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ORIGINAL RESEARCH

Handle decompositions for a class of closed orientable PL 4-manifolds

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Abstract In this article, we study a class of closed connected orientable PL 4-manifolds admitting a semi-simple crystallization and which have an infinite cyclic fundamental group. We show that the manifold in the class admits a handle decomposition in which the number of 2-handles depends upon its second Betti number and other h-handles ($h \leq 4$) are at most 2. More precisely, our main result is the following. For a closed connected orientable PL 4-manifold having a semi-simple crystallization with the fundamental group as $\mathbb{Z}$, we have constructed a handle decomposition for M as one of the following types:

(1) one 0-handle, two 1-handles, $1 + \beta_2(M)$ 2-handles, one 3-handle and one 4-handle,
(2) one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle,

where $\beta_2(M)$ denotes the second Betti number of manifold M with $\mathbb{Z}$ coefficients.

Keywords PL-manifolds · Crystallizations · Regular genus · Handle decomposition

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1 Introduction

A crystallization (Γ, γ) of a closed connected PL d-manifold is a certain type of edge colored graph which represents the manifold (details provided in Subsection 2.1). The journey of crystallization theory has begun due to Pezzana who gives the existence of a crystallization for every closed connected PL d-manifold (see [12]).

Extending the notion of genus in 2 dimension, the term regular genus for a closed connected PL d-manifold has been introduced in [10], which is related to the existence of regular embeddings of graphs representing the manifold into surfaces (cf. Subsection 2.2 for details).

In [3], the term semi-simple gems is introduced for closed 4-manifolds. In this paper, we particularly work on closed connected PL 4-manifolds admitting semi-simple crystallizations.

A problem for closed 4-manifolds was posed by Kirby and which can be formulated as: “Does every simply connected closed 4-manifold have a handlebody decomposition without 1-handles?" Many researchers worked on it for the decades, in manifold with boundary as well.

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In this paper, we extend the earlier known work to the closed connected 4-manifolds with the fundamental group $\mathbb{Z}$ and precisely take a large class of manifolds admitting semi-simple crystallizations. Also, the class of PL 4-manifolds admitting semi-simple crystallizations is not completely known by now. Recently in [8], the authors gave a class of compact 4-manifolds with empty or connected boundary which admit a special handle decomposition lacking in 1-handles and 3-handles. In this article, we show the following: for a closed connected orientable PL 4-manifold having a semi-simple crystallization with fundamental group as $\mathbb{Z}$, we construct a handle decomposition for M as one of the following types:

(1) one 0-handle, two 1-handles, $1 + \beta_2(M)$ 2-handles, one 3-handle and one 4-handle,
(2) one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle.

2 Preliminaries

Crystallization theory provides a combinatorial tool for representing piecewise-linear (PL) manifolds of arbitrary dimension via colored graphs and is used to study geometrical and topological properties of manifolds.

2.1 Crystallization

For a multigraph $\Gamma = (V(\Gamma), E(\Gamma))$ without loops, a surjective map $\gamma : E(\Gamma) \to \Delta_d := \{0, 1, \dots, d\}$ is called a proper edge-coloring if $\gamma(e) \neq \gamma(f)$ for any two adjacent edges e and f. The elements of the set Δ_d are called the *colors* of Γ. A graph (Γ, γ) is called $(d+1)$-*regular* if degree of each vertex is $d + 1$. We refer to [6] for standard terminology on graphs. All spaces and maps will be considered in PL-category.

A regular $(d+1)$-*colored graph* is a pair (Γ, γ), where Γ is $(d+1)$-regular and γ is a proper edge-coloring. If there is no confusion with coloration, one can use Γ for $(d+1)$-colored graphs instead of (Γ, γ). For each $B \subseteq \Delta_d$ with h elements, the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$ is an h-colored graph with edge-coloring $\gamma|_{\gamma^{-1}(B)}$. For a color set $\{j_1, j_2, \dots, j_k\} \subset \Delta_d$, $g(\Gamma_{\{j_1, j_2, \dots, j_k\}})$ or $g_{j_1 j_2 \dots j_k}$ denotes the number of connected components of the graph $\Gamma_{\{j_1, j_2, \dots, j_k\}}$.. A graph (Γ, γ) is called *contracted* if subgraph $\Gamma_{\hat{c}} := \Gamma_{\Delta_d \setminus c}$ is connected for all c.

Let $\mathbb{G}_d$ denote the set of regular $(d+1)$-colored graphs (Γ, γ). For each $(\Gamma, \gamma) \in \mathbb{G}_d$, a corresponding d-dimensional simplicial cell-complex $\mathcal{K}(\Gamma)$ is determined as follows:

- for each vertex $u \in V(\Gamma)$, take a d-simplex $\sigma(u)$ and label its vertices by Δ_d;
- corresponding to each edge of color j between $u, v \in V(\Gamma)$, identify the $(d-1)$-faces of $\sigma(u)$ and $\sigma(v)$ opposite to j-labeled vertices such that the vertices with same label coincide.

The geometric carrier $|\mathcal{K}(\Gamma)|$ is a d-pseudomanifold and (Γ, γ) is said to be a gem (graph encoded manifold) of any d-pseudomanifold homeomorphic to $|\mathcal{K}(\Gamma)|$ or simply is said to represent the d-pseudomanifold. We refer to [5] for CW-complexes and related notions. It is known via the construction that for $\mathcal{B} \subset \Delta_d$ of cardinality $h + 1$, $\mathcal{K}(\Gamma)$ has as many h-simplices with vertices labeled by $\mathcal{B}$ as many connected components of $\Gamma_{\Delta_d \setminus \mathcal{B}}$ are (cf. [9]).

For a k-simplex λ of $\mathcal{K}(\Gamma)$, $0 \leq k \leq d$, the star of λ in $\mathcal{K}(\Gamma)$ is the pseudocomplex obtained by taking the d-simplices of $\mathcal{K}(\Gamma)$ which contain λ and identifying only their $(d-1)$-faces containing λ as per gluings in $\mathcal{K}(\Gamma)$. The link of λ in $\mathcal{K}(\Gamma)$ is the subcomplex of its star obtained by the simplices that do not contain λ. It is known that the $|\mathcal{K}(\Gamma_{\hat{c}})|$ is homeomorphic to the link of vertex c of $\mathcal{K}(\Gamma)$ in the first barycentric subdivision of $\mathcal{K}(\Gamma)$. And from the correspondence between $(d+1)$-regular colored graphs and d-pseudomanifolds, we have that $|\mathcal{K}(\Gamma)|$ is a closed connected PL d-manifold if and only if for each $c \in \Delta_d$, $\Gamma_{\hat{c}}$ represents $\mathbb{S}^{d-1}$.

Definition 1 A $(d+1)$-colored graph (Γ, γ) which is a gem of a closed d-manifold M is called a *crystallization* of M if it is contracted.
In this case, there are exactly $d + 1$ number of vertices in the corresponding colored triangulation.

The initial point of the crystallization theory is the Pezzana's existence theorem (cf. [12]) which gives existence of a crystallization for a closed connected PL d-manifold.

2.2 Regular Genus of closed PL d-manifolds

In [10], the author extended the notion of genus to arbitrary dimension as regular genus. Roughly, if $(\Gamma, \gamma) \in \mathbb{G}_d$ is a bipartite (resp. non bipartite) $(d+1)$-regular colored graph which represents a closed connected PL d-manifold M then for each cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$ of Δ_d, there exists a regular imbedding of Γ into an orientable (resp. non orientable) surface F_ε. Moreover, the Euler characteristic $\chi_\varepsilon(\Gamma)$ of F_ε satisfies

$$\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_{d+1}} g_{\varepsilon_i \varepsilon_{i+1}} + (1-d)p.$$

and the genus (resp. half of genus) ρ_ε of F_ε satisfies

$$\rho_\varepsilon(\Gamma) = 1 - \frac{\chi_\varepsilon(\Gamma)}{2}$$

where $2p$ is the total number of vertices of Γ.

The regular genus $\rho(\Gamma)$ of (Γ, γ) is defined as

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d) \text{ is a cyclic permutation of } \Delta_d\}.$$

The regular genus of M is defined as

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \in \mathbb{G}_d \text{ is a crystallization which represents } M\}.$$

We need the concept of regular genus of graphs $\rho(\Gamma)$ throughout the paper. We will use the result by Montesinos ([11]) which ensures that the 3-handles (if any) and the 4-handle are added in a unique way to obtain the closed 4-manifold. It is known that each closed, orientable PL 4-manifold M admits a handle presentation $M = H_0 \cup pH_1 \cup qH_2 \cup rH_3 \cup H_4$. From [11], we get that M is uniquely deterrmined by the cobordism between $\partial(H_0 \cup pH_1)$ and $\partial(H_0 \cup pH_1 \cup qH_2)$, defined by the 2-handles qH_2.

Proposition 2 *([11]) Each closed, orientable 4-manifold with handle presentation*

$$M = H^{(0)} \cup \left(\cup_p H_p^{(1)}\right) \cup \left(\cup_q H_q^{(2)}\right) \cup \left(\cup_r H_r^{(3)}\right) \cup H^{(4)}.$$

is completely deterimined by $H^{(0)} \cup (\cup_p H_p^{(1)}) \cup (\cup_q H_q^{(2)})$.

3 Semi simple crystallizations of closed 4-manifolds

In [3], semi-simple crystallizations of closed 4-manifolds have been introduced and they are minimal with respect to regular genus among the graphs representing the same manifold. The notion of semi-simple crystallizations is generalisation of the simple crystallizations of closed simply-connected 4-manifolds (see [4]).

Definition 3 Let M be a closed 4-manifold. A 5-colored graph Γ representing M is called semi-simple if $g_{ijk} = m + 1\ \forall i, j, k \in \Delta_4$, where m is the rank of fundamental group of M. In other words, the 1-skeleton of the associated colored triangulation contains exactly $m+1$ number of 1-simplices for each pair of 0-simplices.

From [7], we have the following result on the number of components of crystallization representing closed 4-manifolds and a relation between Euler characteristic and regular genus of crystallizations.

Proposition 4 *([7]) Let M be a closed 4-manifold and (Γ, γ) be a crystallization of M. Then*

$$g_{j-1,j+1} = g_{j-1,j,j+1} + \rho - \rho_{\hat{j}}\ \forall j \in \Delta_4, \tag{1}$$

$$g_{\hat{j-1}\hat{j+1}} = 1 + \rho - \rho_{\hat{j-1}} - \rho_{\hat{j+1}}\ \forall j \in \Delta_4, \tag{2}$$

and

$$\chi(M) = 2 - 2\rho + \sum_{i \in \Delta_4} \rho_{\hat{i}}, \tag{3}$$

where ρ and $\rho_{\hat{i}}$ denote the regular genus of Γ and $\Gamma_{\hat{i}}$ respectively, and $\chi(M)$ is the Euler characteristic of M.

Lemma 5 *Let M be a closed connected orientable 4-manifold. Let (Γ, γ) be a 5-colored semi-simple crystallization for M. Let $\beta_i(M)$ denotes the i^{th} Betti number of manifold M with $\mathbb{Z}$ coefficients. Then $g_{j-1,j+1} = 4m + \beta_2 - 2\beta_1 + 1, \forall j \in \Delta_4$.*

Proof Let (Γ, γ) be a semi-simple crystallization representing M. From Equation (2), for $j = k, k+2$ (mod 5), we get $\rho_{\widehat{k-1}} = \rho_{\widehat{k+3}}$. This is true for each $k \in \Delta_4$ which implies $\rho_{\hat{i}} = \rho_{\hat{0}}$ $\forall i \in \Delta_4$. Then, by adding all the equations in (2) for each $j \in \Delta_4$, we have

$$5m = 5\rho - 10\rho_{\hat{0}} \Rightarrow \rho = m + 2\rho_{\hat{0}}.$$

From Equation (3), $\chi(M) = 2 - 2\rho + 5\rho_{\hat{0}}$. This implies $\chi(M) = 2 - 2m + \rho_{\hat{0}}$. Further, from Equation (1) for $j = 0$, we have $g_{14} = 2m + \rho_{\hat{0}} + 1$ And $g_{14} = 4m + \chi(M) - 1$. It follows from Poincaré duality that $\chi(M) = 2 + \beta_2 - 2\beta_1$. This follows the result. □

From now onwards, we particularly take the manifolds admitting semi-simple crystallizations and with fundamental group $\mathbb{Z}$. This implies, $\beta_1 = 1$, $m = 1$ and $g_{ijk} = 2$. It follows from Lemma 5 that $g_{14} = \beta_2 + 3$.

It is known that every closed 4-manifold M admits a handle decomposition, i.e.,

$$M = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{d_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{d_2}^{(2)}) \cup (H_1^{(3)} \cup \cdots \cup H_{d_3}^{(3)}) \cup H^{(4)},$$

where $H^{(0)} = \mathbb{D}^4$ and each k-handle $H_i^{(k)} = \mathbb{D}^k \times \mathbb{D}^{4-k}$ (for $1 \leq k \leq 4$, $1 \leq i \leq d_k$), is attached with a map $f_i^{(k)} : \partial\mathbb{D}^k \times \mathbb{D}^{4-k} \to \partial(H^{(0)} \cup \cdots \cup (H_1^{(k-1)} \cup \cdots \cup H_{d_{k-1}}^{(k-1)}))$.

The following arguement is known in Crystallization theory (See [10]). Let (Γ, γ) be a crystallization of a closed PL 4-manifold M and $\mathcal{K}(\Gamma)$ be the corresponding triangulation with the vertex set Δ_4. If $B \subset \Delta_4$, then $\mathcal{K}(B)$ denotes the subcomplex of $\mathcal{K}(\Gamma)$ generated by the vertices $i \in B$. Let Sd $\mathcal{K}(\Gamma)$ be the first barycentric subdivision of $\mathcal{K}(\Gamma)$ and $F(i, j)$ (resp. $F(i, j, k)$) be the largest subcomplex of Sd $\mathcal{K}(\Gamma)$, disjoint from Sd $\mathcal{K}(i, j) \cup \mathrm{Sd}\, \mathcal{K}(\Delta_4 \backslash \{i, j\})$ (resp. Sd $\mathcal{K}(i, j, k) \cup \mathrm{Sd}\, \mathcal{K}(\Delta_4 \setminus \{i, j, k\})$). Then, $F(i, j)$ (resp. $F(i, j, k)$) partitions $\mathcal{K}(\Gamma)$ into two subcomplexes $N(i, j)$ and $N(\Delta_4 \backslash \{i, j\})$ (resp. $N(i, j, k)$ and $N(\Delta_4 \backslash \{i, j, k\})$) with $F(i, j)$ (resp. $F(i, j, k)$) as common boundary. Moreover, the polyhedron $|F(i, j)|$ (resp. $|F(i, j, k)|$) is a closed 3-manifold and $|N(i, j)|$ (resp. $|N(i, j, k)|$) is a regular neighbourhood of the subcomplex $|\mathcal{K}(i, j)|$ (resp. $|\mathcal{K}(i, j, k))|$ in $|\mathcal{K}(\Gamma)|$. Thus, M has a decomposition of type $M = N(i, j) \cup_\phi N(\Delta_4 \setminus \{i, j\})$, where ϕ is a boundary identification.

Remark 6 Let M be a closed connected orientable 4-manifold with fundamental group $\mathbb{Z}$ and which admits semi simple crystallization Γ. This implies $g_{ijk} = 2$ $\forall i, j, k$ in Γ and thus number of $\{ij\}$-colored edges in $\mathcal{K}(\Gamma)$ is 2. Without loss of generality, we write $M = N(1, 4) \cup N(0, 2, 3)$. We denote $N(1, 4)$ and $N(0, 2, 3)$ by V and V' respectively. Since number of $\{14\}$-colored edges is 2, V is either $\mathbb{S}^1 \times \mathbb{B}^3$ or $\mathbb{S}^1 \tilde{\times} \mathbb{B}^3$, where $\mathbb{S}^1 \times \mathbb{B}^3$ and $\mathbb{S}^1 \tilde{\times} \mathbb{B}^3$ denote direct and twisted product of spaces $\mathbb{S}^1$, $\mathbb{B}^3$ respectively. From Mayer Vietoris exact sequence of the triples (M, V, V'), we have

$$0 \to H_4(M) \to H_3(\partial V) \to 0.$$

This implies M is orientable if and only if ∂V is orientable. Thus, $V = \mathbb{S}^1 \times \mathbb{B}^3$ and $\partial V = \mathbb{S}^1 \times \mathbb{S}^2$.

Lemma 7 *Let M, V and V' be the spaces as in remark 6. Then*

$$\beta_2(V') - \beta_2(M) - \beta_1(V') + 1 = 0. \tag{4}$$

Proof Since V' collapses onto the 2-dimensional complex $|\mathcal{K}(0, 2, 3)|$, the Mayer Vietoris sequence of the triple (M, V, V') gives the following long exact sequence.

$$\begin{array}{ccccccccc} 0 \longrightarrow H_3(M) \longrightarrow H_2(\partial V) \longrightarrow H_2(V) \oplus H_2(V') \longrightarrow H_2(M) \longrightarrow H_1(\partial V) \\ \downarrow \\ 0 \longleftarrow H_1(M) \longleftarrow H_1(V) \oplus H_1(V') \end{array}$$

By assumption $\pi_1(M) \cong \mathbb{Z}$ which implies $H_1(M) \cong \mathbb{Z}$. By Poincaré duality and Universal Coefficient theorem, $H_3(M) \cong H^1(M) \cong FH_1(M) \cong \mathbb{Z}$. Remark 6 gives $V = \mathbb{S}^1 \times \mathbb{B}^3$ and $\partial V = \mathbb{S}^1 \times \mathbb{S}^2$. Now, $H_2(\partial V) \cong H_1(\partial V) \cong \mathbb{Z}$, $H_2(V) \cong 0$ and $H_1(V) \cong Z$. Thus above exact sequence reduces to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_2(V') \longrightarrow H_2(M) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus H_1(V') \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since the alternate sum of the rank of finitely generated abelian groups in an exact sequence is zero, the result follows. □

Lemma 8 *Let M, V and V' be as in remark 6, where V and V' are regular neighbourhoods $N(1, 4)$ and $N(0, 2, 3)$ respectively. Let $\pi_1(M) = \mathbb{Z}$. Then the the fundamental group of V' is neither trivial nor $\mathbb{Z}_k$ for any k.*

Proof We have $M = V \cup V' = (\mathbb{S}^1 \times \mathbb{B}^3) \cup V'$ with $V \cap V' = \partial V = \partial V'$ from Remark 6. We can extend the spaces V and V' by open simplices. Without loss of generality, we can assume that V and V' in the given hypothesis are open. Let $i_1 : \pi_1(V \cap V') \to \pi_1(V)$ and $i_2 : \pi_1(V \cap V') \to \pi_1(V')$ be the maps induced from inclusion maps $j_1 : V \cap V' \to V$ and $j_2 : V \cap V' \to V'$ respectively. Since $V = \mathbb{S}^1 \times \mathbb{B}^3$ and $V \cap V' = \mathbb{S}^1 \times \mathbb{S}^2$, if we let $\pi_1(V \cap V') = \langle \alpha \rangle$ then $\pi_1(V) = \langle \alpha \rangle$ and $i_1(\alpha) = \alpha$.

If we assume to the contrary that $\pi_1(V') = \langle e \rangle$(or $\langle \beta | \beta^k \rangle$) then Seifert-van Kampen Theorem implies $\pi_1(M) = \langle e \rangle$(or $\langle \beta | \beta^k \rangle$) which is a contradiction as fundamental group of M is $\mathbb{Z}$. Hence, the lemma follows. □

Lemma 9 *Let V and V' be as in remark 6. Then, $0 \leq \beta_1(V') \leq 2$.*

Proof If $\beta_1(V') = k$ then $g_{14} = \beta_2(V') + 4 - k$ using Equation (4) and Lemma 5 for orientable case. Since each edge is a face of at least one triangle, the result follows. □

Proposition 10 *([13]) Let M be a manifold and $X \subset$ int M be a polyhedron. If X collapses onto Y then a regular neighbourhood of X is PL-homeomorphic to a regular neighbourhood of Y.*

Theorem 11 *Let M be a closed orientable 4-manifold with fundamental group $\mathbb{Z}$. Let (Γ, γ) be a semi-simple crystallization representing M. Then, there exists a handle decomposition of M as follows:*

(1) *one 0-handle, two 1-handles, $1 + \beta_2(M)$ 2-handles, one 3-handle and one 4-handle,*
(2) *one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle.*

Proof Let (Γ, γ) be a semi-simple crystallization representing M. We write $M = N(1, 4) \cup N(0, 2, 3) = V \cup V'$, where $V = \mathbb{S}^1 \times \mathbb{B}^3$ by Remark 6. Now, we have to analyse V'. For $i \geq 1$, let A_i be the set of all triangles which have same boundary. Then, the triangles in A_i and A_j do not have all the edges same for $i \neq j$. Since the number of edges with same labeled end vertices is 2, there exist 2^3 triangles such that none of them shares the same boundary. Thus, we have $1 \leq i \leq 8$. Let k_i be the cardinality of A_i for each i. Let B_i denote the set of triangles on collapsing one triangle from A_i. Then $|B_i| = k_i - 1$.

Consider $\mathcal{K}(0, 2, 3)$. Now, each $A_i (1 \leq i \leq 8)$ may not be there in $\mathcal{K}(0, 2, 3)$. Let $A_{j_1}, A_{j_2}, \ldots, A_{j_q}$ be the subcollection of $\{A_i : 1 \leq i \leq 8\}$ composing the cell complex $\mathcal{K}(0, 2, 3)$. In other words, $\mathcal{K}(0, 2, 3) = \cup_{r=1}^{q} A_{j_r}$. Now, $m + 1$ number of triangles with same boundary contribute m number of 2-dimensional holes. This implies that 2-holes in $\mathcal{K}(0, 2, 3)$ should be at least the sum of all the 2-holes in $A_{j_r} (1 \leq r \leq q)$. That is,

$$\Big(\sum_{r=1}^{q} k_{j_r}\Big) - q \leq \beta_2(\mathcal{K}(0, 2, 3)) = \beta_2(V').$$

The last equality follows because V' being regular neighbourhood of $\mathcal{K}(0, 2, 3)$ collapses onto $\mathcal{K}(0, 2, 3)$. We also note that $2 \leq q \leq 8$ as each edge must be a face of at least one triangle. **Case A.** Let us first consider

$$\Big(\sum_{r=1}^{q} k_{j_r}\Big) - q = \beta_2(V'). \tag{5}$$

In this case, $\mathcal{K}(0, 2, 3)$ consists of such a subcollection $A_{j_r} (1 \leq r \leq q)$, where only the triangles with same boundary contribute to $\beta_2(\mathcal{K}(0, 2, 3))(= \beta_2(V'))$. Equivalently, If we choose one triangle T_{j_r} from each A_{j_r} then $\beta_2(\cup_r T_{j_r}) = 0$. It follows from Lemma 9 that $\beta_1(V') = 0$, 1 or 2.

Case 1. Suppose $\beta_1(V') = 2$. By the proof of Lemma 9, we have $g_{14} = \beta_2(V') + 2$. Using Equation (5), we get $q = 2$. This implies that any triangle in A_{j_1} does not have any common edge with any triangle in A_{j_2} in $\mathcal{K}(0, 2, 3)$ because each edge must be a face of at least one triangle. By collapsing two triangles one from each A_{j_1} and A_{j_2}, we observe that $|\mathcal{K}(0, 2, 3)|$ collapses onto a CW complex K'; with single 0-cell, two 1-cells and 2-cells consisting of sets B_{j_1} and B_{j_2} with cardinality $k_{j_1} - 1$ and $k_{j_2} - 1$ respectively.

Now, small regular neighbourhood of single 0-cell in geometric carrier of K' is a 0-handle $H^{(0)} = \mathbb{D}^4$, two 1- cells contribute two 1-handles and 2-cells give $\beta_2(V')$ number of 2-handles by Equation (5). Thus, by using Proposition 10,

$$V' = H^{(0)} \cup \left(H_1^{(1)} \cup H_2^{(1)} \right) \cup \left(H_1^{(2)} \cup \cdots \cup H_{\beta_2(V')}^{(2)} \right).$$

Now, the boundary identification between V and V' is attachment of one 3- and one 4-handle, that is done uniquely by Proposition 2 in [11]. Further, $\beta_2(V')$ equals $1 + \beta_2(M)$ from Equation (4). Thus,

$$M = H^{(0)} \cup \left(H_1^{(1)} \cup H_2^{(1)} \right) \cup \left(H_1^{(2)} \cup \cdots \cup H_{1+\beta_2(M)}^{(2)} \right) \cup H^{(3)} \cup H^{(4)}.$$

Case 2. Suppose $\beta_1(V') = 1$. By the proof of Lemma 9, we have $g_{14} = \beta_2(V') + 3$. Using Equation (5), we get $q = 3$. As we approached in Case (1), we collapse three triangles one from each A_{j_r}, $r \in \{1, 2, 3\}$ and observe that $|\mathcal{K}(0, 2, 3)|$ collapses onto a CW complex K'; with one 0-cell, one 1-cell and 2-cells consisting of sets B_{j_r} with cardinality $k_{j_r} - 1$, $r \in \{1, 2, 3\}$.

Now, small regular neighbourhood of single 0-cell in geometric carrier of K' is a 0-handle $H^{(0)} = \mathbb{D}^4$, one 1- cell contribute one 1-handle and 2-cells give $\beta_2(V')$ number of 2-handles by Equation (5). Thus,

$$V' = H^{(0)} \cup H_1^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_{\beta_2(V')}^{(2)} \right).$$

Now, the boundary identification between V and V' is attachment of one 3- and one 4-handle and using $\beta_2(V')$ equals $\beta_2(M)$ from Equation (4). Thus, by using Proposition 10,

$$M = H^{(0)} \cup H_1^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)} \right) \cup H^{(3)} \cup H^{(4)}.$$

Case 3. Suppose $\beta_1(V') = 0$. Then we have $g_{14} = \beta_2(V') + 4$. Using Equation (5), we get $q = 4$. We claim that the fundamental group of $|\mathcal{K}(0, 2, 3)| = \{A_{i_r} | 1 \leq r \leq 4\}$ is $\mathbb{Z}_2$ or $\langle e \rangle$. To find its fundamental group, it is sufficient to assume that $|A_{i_r}| = 1$. Let l_i, r_i and d_i be the the edges with end points $\{0, 2\}$, $\{0, 3\}$ and $\{2, 3\}$ respectively, for $i \in \{1, 2\}$. If the number of triangles passing through any of the $l_1, l_2, r_1, r_2, d_1, d_2$ edges are two then the space is either projective plane or the space in Fig. 1. Else, without loss of generality, we assume that the number of triangles passing through l_1 is 3. Then there is exactly one triangle with edge l_2. Now, collapsing the triangle from the free side l_2, we reduce the number of triangles to three such that one of r_1, r_2, d_1, d_2 edge becomes free. Again, we collapse the triangle from the free edge, we get the resultant space to be contractible. However, by the equivalent statement for Case A, we note that the space in Fig. 1 does not include in the Case A as $\beta_2(\cup_r A_{j_r}) \neq 0$. Hence, the claim follows.

Now, If the fundamental group of $K(0, 2, 3)$ is $\mathbb{Z}_2$ or $\langle e \rangle$, then the fundamental group of V' is $\mathbb{Z}_2$ or $\langle e \rangle$, which is a contradiction by Lemma 8. Thus, this case is not possible.

Case B. Let $\left(\sum_{r=1}^{q} k_{j_r} \right) - q < \beta_2(V')$. In other words, this case considers the subcollection $\{A_{j_r} : 1 \leq r \leq q\}$ such that not only the triangles with the same boundary contribute to $\beta_2(V')$. In other words, the subcollection consists of the triangles such that the different A_i's also contribute to $\beta_2(V')$. Now, to find such a subcollection explicitly, it is enough to work with the condition that $|A_i| = 1$ for each i. So, we have to find the subset of all 2^3 possible triangles which gives non-zero second Betti number. Let $T = \cup_{r=1}^{q} A_{j_r}$, where $|A_{j_r}| = 1$ for each r.

If $5 \leq q \leq 8$ then we claim that the fundamental group of T is trivial. For, we let l_i, r_i and d_i be the the edges with end points $\{0, 2\}$, $\{0, 3\}$ and $\{2, 3\}$ respectively, for $i \in \{1, 2\}$. It follows from the Pigeonhole principle that the loops $l_1 l_2$, $r_1 r_2$ or $d_1 d_2$ are contractible. Therefore, the fundamental group of T is trivial, which contradicts the Lemma 8. If $q < 4$ then $\beta_2(T)$ is zero by finding second Homology group manually and thus this possibility is not included in Case B. If $q = 4$ then the non-zero $\beta_2(T)$ is given by only the space in Fig. 1 consisting four triangles, say A_{i_1}, A_{i_2}, A_{i_3} and A_{i_4}, which is PL-homeomorphic to a CW-complex consists of one l-cell, for each l, $0 \leq l \leq 2$. Thus, there exists only one subcollection in Case B. Now to write a Handle decomposition we work in the general case, that is, we do not restrict ourselves to the condition with $|A_i| = 1$.

Now, we collapse four triangles one from each A_{i_t}, $t \in \{1, 2, 3, 4\}$ and get that $|\mathcal{K}(0, 2, 3)|$ collapses to a CW complex K'; with one 0-cell, one 1-cell and 2-cells with cardinality one more the cardinality of union of sets $\{B_{i_r} | 1 \leq q \leq 4\}$.

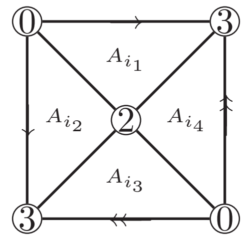

Fig. 1 .

Now, small regular neighbourhood of single 0-cell in geometric carrier of K' is a 0-handle $H^{(0)} = \mathbb{D}^4$, one 1- cell contribute one 1-handle and 2-cells give $\beta_2(V')$ number of 2-handles. Thus,

$$V' = H^{(0)} \cup H_1^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_{\beta_2(V')}^{(2)}\right).$$

Now, the boundary identification between V and V' is attachment of one 3- and one 4-handle that is done in a unique way by Proposition 2 in [11], and using $\beta_2(V')$ equals $\beta_2(M)$ from Equation (4). Thus,

$$M = H^{(0)} \cup H_1^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)}\right) \cup H^{(3)} \cup H^{(4)}.$$

This completes the proof. □

The following is the obvious question to be asked next:
Question: What can be said about the generalisation of the result with no restriction on the fundamental group of PL 4 manifold?

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