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ORIGINAL RESEARCH

Existence of non-standard ideals in the formal power series algebra $\mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$

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Abstract This article serves four purposes: (1) A complete characterization of semigroup ideals in $\mathbb{Z}_+^2$ is given; (2) The concept of standard ideals is defined in most general set up; (3) Unlike the algebra $\mathfrak{F}(\mathbb{Z}_+; \mathbb{C})$, there is a non-standard ideal in the algebra $\mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$; and (4) This is a first step in the direction of studying standard closed ideals in $\ell^1(\mathbb{Z}_+^2, \omega)$. It is also proved that, under a certain condition on $f \in \mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$, the ideal $I_f = f * \mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$ is always a standard ideal. Though the proofs are elementary, the results will give more clarity about standard ideals in the formal power series algebras.

Keywords Semigroup · Semigroup ideal · Algebra · Standard ideals · And Non-standard ideals

Mathematics Subject Classification Primary 16D25 · Secondary 20M12 · 46H10

1 Introduction

Let $(S, +, \leq)$ be a unital, totally ordered, abelian semigroup; for example, $\mathbb{Z}_+$ and $\mathbb{Z}_+^2$ are such semigroups. Consider the linear space $\mathfrak{F}(S; \mathbb{C}) = \{f : S \longrightarrow \mathbb{C} : \operatorname{supp} f \text{ is well-ordered}\}$ with pointwise linear operations. Then $\mathfrak{F}(S; \mathbb{C})$ is a unital, commutative algebra [2, P.41] with respect to the *convolution product* $f * g$, which is defined as $f * g(s) = \sum\{f(u)g(v) : u + v = s \text{ and } u, v \in S\} \quad (s \in S)$. The algebra $\mathfrak{F}(S; \mathbb{C})$ is called the *formal power series algebra* over S. It is extensively studied in [2, Chapter-2]. Let $T \subset S$ and let $\mathfrak{F}(S; T; \mathbb{C}) = \{f \in \mathfrak{F}(S; \mathbb{C}) : \operatorname{supp} f \subset T\}$. Then $\mathfrak{F}(S; T; \mathbb{C})$ is a linear subspace of $\mathfrak{F}(S; \mathbb{C})$. Moreover, T is a semigroup ideal in S if and only if $\mathfrak{F}(S; T; \mathbb{C})$ is an ideal in $\mathfrak{F}(S; \mathbb{C})$; such ideals are called *standard ideals* in $\mathfrak{F}(S; \mathbb{C})$. The standard closed ideals are also defined in the *weighted discrete semigroup algebra* $\ell^1(\mathbb{Z}_+, \omega)$ in [1, Definition 4.6.18] and it is studied by many people (see [3–5] and references therein). In this paper, we set up the most general definition of a standard ideal in any subalgebra of $\mathfrak{F}(S; \mathbb{C})$. It is remarked in [1, P. 89] that there are various types of ideals in $\mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$. The example of an ideal, which is mentioned on Page-89 in [1], is a standard ideal as per our new definition. However, still we are able to prove the existence of a non-standard ideal in $\mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$ as per our most general definition.

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2 Semigroup ideals in $\mathbb{Z}_+^2$

A triplet $(S, +, \leq)$ is an *ordered semigroup* if: (i) $(S, +)$ is a semigroup; (ii) $(S, \leq)$ is a partially ordered set; and (iii) whenever $s_1, s_2 \in S$ and $s_1 < s_2$, then $s_1 + t < s_2 + t$ and $t + s_1 < t + s_2$ for all $t \in S$. An ordered semigroup $(S, +, \leq)$ is a *totally ordered semigroup* if $(S, \leq)$ is a totally ordered set. A totally ordered semigroup is a *well-ordered semigroup* if every non-empty subset of S admits a minimum element. Throughout this paper, S is a unital, well-ordered, abelian semigroup with binary operation $+$ and with total order $\leq$. The readers should refer to [2] for basic results on well-ordered semigroups.

Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}_+$ denote the set of all non-negative integers, and $\mathbb{Z}_m = \{n \in \mathbb{Z} : n \geq m\}$ for $m \in \mathbb{Z}$; thus $\mathbb{Z}_+ = \mathbb{Z}_0$ as per our notation. It is clear that, under the natural order, $\mathbb{Z}_+$ is a well-ordered semigroup. On the other hand, the set of all non-negative rational numbers $\mathbb{Q}_+$ is a totally ordered but it is not a well-ordered semigroup. Further, the semigroup $\mathbb{Z}_+^2$ is a well-ordered semigroup with co-ordinatewise addition and with the *dictionary order* in the sense that: $(k, l) \leq (m, n)$ if $k < m$ or "$k = m$ and $l \leq n$". Throughout, we shall consider this dictionary order on $\mathbb{Z}_+^2$.

A subset T of S is a semigroup ideal in S if $s + t \in T$ $(s \in S; t \in T)$. In this section, we characterize semigroup ideals in $\mathbb{Z}_+^2$, which will be used in next section. The following two sets $U(c, d)$ and $V(c, d)$ are basic semigroup ideals in $\mathbb{Z}_+^2$.

Definition 2.1 Let $c, d \in \mathbb{Z}_+$. Define

$$U(c, d) = \{s = (s_1, s_2) \in \mathbb{Z}_+^2 : c \leq s_1 \text{ and } d \leq s_2\};$$
$$V(c, d) = \{s = (s_1, s_2) \in \mathbb{Z}_+^2 : c \leq s_1 \text{ or } d \leq s_2\}.$$

Basic properties of these two sets are listed out in the following lemma.

Lemma 2.2 *Let $c, d \in \mathbb{Z}_+$. Then*

(1) $U(c, d) = \mathbb{Z}_+^2 + (c, d)$*;*
(2) $U(c, d) \subseteq V(c, d)$*;*
(3) $U(c, d) = \mathbb{Z}_+^2$ *iff* $c = d = 0$*;*
(4) $V(c, d) = \mathbb{Z}_+^2$ *iff either* $c = 0$ *or* $d = 0$*;*
(5) $U(c, d) = \mathbb{Z}_c \times \mathbb{Z}_d$ *and* $V(c, d) = [\mathbb{Z}_c \times \mathbb{Z}_+] \cup [\mathbb{Z}_+ \times \mathbb{Z}_d]$*;*
(6) $U(c, d)$ *and* $V(c, d)$ *are semigroup ideals in* $\mathbb{Z}_+^2$*;*
(7) $U(c, d) = U(c, 0) \cap U(0, d)$ *and* $V(c, d) = U(c, 0) \cup U(0, d)$*;*
(8) If T is a semigroup ideal in $\mathbb{Z}_+^2$, then T admits a minimum element (c_0, d_0) and $U(c_0, d_0) \subseteq T \subset V(c_0, d_0)$.

Proof Proofs of Statements (1) to (7) are trivial. So we prove only Statement (8). Since $\mathbb{Z}_+^2$ is a well-ordered semigroup, T has a minimum element. Let (c_0, d_0) be the minimum element of T. Now let $(k, l) \in U(c_0, d_0)$. Then $c_0 \leq k$ and $d_0 \leq l$. Take $m = k - c_0 \in \mathbb{Z}_+$ and $n = l - d_0 \in \mathbb{Z}_+$. Then $(m, n) \in \mathbb{Z}_+^2$ Since T is an ideal, $(k, l) = (c_0, d_0) + (m, n) \in T$. Thus $U(c_0, d_0) \subset T$. Now let $(k, l) \in T$. Then $(c_0, d_0) \leq (k, l)$ and hence either $c_0 < k$ or $c_0 = k$ and $d_0 \leq l$. If $c_0 < k$, then $(k, l) \in V(c_0, d_0)$. If $c_0 = k$ and $d_0 \leq l$, then also $(k, l) \in V(c_0, d_0)$. □

It is clear that a subset T of $\mathbb{Z}_+$ is a semigroup ideal in $\mathbb{Z}_+$ if and only if $T = \mathbb{Z}_m$ for some $m \in \mathbb{Z}_+$. The following result is its analogue in $\mathbb{Z}_+^2$; it gives a complete characterization of semigroup ideals in $\mathbb{Z}_+^2$.

Theorem 2.3 *Let T be a semigroup ideal in $\mathbb{Z}_+^2$. Then there exists a finite subset $\{(c_1, d_1), \ldots, (c_n, d_n)\}$ of $\mathbb{Z}_+^2$ such that*

(1) $T = \bigcup_{i=1}^{n} U(c_i, d_i)$*;*
(2) (c_1, d_1) *is the minimum element of* T*;*
(3) (c_k, d_k) *is the minimum element of* $T \setminus \bigcup_{i=1}^{k-1} U(c_i, d_i)$ $(1 \leq k \leq n)$*;*
(4) $c_i < c_{i+1}$ *and* $d_{i+1} < d_i$ $(1 \leq i \leq n)$.

Proof Let (c_1, d_1) be the minimum element of T. By Lemma 2.2, $U(c_1, d_1) \subseteq T$. If $T = U(c_1, d_1)$, then the proof is complete. Otherwise, take the minimum element (c_2, d_2) of $T \setminus U(c_1, d_1)$. Since (c_1, d_1) is the minimum element of T, we must have $c_1 < c_2$ and $d_2 < d_1$. Again, if $T = U(c_1, d_1) \cup U(c_2, d_2)$, then the proof is finished.

Otherwise, choose the minimum element (c_3, d_3) of $T \setminus (U(c_1, d_1) \cup U(c_2, d_2))$. As above we can verify that $c_2 < c_3$ and $d_3 < d_2$. This process will stop after a finite number of steps because $0 \leq \ldots < d_3 < d_2 < d_1$. Hence the proof is complete. □

Corollary 2.4 *Let $k \in \mathbb{Z}_+$ and $T = \{(m, n) \in \mathbb{Z}_+^2 : m + n \geq k\}$. Then $(0, k)$ is the minimum element of T and $T = \cup_{i=0}^{k} U(k - i, i)$.*

3 The ideals in $\mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})$

First we introduce our notation in the following definition.

Definition 3.1 Throughout let S be a unital, totally ordered, abelian semigroup, and $T \subset S$. Then

$$\begin{aligned}
\mathfrak{F}(S; \mathbb{C}) &= \{f : S \longrightarrow \mathbb{C} : \operatorname{supp} f \text{ is a well-ordered set}\};\\
\mathfrak{F}(S; \mathbb{C}) &= \mathfrak{F} \quad (\text{if } S = \mathbb{Z}_+);\\
\mathfrak{F}(S; \mathbb{C}) &= \mathfrak{F}_2 \quad (\text{if } S = \mathbb{Z}_+^2);\\
\mathfrak{F}(S; T; \mathbb{C}) &= \{f \in \mathfrak{F}(S; \mathbb{C}) : \operatorname{supp} f \subset T\};\\
\mathfrak{F}(S; T; \mathbb{C}) &= \mathfrak{F}(T) \quad (\text{if } S = \mathbb{Z}_+);\\
\mathfrak{F}(S; T; \mathbb{C}) &= \mathfrak{F}_2(T) \quad (\text{if } S = \mathbb{Z}_+^2).
\end{aligned}$$

In this article, we are mainly concerned with $\mathfrak{F}_2$ and $\mathfrak{F}_2(T)$.

Let A be an algebra. A linear subspace I of A is an *(algebra) ideal* in A if ab and ba belong to I whenever $a \in A$ and $b \in I$. The following lemma is very basic.

Lemma 3.2 *Let $T \subseteq S$. Then*

(1) $\mathfrak{F}(S; \mathbb{C})$ is an algebra with respect to the convolution product.
(2) $\mathfrak{F}(S; T; \mathbb{C})$ is a linear subspace of $\mathfrak{F}(S; \mathbb{C})$.
(3) $\mathfrak{F}(S; T; \mathbb{C})$ is a subalgebra of $\mathfrak{F}(S; \mathbb{C})$ iff T is a subsemigroup of S.
(4) $\mathfrak{F}(S; T; \mathbb{C})$ is an ideal in $\mathfrak{F}(S; \mathbb{C})$ iff T is a semigroup ideal in S.

Proof Statement (1) can be proved as per the details given in [2, P.41]. Other statements are easy to prove. □

Notation 3.3 *Let $T \subseteq S$ and let $A \subseteq \mathfrak{F}(S; \mathbb{C})$. Then we shall write*

$$\begin{aligned}
\mathfrak{F}(A; S; T; \mathbb{C}) &= A \cap \mathfrak{F}(S; T; \mathbb{C}) = \{f \in A : \mathit{supp} f \subset T\};\\
\mathfrak{F}(A; S; T; \mathbb{C}) &= \mathfrak{F}_2(A; T; \mathbb{C}) \quad (\mathit{if}\ S = \mathbb{Z}_+^2);\\
\mathfrak{F}(A; S; T; \mathbb{C}) &= \mathfrak{F}(S; T; \mathbb{C}) \quad (\mathit{if}\ A = \mathfrak{F}(S; \mathbb{C}));\\
\mathfrak{F}(A; S; T; \mathbb{C}) &= \mathfrak{F}_2(T) \quad (\mathit{if}\ S = \mathbb{Z}_+^2\ \mathit{and}\ A = \mathfrak{F}_2 = \mathfrak{F}(\mathbb{Z}_+^2; \mathbb{C})).
\end{aligned}$$

Now we propose the most general definition of standard ideals and standard closed ideals.

Definition 3.4 Let A be a subalgebra of $\mathfrak{F}(S; \mathbb{C})$. Then an ideal I in A is a *standard ideal* if either $I = \{0\}$ or $I = \mathfrak{F}(A; S; T; \mathbb{C})$ for some semigroup ideal T in S. An ideal I in A is a *non-standard ideal* if it is not a standard ideal.

Definition 3.5 A *Banach algebra of power series (BAPS)* on a totally ordered abelian semigroup S is a subalgebra A of $\mathfrak{F}(S; \mathbb{C})$, which is also a Banach algebra with some norm.

For example, $\ell^1(\mathbb{Z}_+, \omega)$ and $\ell^1(\mathbb{Z}_+^2, \omega)$ are BAPS on $\mathbb{Z}_+$ and $\mathbb{Z}_+^2$, respectively. On the other hand, $\ell^1(\mathbb{Q}_+, \omega)$ is not a BAPS on $\mathbb{Q}_+$.

Definition 3.6 Let A be a BAPS on a totally ordered abelian semigroup S. A closed ideal I in A is a *standard closed ideal* in A if either $I = \{0\}$ or $I = \mathfrak{F}(A; S; T; \mathbb{C})$ for some semigroup ideal T in S; others are *non-standard closed ideals* in A.

Remark 3.7 Note that each semigroup ideal in $\mathbb{Z}_+$ is precisely of the form $\mathbb{Z}_k = \{n \in \mathbb{Z}_+ : n \geq k\}$ for some $k \in \mathbb{Z}_+$. Hence, if A is a BAPS on $\mathbb{Z}_+$, then the closed ideal $M_k(A)$ defined on Page 508 in [1] is same as our closed ideal $\mathfrak{F}(A; \mathbb{Z}_+; \mathbb{Z}_k; \mathbb{C})$. Thus, Definition 3.6 is consistent with [1, Definition 4.6.18]. Hence, because of Theorem 2.3, we should be able to prove analogous results of Domar [1, Theorem 4.6.28] and Thomas [1, Theorem 4.6.29] in $\ell^1(\mathbb{Z}_+^2, \omega)$.

First we introduce the following two basic standard ideals in $\mathfrak{F}_2$.

Definition 3.8 Let $c, d \in \mathbb{Z}_+$. Define

$$\begin{aligned} M_{(c,d)}(\mathfrak{F}_2) &= \{f \in \mathfrak{F}_2 : \operatorname{supp} f \subseteq U(c,d)\}; \\ &= \{f \in \mathfrak{F}_2 : f(s) = 0 \, (s = (s_1, s_2) \in \mathbb{Z}_+^2;\ s_1 < c \text{ or } s_2 < d)\}; \\ N_{(c,d)}(\mathfrak{F}_2) &= \{f \in \mathfrak{F}_2 : \operatorname{supp} f \subseteq V(c,d)\}; \\ &= \{f \in \mathfrak{F}_2 : f(s) = 0 \, (s = (s_1, s_2) \in \mathbb{Z}_+^2;\ s_1 < c \text{ and } s_2 < d)\}. \end{aligned}$$

Basic properties of these two sets are given in the following lemma. For $(c, d) \in \mathbb{Z}_+^2$, we define $\delta_{(c,d)}((c, d)) = 1$ and $\delta_{(c,d)}((m, n)) = 0$ otherwise.

Lemma 3.9 *Let $c, d \in \mathbb{Z}_+$. Then*

(1) $M_{(c,d)}(\mathfrak{F}_2) = \delta_{(c,d)} * \mathfrak{F}_2$;
(2) $M_{(c,d)}(\mathfrak{F}_2) \subset N_{(c,d)}(\mathfrak{F}_2)$;
(3) $M_{(c,d)}(\mathfrak{F}_2) = \mathfrak{F}_2$ iff $c = d = 0$;
(4) $N_{(c,d)}(\mathfrak{F}_2) = \mathfrak{F}_2$ iff either $c = 0$ or $d = 0$;
(5) $M_{(c,d)}(\mathfrak{F}_2)$ and $N_{(c,d)}(\mathfrak{F}_2)$ are ideals in $\mathfrak{F}_2$;
(6) $\mathfrak{F}_2(U(c, d)) = M_{(c,d)}(\mathfrak{F}_2)$, i.e., $M_{(c,d)}(\mathfrak{F}_2)$ is a standard ideal;
(7) $\mathfrak{F}_2(V(c, d)) = N_{(c,d)}(\mathfrak{F}_2)$ i.e., $N_{(c,d)}(\mathfrak{F}_2))$ is a standard ideal;
(8) If T is a semigroup ideal in $\mathbb{Z}_+^2$, then there are $c_0, d_0 \in \mathbb{Z}_+$ such that (c_0, d_0) is the minimum element of T and $M_{(c_0,d_0)}(\mathfrak{F}_2) \subseteq \mathfrak{F}_2(T) \subseteq N_{(c_0,d_0)}(\mathfrak{F}_2)$.

Proof Proofs are straightforward. □

Proposition 3.10 *Let T be a semigroup ideal in $\mathbb{Z}_+^2$. Then there is a finite sequence $(c_1, d_1) < \cdots < (c_n, d_n)$ in $\mathbb{Z}_+^2$ such that $\mathfrak{F}_2(T) = M_{(c_1,d_1)}(\mathfrak{F}_2) + \cdots + M_{(c_n,d_n)}(\mathfrak{F}_2)$.*

Proof By Theorem 2.3, there exists a finite sequence $(c_1, d_1) < \cdots < (c_n, d_n)$ such that $T = \bigcup_{i=1}^{n} U(c_i, d_i)$. Clearly, $M_{(c_i,d_i)}(\mathfrak{F}_2) \subset \mathfrak{F}_2(T)$ for each $1 \leq i \leq n$. Since $\mathfrak{F}_2(T)$ is an ideal, $M_{(c_1,d_1)}(\mathfrak{F}_2) + \cdots M_{(c_n,d_n)}(\mathfrak{F}_2) \subset \mathfrak{F}_2(T)$. Conversely, let $f \in \mathfrak{F}_2(T)$. Then supp$f \subset T$. Since $T = \bigcup_{i=1}^{n} U(c_i, d_i)$, there exist $f_1, \ldots, f_n \in \mathfrak{F}_2$ such that $f = f_1 + \cdots f_n$, supp$f_1 \subset U(c_1, d_1)$, and supp$f_k \subset U(c_k, d_k) \setminus \bigcup_{i=1}^{k-1} U(c_i, d_i)$ for each $2 \leq k \leq n$. So that $f_k \in M_{(c_k,d_k)}(\mathfrak{F}_2)$ for each $1 \leq k \leq n$. Hence $f = f_1 + \cdots + f_n \in M_{(c_1,d_1)}(\mathfrak{F}_2) + \cdots + M_{(c_n,d_n)}(\mathfrak{F}_2)$. □

Proposition 3.11 *Let I be a non-zero ideal in $\mathfrak{F}_2$. Then there exists a largest integer $c \in \mathbb{Z}_+$ such that $I \subset \mathfrak{F}_2(U(c, 0))$*

Proof Let $P = \bigcup\{\text{supp} f : f \in I \setminus \{0\}\}$ and let (c, d) be the minimum element of P. Then it is clear that $P \subset U(c, 0)$. Hence $I \subset \mathfrak{F}_2(U(c, 0))$. As per the choice of (c, d), it is obvious that c is the largest element in $\mathbb{Z}_+$ such that $I \subseteq \mathfrak{F}_2(U(c, 0))$. □

Proposition 3.12 *Let $f \in \mathfrak{F}_2$ be non-zero, and let $(c, d) = \min\{(k, l) : f(k, l) \neq 0\}$. If supp$f \subset U(c, d)$, then $I_f = f * \mathfrak{F}_2 = \mathfrak{F}_2(U(c, d))$. In particular, I_f is a standard ideal in $\mathfrak{F}_2$.*

Proof First we note that $\mathrm{supp}(f*g) \subseteq \mathrm{supp} f$ for any $g \in \mathfrak{F}_2$. Hence by the assumption, $\mathrm{supp}(f*g) \subset U(c,d)$ for any $g \in \mathfrak{F}_2$. Thus $I_f \subset \mathfrak{F}_2(U(c,d))$. For the reverse inclusion, we have

$$\begin{aligned} f &= \sum\{f(k,l)\delta_{(k,l)} : (k,l) \in \mathbb{Z}_+^2\} \\ &= \sum\{f(k,l)\delta_{(k,l)} : (k,l) \in \mathbb{Z}_+^2, c \le k, d \le l\} \quad (\because \mathrm{supp} f \subset U(c,d)) \\ &= \sum\{f(k,l)(\delta_{(c,d)} * \delta_{(k-c,l-d)}) : (k,l) \in \mathbb{Z}_+^2, c \le k, d \le l\} \\ &= \delta_{(c,d)} * \sum\{f(k,l)\delta_{(k-c,l-d)} : (k,l) \in \mathbb{Z}_+^2, c \le k, d \le l\} \\ &= \delta_{(c,d)} * g, \end{aligned}$$

where $g(k,l) = f(k+c, l+d) \quad ((k,l) \in \mathbb{Z}_+^2)$. Note that $g(0,0) = f(c,d) \neq 0$. By [1, Proposition 1.6.19], g is invertible in $\mathfrak{F}_2$. Let g^{-1} be the inverse of g. Then $\delta_{(c,d)} = \delta_{(c,d)} * g * g^{-1} = f * g^{-1} \in I_f$. Since I_f is an ideal, $\delta_{(k,l)} \in I_f$ $((k,l) \in U(c,d))$. Hence $\mathfrak{F}_2(U(c,d)) \subset I_f$. □

Note that, in one-variable case, it is true that every non-zero ideal in $\mathfrak{F}$ is a standard ideal [1, Proposition 1.6.20]. On the contrary, we show in next result that there does exist a non-standard ideal in $\mathfrak{F}_2$; it is our main theorem.

Theorem 3.13 *There exists a non-standard ideal in* $\mathfrak{F}_2$.

Proof Let $f = \delta_{(0,1)} + \delta_{(1,0)}$ and $I_f = f * \mathfrak{F}_2 = \{f * g : g \in \mathfrak{F}_2\}$. If I_f were a standard ideal, then, by the definition of standard ideals, there should be a semigroup ideal $T \subseteq \mathbb{Z}_+^2$ such that $I_f = \mathfrak{F}_2(T)$; in particular, I_f contains many functions whose support is singleton (namely, $\delta_{(m,n)}$ for any $(m,n) \in T$). However, we *claim* that the above ideal I_f is not a standard ideal. Let $g \in \mathfrak{F}_2$ be an arbitrary non-zero element. By above arguments, it is enough to show that the $\mathrm{supp}(f*g)$ is not a singleton set, i.e., it contains at least two distinct points. Let $(c_1, d_1) = \min\{(k,l) : g(k,l) \neq 0\}$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Now we have two possibilities.
Case-(I): $\mathrm{supp}\, g \subseteq U(c_1, d_1)$. It is clear that $(c_1,d_1) + e_1 = (c_1+1, d_1)$ and $(c_1,d_1) + e_2 = (c_1, d_1+1)$ are different points. First we show that $(f*g)((c_1,d_1)+e_1) \neq 0$. First we note that the equation $e_2 + (k,l) = (c_1,d_1) + e_1$ has no solution in $U(c_1,d_1)$ and $\mathrm{supp}\, g \subseteq U(c_1,d_1)$. Hence, we have

$$\sum\{g(k,l) : e_2 + (k,l) = (c_1,d_1) + e_1\} = 0. \tag{3.1}$$

Now

$$\begin{aligned} (f*g)((c_1,d_1)+e_1) &= [\delta_{e_2} * g + \delta_{e_1} * g]((c_1,d_1)+e_1) \\ &= \delta_{e_2} * g((c_1,d_1)+e_1) + \delta_{e_1} * g((c_1,d_1)+e_1) \\ &= \sum\{\delta_{e_2}(m,n)g(k,l) : (m,n)+(k,l) = (c_1,d_1)+e_1\} \\ &\quad + \sum\{\delta_{e_1}(m,n)g(k,l) : (m,n)+(k,l) = (c_1,d_1)+e_1\} \\ &= \sum\{g(k,l) : e_2 + (k,l) = (c_1,d_1)+e_1\} \\ &\quad + \sum\{g(k,l) : e_1 + (k,l) = (c_1,d_1)+e_1\} \\ &= 0 + \sum\{g(k,l) : (k,l) = (c_1,d_1)\} \quad \text{(By Equation 3.1)} \\ &= g(c_1,d_1) \\ &\neq 0. \end{aligned}$$

Thus $(c_1,d_1)+e_1 \in \mathrm{supp}(f*g)$. Similarly, we can show that $(f*g)((c_1,d_1)+e_2) = g(c_1,d_1) \neq 0$. Thus $(c_1,d_1)+e_2 \in \mathrm{supp}(f*g)$. So $\mathrm{supp}(f*g)$ contains at least two distinct points.
Case-(II): $\mathrm{supp}\, g \nsubseteq U(c_1,d_1)$. Let T be the smallest semigroup ideal in $\mathbb{Z}_+^2$ generated by $\mathrm{supp}\, g$. Then, as per the proof of Theorem 2.3, there exists $n \geq 2$ and a finite set $\{(c_1,d_1), \ldots, (c_n,d_n)\}$ in $\mathbb{Z}_+^2$ such that

(i) $(c_1,d_1) < (c_2,d_2) < \cdots < (c_n,d_n)$;
(ii) $g(c_i,d_i) \neq 0 \quad (1 \le i \le n)$;
(iii) $(c_k,d_k) =$ the minimum element of $\mathrm{supp}\, g \setminus \cup_{j=1}^{k-1} U(c_j,d_j) \quad (2 \le k \le n)$;
(iv) $\mathrm{supp}\, g \subseteq T = \cup_{j=1}^{n} U(c_j,d_j)$.

Then it is clear that $(c_1, d_1)+e_2$ and $(c_n, d_n)+e_1$ are distinct elements in $\mathbb{Z}_+^2$. First we show that $(f*g)((c_1, d_1)+e_2) \neq 0$. First we note that the equation $e_1 + (k, l) = (c_1, d_1) + e_2$ has no solution in $T = \cup_{j=1}^{n} U(c_j, d_j)$ and hence it has no solution in supp g because supp $g \subseteq T$. Thus, we have

$$\sum\{g(k, l) : e_1 + (k, l) = (c_1, d_1) + e_2\} = 0. \tag{3.2}$$

Now

$$\begin{aligned}
(f * g)((c_1, d_1) + e_2) &= [\delta_{e_2} * g + \delta_{e_1} * g]((c_1, d_1) + e_2) \\
&= \delta_{e_2} * g((c_1, d_1) + e_2) + \delta_{e_1} * g((c_1, d_1) + e_2) \\
&= \sum\{\delta_{e_2}(m, n)g(k, l) : (m, n) + (k, l) = (c_1, d_1) + e_2\} \\
&\quad + \sum\{\delta_{e_1}(m, n)g(k, l) : (m, n) + (k, l) = (c_1, d_1) + e_2\} \\
&= \sum\{g(k, l) : e_2 + (k, l) = (c_1, d_1) + e_2\} \\
&\quad + \sum\{g(k, l) : e_1 + (k, l) = (c_1, d_1) + e_2\} \\
&= \sum\{g(k, l) : (k, l) = (c_1, d_1)\} + 0 \quad \text{(By Equation 3.2)} \\
&= g(c_1, d_1) \\
&\quad \neq 0 \quad \text{(By property } (ii) \text{ above)}.
\end{aligned}$$

Thus $(c_1, d_1) + e_2 \in \operatorname{supp}(f * g)$. Similarly, we can show that $(f * g)((c_n, d_n) + e_1) = g(c_n, d_n) \neq 0$ as per the property (ii) above. Thus $(c_n, d_n) + e_1 \in \operatorname{supp}(f * g)$. So $\operatorname{supp}(f * g)$ contains at least two distinct points. □

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