L(s))$.

For those point t, where $\gamma' \neq 0$, we have $L(t) = c(t)\gamma'(t)$. This gives

$$\alpha_2'(t) - \alpha_3'(t) = \frac{1}{2}(\gamma_2'(t) - \gamma_3'(t))(1 - c(t))$$
$$\beta_2'(s) + \beta_3'(s) = \frac{1}{2}(\gamma_2'(s) + \gamma_3'(s))(1 + c(s)).$$

Similarly at those points where $L \neq 0$, $\gamma'(t) = d(t)L(t)$, we get

$$\alpha_2'(t) - \alpha_3'(t) = \frac{1}{2}(L_2(t) - L_3(t))(-1 + d(t))$$
$$\beta_2'(s) + \beta_3'(s) = \frac{1}{2}(L_2(s) + L_3(s))(1 + d(s)).$$

The conditions on the singular Björling data as mentioned in the starting of this section and in the subsection 2.4, imply that for all points $\alpha_2' - \alpha_3' \neq 0$, $\beta_2' + \beta_3' \neq 0$.

Let $\left(g_1, g_2, \hat{\omega_1}, \hat{\omega_2}\right)$ be the Weierstrass quadruple for $\tilde{X}$. $\tilde{X}$ has the representation as in (2.2). By comparing the two representations (2.2) and (3.1), we get components of α' and β' as the following:

$$\begin{aligned}
&\alpha_1'(u) = 2g_1(u)\hat{\omega_1}(u), \quad \alpha_2'(u) = (1 - {g_1}^2(u))\hat{\omega_1}(u),\\
&\alpha_3'(u) = (-1 - {g_1}^2(u))\hat{\omega_1}(u) && (3.2)\\
&\beta_1'(v) = -2g_2(v)\hat{\omega_2}(v), \quad \beta_2'(v) = (1 - {g_2}^2(v))\hat{\omega_2}(v),\\
&\beta_3'(v) = (1 + {g_2}^2(v))\hat{\omega_2}(v). && (3.3)
\end{aligned}$$

From (3.2) and since $\hat{\omega_1} \neq 0$, we get $g_1 = \frac{\alpha_1'}{2\hat{\omega_1}}$. Also $\alpha_2' = (1 - g_1^2)\hat{\omega_1}$. Solving this, we get $\hat{\omega_1} = \frac{\alpha_2' \pm \alpha_3'}{2}$ and then $g_1 = \frac{\alpha_1'}{\alpha_2' \pm \alpha_3'}$.

Now for the case when $g_1 = \frac{\alpha_1'}{\alpha_2' - \alpha_3'}$ and $\hat{\omega_1} = \frac{\alpha_2' - \alpha_3'}{2}$, we get

$$-(1 + g_1^2)\hat{\omega_1} = \frac{-1}{2}\left[\frac{\alpha_2'^2 + \alpha_3'^2 - 2\alpha_2'\alpha_3' + \alpha_1'^2}{\alpha_2' - \alpha_3'}\right] = \alpha_3'.$$

For other choices of $\hat{\omega_1}, g_1$, we see that $-(1 + g_1^2)\hat{\omega_1} \neq \alpha_3'$. Therefore we must have

$$g_1 = \frac{\alpha_1'}{\alpha_2' - \alpha_3'} \text{ and } \hat{\omega_1} = \frac{\alpha_2' - \alpha_3'}{2}.$$

Similarly we have

$$g_2(v) = \frac{-\beta_1'}{\beta_2' + \beta_3'} \text{ and } \hat{\omega_2}(v) = \frac{\beta_2' + \beta_3'}{2}.$$

Now to get the Weierstrass quadruple in terms of the singular Björling data, we replace the values of α and β in terms of γ and L. We will do this case by case.

If at u, $\gamma' \neq 0$, then $L(u) = c(u)\gamma'(u)$ for the real function $c = \frac{L_3}{\gamma_3'}$ and we get,

$$\hat{\omega_1} = \frac{\gamma_2' - \gamma_3'}{4}(1-c), \quad g_1 = -\frac{\gamma_2' + \gamma_3'}{\gamma_1'}, \quad \hat{\omega_2} = \frac{\gamma_2' + \gamma_3'}{4}(1+c), \quad g_2 = \frac{\gamma_2' - \gamma_3'}{\gamma_1'}. \tag{3.4}$$

We recall here the fact that at points where $\gamma' \neq 0$, we have $\gamma_3' \neq 0$, and as $\gamma_2' \neq \gamma_3'$, and γ a null curve, this gives $\gamma_1' \neq 0$.

Similarly when $L \neq 0$, with $d = \frac{\gamma_3'}{L_3}$, we have

$$\hat{\omega_1} = \frac{L_2 - L_3}{4}(-1+d), \quad g_1 = -\frac{L_2 + L_3}{L_1}, \quad \hat{\omega_2} = \frac{L_2 + L_3}{4}(1+d), \quad g_2 = \frac{L_2 - L_3}{L_1}. \tag{3.5}$$

Thus, we found the Weierstrass quadruple of $X_{min,\gamma,L}$ in terms of $\{\gamma, L\}$. In the next subsection we will convert conditions in terms of the singular Björling data so that a time-like minimal surface has various singularities.

3.2 Conditions on the singular Björling data of minface for various singularity

Let $\{\gamma, L\}$ be the singular Björling data for the generalized time-like minimal surface $X_{min,\gamma,L}$, as defined in the section 2.4. We denote

$$\delta_1 := \frac{g_1'}{g_1^2 \hat{\omega_1}}, \delta_2 := \frac{g_2'}{g_2^2 \hat{\omega_2}}, D(\gamma_{23}', \gamma_{23}'') := \gamma_2'\gamma_3'' - \gamma_3'\gamma_2'', D(L_{23}L_{23}') := L_2L_3' - L_3L_2'.$$

We again start with the case when $\gamma' \neq 0$ and $L = c\gamma'$. Using (3.4) we have

$$\begin{aligned}
\delta_1 &= \frac{4g_1'}{g_1^2(1-c)(\gamma_2' - \gamma_3')} \\
&= \frac{4[(\gamma_2'' + \gamma_3'')\gamma_1' - \gamma_1''(\gamma_2' + \gamma_3')]}{(\gamma_1'^2)(1-c)(\gamma_2' + \gamma_3')} \\
&= \frac{4}{\gamma_1'^3(1-c)}\left[-\gamma_1''\gamma_1' - \gamma_2''\gamma_2' + \gamma_3''\gamma_3' + \gamma_2'\gamma_3'' - \gamma_3'\gamma_3''\right] \\
&= \frac{-4}{\gamma_1'^3(1-c)}\left[\langle\gamma', \gamma''\rangle + D(\gamma_{23}', \gamma_{23}'')\right].
\end{aligned}$$

Since γ is a singular curve, $\langle\gamma', \gamma''\rangle = 0$. This gives $\delta_1 = -4\frac{D(\gamma_{23}', \gamma_{23}'')}{\gamma_1'^3(1-c)}$.

Moreover at those points where $c(u) = 0$, we have

$$\delta_1' = -4\frac{\gamma_1' D(\gamma_{23}', \gamma_{23}''') - D(\gamma_{23}', \gamma_{23}'')(3\gamma_1'' - \gamma_1'c')}{\gamma_1'^4}.$$

A similar calculation gives $\delta_2 = -4\frac{D(\gamma_{23}', \gamma_{23}'')}{\gamma_1'^3(1+c)}$, and at those points where $c(u) = 0$,

$$\delta_2' = -4\frac{\gamma_1' D(\gamma_{23}', \gamma_{23}''') - D(\gamma_{23}', \gamma_{23}'')(3\gamma_1'' + \gamma_1'c')}{\gamma_1'^4}.$$

As $c(u)=\frac{L_3(u)}{\gamma_3'(u)}$, we get

$$\begin{aligned}\delta_1+\delta_2&=\frac{g_1'}{g_1^2\hat{\omega}_1}+\frac{g_2'}{g_2^2\hat{\omega}_2}\\&=\frac{-8}{\gamma_1'^3(1-c^2)}\left[\langle\gamma',\gamma''\rangle c+D(\gamma_{23}',\gamma_{23}'')\right]\\&=\frac{-8\gamma_3'}{\gamma_1'^3(\gamma_3'^2-L_3'^2)}\left[\langle\gamma',\gamma''\rangle L_3+\gamma_3' D(\gamma_{23}',\gamma_{23}'')\right]\\&=\frac{-8\gamma_3'^2 D(\gamma_{23}',\gamma_{23}'')}{\gamma_1'^3(\gamma_3'^2-L_3'^2)},\ \text{and}\\\delta_1-\delta_2&=\frac{g_1'}{g_1^2\hat{\omega}_1}-\frac{g_2'}{g_2^2\hat{\omega}_2}=\frac{-8\gamma_3' L_3 D(\gamma_{23}',\gamma_{23}'')}{\gamma_1'^3(\gamma_3'^2-L_3'^2)}.\end{aligned}$$

Below we find the conditions for a generalized time-like minimal surface having singularities like cuspidal edge, cuspidal crosscaps and swallowtails in term of singular Björling data $\{\gamma, L\}$.

3.2.1 Cuspidal edge

Using (2.4), the time-like minimal surface $X_{min,\gamma,L}$ for the data $\{\gamma, L\}$ has cuspidal edge at u if and only if $\gamma'\neq 0$, $L\neq 0$, $D(\gamma_{23}',\gamma_{23}'')\neq 0$.

3.2.2 Cuspidal crosscaps

At those point where $\gamma'\neq 0$, $\delta_1-\delta_2=0$, and $\delta_1+\delta_2\neq 0$ give

$$\gamma_3'\neq 0,\quad L_3=0,\quad D(\gamma_{23}',\gamma_{23}'')\neq 0. \tag{3.6}$$

Moreover at those point where (3.6) satisfies, we have

$$\delta_1'\frac{g_2'}{g_2}+\delta_2'\frac{g_1'}{g_1}=\frac{g_1'}{g_1}(\delta_2'-\delta_1')=-\frac{4}{\gamma_1'^6}D(\gamma_{23}',\gamma_{23}'')\left[D(\gamma_{23}',\gamma_{23}'')(2c'\gamma_1')\right]$$

and $c'=\frac{L_3'\gamma_3'-L_3\gamma_3''}{\gamma_3'^2}$. Therefore

$$\delta_1'\frac{g_2'}{g_2}+\delta_2'\frac{g_1'}{g_1}\neq 0 \iff c'\neq 0 \iff L'\neq 0.$$

Using (2.5), at u, where $\gamma'\neq 0$, the time-like minimal surface $X_{min,\gamma,L}$ for the data $\{\gamma, L\}$ has cuspidal crosscap if and only if $\gamma'\neq 0$, $L=0$, $L'\neq 0$, and $D(\gamma_{23}',\gamma_{23}'')\neq 0$.

3.2.3 Swallowtails

Similar to the cuspidal crossscaps, at u, where $\gamma'\neq 0$, the time-like minimal surface $X_{min,\gamma,L}$ for the data $\{\gamma, L\}$ has swallowtails if and only if $\gamma'(u)=0$, $L(u)\neq 0$, $\gamma''(u)\neq 0$ and $D(L_{23}, L_{23}')\neq 0$.

For a non-degenerate curve (the curve is non-degenerate if and only if for all $u\in I$, $D(\gamma_{23}',\gamma_{23}'')\neq 0$ or $D(\gamma_{12}',\gamma_{12}'')\neq 0$), these conditions turn out to be similar to the one given in the table (1) for the maxface.

4 Family of maxfaces and minfaces

We start this section with an example: Helicoid. This is an example of how the time-like and space-like Helicoid patches smoothly along the folded singularity. For details, we refer to [11].

4.1 Mix-type Helicoid and a sequence

Let $\gamma : I \to \mathbb{E}_1^3;\ \ \gamma(t) := (\cos t, \sin t, t)$. With $L = 0$, $\{\gamma, L\}$ turns out to be the singular Björling data for both generalized time-like minimal surface and generalized space-like maximal surface.

For this data, we have the following generalized space-like maximal surface

$$X_{max,\gamma,L}(u, v) = (\cos u \cosh v, \sin u \cosh v, u) \tag{4.1}$$

on Ω and γ is its singular curve corresponding to the points $v = 0$. Also, for the same data, we have a generalized time-like minimal surface as following

$$X_{min,\gamma,L}(u, v) = (\cos u \cos v, \sin u \cos v, u). \tag{4.2}$$

The generalized immersions as in (4.1), (4.2) are parametrizations of the helicoid of type space-like and time-like, respectively. We see that both these surfaces contain γ and have folded singularity on I.

Without loss of generality we assume that $I = (0, 2\pi)$ and let $r > 0$ be such that $\overline{\Omega}, \overline{\Diamond} \subset B(0, r)$. With this r and $n \geq 1$, we take the singular Björling data as

$$\left\{\gamma, \gamma' \frac{1}{n(r+1)^n} \prod_{i=1}^{n} (w - \frac{1}{i})\right\}. \tag{4.3}$$

For each $n \in \mathbb{N}$, let $X_{max,\gamma,L,n}$, be the generalized maximal immersion as in (2.1) and $X_{min,\gamma,L,n}$ be the generalized time-like minimal immersion as in (2.3). This defines a pair $\{X_{max,\gamma,L,n}, X_{min,\gamma,L,n}\}$ of generalized space-like maximal surfaces and the generalized time-like minimal surface containing γ as a singular curve at $v = 0$.

Using the condition we found in section 3, it is easy to verify that both the generalized maximal immersions and generalized minimal immersions $\{X_{max,\gamma,L,n}, X_{min,\gamma,L,n}\}$ has n cuspidal crosscaps at $t = \frac{1}{k}$. Moreover corresponding cuspidal crosscaps are $\{(\cos\frac{1}{k}, \sin\frac{1}{k}, \frac{1}{k})\}_{k=1}^{n} \subset \gamma(I)$.

Helicoid is an example of a mix-type zero mean curvature surface, where the casual character changes along the folded singularity (see [9]). We have generated a sequence of immersions $\{X_{max,\gamma,L,n}, X_{min,\gamma,L,n}\}_{n\in\mathbb{N}}$ with the increasing number of cuspidal crosscaps. For each n, $X_{max,\gamma,L,n}$ and $X_{min,\gamma,L,n}$ are joined along γ. Moreover the sequence $\{X_{max,\gamma,L,n}\}$ 'converges' towards the mix-type helicoid (that is smoothly patched along a folded singularity) as $n \to \infty$.

In the next subsection, we will start with a generalized mixed-type surface that is smoothly joined along a folded singularity (for example: helicoid). Next, we will find a sequence of mixed-type surfaces joined along a null curve with increasing cuspidal crosscaps and this sequence 'converges' to the surface we started with.

4.2 Family for a general mix-type zero mean curvature surfaces

Authors in [3,13] have defined and discussed the convergence of maxfaces in a particular norm. We would recall the norm here. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain, $\overline{\Omega}$ be its closure. Let $X \in C(\overline{\Omega}, \mathbb{R}^3)$, the space of continuous maps. For each $z \in \overline{\Omega}$, we define

$$\|X(z)\| := \max\{|X_1(z)|, |X_2(z)|, |X_3(z)|\} \text{ and } \|X\|_{\Omega} := \sup_{z\in\overline{\Omega}} \|X(z)\|.$$

We start with the singular Björling data $\{\gamma, L = 0\}$, defined on a closed and bounded interval I, without loss of generality we will assume $I = [0, 1]$. Moreover, $X_{max,\gamma,0}$ and $X_{min,\gamma,0}$ be the generalized maximal immersion and generalized time-like minimal immersion, respectively, as in (2.1) and (2.3) defined on some bounded open subsets Ω_1 and Ω_2 (of $\mathbb{R}^2$) that contains I. We also assume both have folded singularity along γ. Since $L = 0$, $X_{max,\gamma,0}$ and $X_{min,\gamma,0}$ have folded singularity at $v = 0$. Therefore γ is non-degenerate (see definition 2.1 and the curve is non-degenerate if and only if for all $u \in I$, $D(\gamma_{23}', \gamma_{23}'') \neq 0$ or $D(\gamma_{12}', \gamma_{12}'') \neq 0$).

As both Ω_1 and Ω_2 are bounded and $I = [0, 1] \subset \Omega_1 \cap \Omega_2$, there exists some $r > 0$ and $\Omega, \lozenge$ such that $I \subset \Omega \subset \overline{\Omega_1} \subset B(0, r)$, and $I \subset \lozenge \subset \overline{\Omega_2} \subset B(0, r)$. With this r and $n \geq 1$, we take the singular Björling data as in (4.3). For this data, we have corresponding immersions as follows.

$$X_{max,\gamma,L,n}(u, v) = \frac{1}{2}(\gamma(u + iv) + \gamma(u - iv)) - \frac{i}{2n(r + 1)^n} \int_{u-iv}^{u+iv} \gamma'(s) \prod_{k=1}^{n}(s - \frac{1}{k})ds \tag{4.4}$$

on Ω_1.

$$X_{min,\gamma,L,n}(u, v) = \frac{1}{2}(\gamma(u + v) + \gamma(u - v)) + \frac{1}{2n(r + 1)^n} \int_{u-v}^{u+v} \gamma'(s) \prod_{k=1}^{n}(s - \frac{1}{k})ds \tag{4.5}$$

on Ω_2.

Both the generalized immersions have cuspidal-crosscap singularities at $\{\frac{1}{k} : k \in \{1, 2, ..., n\}\}$ for each n. Moreover following the similar calculations as in [3,13], we see that

$$\|X_{max,\gamma,L,n} - X_{max,\gamma,0}\|_{\Omega_1} \leq M(\|\gamma_n^{k\prime} - \gamma^{k\prime}\|_{\Omega_1} + \|L_n^k - L^k\|_{\Omega_1}) \leq M\|L_n\|_{\Omega_1} \to zero.$$

Similarly, we have $\|X_{min,\gamma,L,n} - X_{min,\gamma,0}\|_{\Omega_2} \leq M'\|L_n\|_{\Omega_2} \to zero.$

Thus we summarize all our discussions in this section in the following theorem.

Theorem 4.1 *Let $\{X_{max}, X_{min}\}$ be a maxface and a minface having a folded singularity along γ. Then there exists two sequences $X_{max,\gamma,L,n}$ and $X_{min,\gamma,L,n}$ of maxfaces and minfaces respectively such that*

(1) On the trace of γ both immersions have an increasing number of cuspidal crosscaps at the same points.
(2) $X_{max,\gamma,L,n}$ converges to X_{max} in $\|.\|_{\Omega_1}$ and $X_{min,\gamma,L,n}$ converges to X_{min} in $\|.\|_{\Omega_2}$.

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