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**ORIGINAL RESEARCH**

# Prime numbers of the form $[\mathbf{n}^c \tan^\theta(\log \mathbf{n})]$

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**Abstract** Let $[\,\cdot\,]$ be the floor function. In the present paper we prove that when $1 < c < \frac{12}{11}$ and $\theta > 1$ is a fixed, then there exist infinitely many prime numbers of the form $[n^c \tan^\theta(\log n)]$.

**Keywords** Primes · Exponential sums · Lower bound

## 1 Notations

Let $x$ be a sufficiently large positive number. The letter $p$ with or without subscript will always denote prime number. By $\varepsilon$ we denote an arbitrary small positive constant, not the same in all appearances. We denote by $\Lambda(n)$ von Mangoldt's function. Moreover $e(y) = e^{2\pi i y}$. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of $t$. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \dots m_k = n$ in natural numbers $m_1, \dots, m_k$. Throughout this paper we suppose that $1 < c < \frac{12}{11}$. Assume that $\theta > 1$ is a fixed. Denote

$$\gamma = \frac{1}{c}\,; \tag{1}$$

$$\psi(t) = \{t\} - 1/2\,; \tag{2}$$

$$\Delta_1 = e^{\pi\left[\frac{\log x}{\pi}\right] + \arctan 1}\,; \tag{3}$$

$$\Delta_2 = e^{\pi\left[\frac{\log x}{\pi}\right] + \arctan 2}\,; \tag{4}$$

$$S_c(x) = \sum_{\substack{\Delta_1 \le n < \Delta_2 \\ [n^c \tan^\theta(\log n)] = p}} 1\,. \tag{5}$$

## 2 Introduction and statement of the result

The problem of the existence of infinitely many prime numbers of a special form is as interesting as it is difficult in prime number theory. It is not known if there exists any quadratic polynomial that takes infinitely many prime

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values. There is a conjecture that there exist infinitely many prime numbers of the form $n^2+1$, which is out of reach of the current state of the mathematics. For this reason, the mathematical world deals with the accessible problem of primes of the form $[n^c]$.

Let $\mathbb{P}$ denotes the set of all prime numbers. In 1953 Piatetski-Shapiro [14] has shown that for any fixed $1 < c < \frac{12}{11}$ the set

$$\mathbb{P}_c = \{p \in \mathbb{P} \mid p = [n^c] \quad \text{for some } n \in \mathbb{N}\}$$

is infinite. The prime numbers of the form $p = [n^c]$ are called Piatetski-Shapiro primes. Denote

$$\pi_c(x) = \sum_{\substack{n \le x \\ [n^c] = p}} 1.$$

Piatetski-Shapiro's result states that

$$\pi_c(x) \sim \frac{x}{c \log x}$$

for

$$1 < c < \frac{12}{11}.$$

Subsequently the interval for $c$ was sharpened many times [1,6–13,15]. To achieve a longer interval for $c$ the authors used the fact that the upper bound for $c$ is closely connected with the estimate of an exponential sum over primes. The best results up to now belongs to Rivat and Sargos [16] with

$$\pi_c(x) \sim \frac{x}{c \log x}$$

for

$$1 < c < \frac{2817}{2426}$$

and to Rivat and Wu [17] with

$$\pi_c(x) \gg \frac{x}{\log x}$$

for

$$1 < c < \frac{243}{205}.$$

In this connection it is worth mentioning the theorems of Cao and Zhai [2], [3], concerning the study of square-free numbers of the form $[n^c]$.

Motivated by these results we investigate the existence of infinitely many prime numbers of the form associated with the form of Piatetski-Shapiro primes. We show that when $1 < c < \frac{12}{11}$ and $\theta > 1$ is a fixed, then the set

$$\mathbb{T}_c = \{p \in \mathbb{P} \mid p = [n^c \tan^\theta(\log n)] \quad \text{for some } n \in \mathbb{N}\}$$

is infinite. More precisely we prove the following theorem.

**Theorem 1** *Let $1 < c < \frac{12}{11}$ and $\theta > 1$ is a fixed. Then for the sum (5) the lower bound*

$$S_c(x) \gg \frac{x}{\log x} \tag{6}$$

*holds.*

## 3 Preliminary lemmas

**Lemma 1** *Let $I$ be a subinterval of $(X, 2X]$. For any complex numbers $z(n)$ we have*

$$\left|\sum_{n\in I} z(n)\right|^2 \leq \left(1+\frac{X}{Q}\right)\sum_{|q|<Q}\left(1-\frac{|q|}{Q}\right)\sum_{n,\, n+q\,\in I} z(n+q)\overline{z(n)},$$

*where $Q \geq 1$.*

*Proof* See ( [6], Lemma 5). □

**Lemma 2** *Let $k \geq 0$ be an integer. Suppose that $f(t)$ has $k+2$ continuous derivatives on $I$, and that $I \subseteq (N, 2N]$. Assume also that there is some constant $F$ such that*

$$|f^{(r)}(t)| \asymp FN^{-r} \tag{7}$$

*for $r = 1, \ldots, k+2$. Let $Q = 2^k$. Then*

$$\left|\sum_{n\in I} e(f(n))\right| \ll F^{\frac{1}{4Q-2}} N^{1-\frac{k+2}{4Q-2}} + F^{-1}N.$$

*The implied constant depends only upon the implied constants in (7).*

*Proof* See ( [4], Theorem 2.9). □

**Lemma 3** *Let $G(n)$ be a complex valued function. Assume further that*

$$P > 2\,,\quad P_1 \leq 2P\,,\quad 2 \leq U < V \leq Z \leq P\,,$$
$$U^2 \leq Z\,,\quad 128UZ^2 \leq P_1\,,\quad 2^{18}P_1 \leq V^3\,.$$

*Then the sum*

$$\sum_{P<n\leq P_1} \Lambda(n)G(n)$$

*can be decomposed into $O\Big(\log^6 P\Big)$ sums, each of which is either of Type I*

$$\sum_{\substack{M<m\leq M_1}} a(m) \sum_{\substack{L<l\leq L_1 \\ P<ml\leq P_1}} G(ml)$$

*and*

$$\sum_{\substack{M<m\leq M_1}} a(m) \sum_{\substack{L<l\leq L_1 \\ P<ml\leq P_1}} G(ml)\log l,$$

*where*

$$L \geq Z,\quad M_1 \leq 2M,\quad L_1 \leq 2L,\quad a(m) \ll \tau_5(m)\log P$$

*or of Type II*

$$\sum_{\substack{M<m\leq M_1}} a(m) \sum_{\substack{L<l\leq L_1 \\ P<ml\leq P_1}} b(l)G(ml)$$

*where*

$$U \leq L \leq V,\quad M_1 \leq 2M,\quad L_1 \leq 2L,\quad a(m) \ll \tau_5(m)\log P,\quad b(l) \ll \tau_5(l)\log P.$$

*Proof* See ( [5]). □

**Lemma 4** *For every $M \geq 2$, we have*

$$\psi(t) = \sum_{1\leq|h|\leq M} a(h)e(ht) + \mathcal{O}\Bigg(\sum_{|h|\leq M} b(h)e(ht)\Bigg),\quad a(h) \ll 1/|h|,\quad b(h) \ll 1/M.$$

*Proof* See [18]. □

## 4 Proof of the Theorem

We have that

$$[n^c \tan^\theta(\log n)] = p$$

if and only if

$$p \le n^c \tan^\theta(\log n) < p + 1. \tag{8}$$

From (5) and (8) we write

$$\begin{aligned}
S_c(x) &= \sum_{\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < p < \Delta_2^c \tan^\theta(\log \Delta_2)} \sum_{\substack{\Delta_1 \le n < \Delta_2 \\ p \le n^c \tan^\theta(\log n) < p+1}} 1 \\
&= \sum_{\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < p < \Delta_2^c \tan^\theta(\log \Delta_2)} \sum_{m'_p \le n < m''_p} 1 + \mathcal{O}(1) \\
&= \sum_{\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < p < \Delta_2^c \tan^\theta(\log \Delta_2)} \Big( [-m'_p] - [-m''_p] \Big) + \mathcal{O}(1) \\
&= \Gamma + \Sigma + \mathcal{O}(1)\,,
\end{aligned} \tag{9}$$

where

$$\big[m'_p, m''_p\big) \subset [\Delta_1, \Delta_2) \tag{10}$$

and the interval $\big[m'_p, m''_p\big)$ is a solution of the system inequalities

$$\left| \begin{array}{l} n^c \tan^\theta(\log n) \ge p \\ n^c \tan^\theta(\log n) < p + 1 \end{array} \right. \tag{11}$$

and where

$$\Gamma = \sum_{\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < p < \Delta_2^c \tan^\theta(\log \Delta_2)} \big( m''_p - m'_p \big)\,, \tag{12}$$

$$\Sigma = \sum_{\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < p < \Delta_2^c \tan^\theta(\log \Delta_2)} \big( \psi(-m''_p) - \psi(-m'_p) \big)\,. \tag{13}$$

### 4.1 Estimation of $\Gamma$

Consider the function $t(y)$ defined by

$$t = y^c \tan^\theta(\log y), \tag{14}$$

for

$$y \in \big[m'_p, m''_p\big]. \tag{15}$$

The first derivative of $y$ as implicit function of $t$ is

$$y' = \frac{y^{1-c}}{\big(c \tan(\log y) + \theta \sec^2(\log y)\big) \tan^{\theta-1}(\log y)}. \tag{16}$$

By (1), (3), (4), (10), (11), (12), (14), (15) and the mean-value theorem we obtain

$$\Gamma \gg \sum_{\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < p < \Delta_2^c \tan^\theta(\log \Delta_2)} y'\big(\xi_p\big), \tag{17}$$

where

$$\left| \begin{array}{l} p < \xi_p < p+1 \\ y(\xi_p) \asymp \xi_p^{\gamma} \end{array} \right. . \tag{18}$$

Using (1), (3), (4), (10), (15), (16), (17), (18) and the prime number theorem we deduce

$$\Gamma \gg \sum_{\Delta_1^c \tan^\theta(\log \Delta_1)-1<p<\Delta_2^c \tan^\theta(\log \Delta_2)} p^{\gamma-1} \gg \frac{x}{\log x}. \tag{19}$$

4.2 Estimation of $\Sigma$

Using (13) and Abel's summation formula in a standard way we get

$$\Sigma \ll \frac{x}{\log^2 x} + |\Sigma_1(N_0)|, \tag{20}$$

where

$$\Sigma_1(N_0) = \sum_{\Delta_1^c \tan^\theta(\log \Delta_1)-1<n\leq N_0} \Lambda(n)\Psi(n) \tag{21}$$

for some number $N_0 \in \big(\Delta_1^c \tan^\theta(\log \Delta_1) - 1,\ \Delta_2^c \tan^\theta(\log \Delta_2)\big]$ and where

$$\Psi(n) = \psi(-m_n'') - \psi(-m_n'). \tag{22}$$

It is easy to see that (3) and (4) imply

$$2\big(\Delta_1^c \tan^\theta(\log \Delta_1) - 1\big) < \Delta_2^c \tan^\theta(\log \Delta_2).$$

We assume that

$$2\big(\Delta_1^c \tan^\theta(\log \Delta_1) - 1\big) < N_0 \leq \Delta_2^c \tan^\theta(\log \Delta_2). \tag{23}$$

When

$$\Delta_1^c \tan^\theta(\log \Delta_1) - 1 < N_0 < 2\big(\Delta_1^c \tan^\theta(\log \Delta_1) - 1\big)$$

we proceed in a similar way.
Bearing in mind (21) and (23) we split the range of $n$ into dyadic subintervals to derive

$$\Sigma_1(N_0) \ll (\log x)|\Sigma_2(N)|, \tag{24}$$

where

$$\Sigma_2(N) = \sum_{N<n\leq 2N} \Lambda(n)\Psi(n) \tag{25}$$

and where

$$\Delta_1^c \tan^\theta(\log \Delta_1) - 1 \leq N \leq \frac{N_0}{2}. \tag{26}$$

Having in mind (20), (23), (24) and (26) we deduce

$$\Sigma \ll \frac{x}{\log^2 x} + (\log x) \max_{\Delta_1^c \tan^\theta(\log \Delta_1)-1\leq N\leq \frac{1}{2}\Delta_2^c \tan^\theta(\log \Delta_2)} |\Sigma_2(N)|. \tag{27}$$

Hence fort we shall use that

$$\left| \begin{array}{c} N \asymp x^c \\ \Delta_1^c \tan^\theta(\log \Delta_1) - 1 \leq N \leq \frac{1}{2}\Delta_2^c \tan^\theta(\log \Delta_2) \end{array} \right. . \tag{28}$$

It remains to find a non-trivial estimate for $\Sigma_2(N)$.

Let $M \geq 2$ is a real parameter, we shall choose latter depending on $N$. From (22) and Lemma 4 it follows

$$\Psi(n) = \sum_{1\leq|h|\leq M} a(h)\omega(n,h) + \mathcal{O}\big(\omega_1(n)\big) + \mathcal{O}\big(\omega_2(n)\big), \tag{29}$$

where

$$\omega(n,h) = e(-hm''_n) - e(-hm'_n)\,, \tag{30}$$

$$\omega_1(n) = \sum_{|h|\leq M} b(h)e(-hm'_n)\,, \tag{31}$$

$$\omega_2(n) = \sum_{|h|\leq M} b(h)e(-hm''_n)\,, \tag{32}$$

$$a(h) \ll 1/|h|\,, \quad b(h) \ll 1/M\,. \tag{33}$$

Now (25), (29), (31) and (32) and yields

$$\Sigma_2(N) = \sum_{1\leq|h|\leq M} a(h) \sum_{N<n\leq 2N} \Lambda(n)\omega(n,h) + \mathcal{O}\big(\Omega_1 \log N\big) + \mathcal{O}\big(\Omega_2 \log N\big), \tag{34}$$

where

$$\Omega_1 = \sum_{N<n\leq 2N} \sum_{|h|\leq M} b(h)e(-hm'_n)\,, \tag{35}$$

$$\Omega_2 = \sum_{N<n\leq 2N} \sum_{|h|\leq M} b(h)e(-hm''_n)\,. \tag{36}$$

We first estimate $\Omega_1$. According to (10), (11), (28) and (35) the variable $m'_n$ is an implicit function of $n$ defined by

$$y^c \tan^\theta(\log y) = n \quad \text{for} \quad n \in (N, 2N] \tag{37}$$

and

$$y \subset [\Delta_1, \Delta_2). \tag{38}$$

Taking into account (1), (3), (4), (14), (16), (28), (37) and (38) we find

$$y' \asymp N^{\gamma-1}. \tag{39}$$

Proceeding in the same way we get

$$y'' \asymp N^{\gamma-2}. \tag{40}$$

Now (28), (33), (35), (39), (40) and Lemma 2 with $k = 0$ imply

$$\begin{aligned}\Omega_1 &\ll \frac{1}{M} \sum_{|h|\leq M} \Bigg|\sum_{N<n\leq 2N} e(-hm'_n)\Bigg| \\ &\ll \frac{N}{M} + \frac{1}{M} \sum_{1\leq h\leq M} \Bigg|\sum_{N<n\leq 2N} e(-hm'_n)\Bigg| \\ &\ll \frac{N}{M} + \frac{1}{M} \sum_{1\leq h\leq M} \Big(h^{\frac{1}{2}} N^{\frac{\gamma}{2}} + h^{-1}N^{1-\gamma}\Big) \\ &\ll NM^{-1} + N^{\frac{\gamma}{2}} M^{\frac{1}{2}} + N^{1-\gamma}M^{-1}\log M\,. \end{aligned}\tag{41}$$

Take

$$M = N^{1-\gamma}\log^4 N. \tag{42}$$

By (41) and (42) we obtain

$$\Omega_1 \ll \frac{N^\gamma}{\log^4 N} + N^{\frac{1}{2}} \log^2 N.$$

The last estimate and (28) give us

$$\Omega_1 \ll \frac{x}{\log^4 x} + x^{\frac{c}{2}} \log^2 x \ll \frac{x}{\log^4 x}. \tag{43}$$

Arguing in the same way we get

$$\Omega_2 \ll \frac{x}{\log^4 x}. \tag{44}$$

From (28), (34), (43) and (44) it follows

$$\Sigma_2(N) \ll |\Sigma_3(N)| + \frac{x}{\log^3 x}, \tag{45}$$

where

$$\Sigma_3(N) = \sum_{1\le|h|\le M} a(h) \sum_{N<n\le 2N} \Lambda(n)\omega(n,h). \tag{46}$$

By (30) we have

$$\omega(n,h) = \Phi_h(n)e(-hm'_n), \tag{47}$$

where

$$\Phi_h(n) = e\big(h(m'_n - m''_n)\big) - 1. \tag{48}$$

Using (47), (48) and Abel's summation formula we find

$$\begin{aligned}\sum_{N<n\le 2N} \Lambda(n)\omega(n,h) &= \Phi_h(2N) \sum_{N<n\le 2N} \Lambda(n)e(-hm'_n) \\ &\quad - \int\limits_N^{2N} \Phi'_h(t) \sum_{N<n\le t} \Lambda(n)e(-hm'_n)\, dt \\ &\ll \left(|\Phi_h(2N)| + \int\limits_N^{2N} |\Phi'_h(t)|\, dt\right) \max_{N_1\in[N,2N]} |\Sigma_4(N,N_1)|,\end{aligned} \tag{49}$$

where

$$\Sigma_4(N,N_1) = \sum_{N<n\le N_1} \Lambda(n)e(hm'_n). \tag{50}$$

Consider $\Phi_h(2N)$. By (39), (48) and the mean-value theorem we get

$$|\Phi_h(2N)| \le 2\big|\sin\big(\pi h(m'_n - m''_n)\big)\big| \ll |h|\big|m'_n - m''_n\big| \ll |h|N^{\gamma-1}. \tag{51}$$

On the other hand (40), (48) and the mean-value theorem yield

$$\Phi'_h(t) \ll |h|N^{\gamma-2} \quad \text{for} \quad t \in [N, 2N]. \tag{52}$$

Bearing in mind (49) – (52) we obtain

$$\sum_{N<n\le 2N} \Lambda(n)\omega(n,h) \ll |h|N^{\gamma-1} \max_{N_1\in[N,2N]} |\Sigma_4(N,N_1)|. \tag{53}$$

Thus from (27), (33), (45), (46) and (53) it follows

$$\Sigma \ll (\log x) \max_{\Delta_1^c \tan^\theta(\log \Delta_1)-1\leq N\leq \frac{1}{2}\Delta_2^c \tan^\theta(\log \Delta_2)} \Big(N^{\gamma-1}\Sigma_5(N,N_1)\Big) + \frac{x}{\log^2 x}, \tag{54}$$

where

$$\Sigma_5(N,N_1) = \sum_{1\leq h\leq M} \max_{N_1\in[N,2N]} |\Sigma_4(N,N_1)|. \tag{55}$$

It remains to apply Heath-Brown's identity to the sum $\Sigma_4(N,N_1)$ defined by (50).

**Lemma 5** *Set*

$$S_I = \sum_{\substack{D<d\leq D_1\\ N<dl\leq N_1}} a(d) \sum_{L<l\leq L_1} e\big(hm'_{dl}\big) \tag{56}$$

*and*

$$S'_I = \sum_{\substack{D<d\leq D_1\\ N<dl\leq N_1}} a(d) \sum_{L<l\leq L_1} e\big(hm'_{dl}\big)\log l, \tag{57}$$

*where*

$$L \geq 2^{-10}N^{\frac{1}{2}}, \quad D_1\leq 2D, \quad L_1\leq 2L, \quad N_1\leq 2N, \quad a(d)\ll \tau_5(d)\log N. \tag{58}$$

*Then*

$$S_I,\ S'_I \ll h^{\frac{1}{6}} N^{\frac{2\gamma+9}{12}+\varepsilon}.$$

*Proof* First we notice that (56) and (58) imply

$$DL \asymp N. \tag{59}$$

Denote

$$f(d,l) = m'_{dl}. \tag{60}$$

By (56), (58) and (60) we write

$$S_I \ll N^\varepsilon \sum_{D<d\leq D_1} \Bigg|\sum_{L'<l\leq L'_1} e\big(hf(d,l)\big)\Bigg|, \tag{61}$$

where

$$L' = \max\left\{L, \frac{N}{d}\right\}, \quad L'_1 = \min\left\{L_1, \frac{N_1}{d}\right\}. \tag{62}$$

From (58) and (62) for the sum in (61) it follows

$$\left|\begin{array}{c} D<d\leq D_1\\ L'<l\leq L'_1\\ N<dl\leq N_1\\ (L',L'_1]\subseteq(L,2L]\end{array}\right. . \tag{63}$$

According to (10), (11), (28), (58), (60) and (63) the function $f(d,l)$ is an implicit function of $d$ and $l$ defined by

$$y^c\tan^\theta(\log y) = dl \quad \text{for} \quad dl\in(N,N_1] \tag{64}$$

and

$$y \subset [\Delta_1, \Delta_2). \tag{65}$$

On the other hand for the function defined by (60) and (64) we find

$$\frac{\partial f(d,l)}{\partial l} = \frac{dy^{1-c}}{\big(c\tan(\log y) + \theta \sec^2(\log y)\big)\tan^{\theta-1}(\log y)}. \tag{66}$$

Now (3), (4), (28), (59), (63), (65) and (66) yields

$$\frac{\partial f(d,l)}{\partial l} \asymp N^{\gamma} L^{-1}. \tag{67}$$

Proceeding in the same way we get

$$\frac{\partial^2 f(d,l)}{\partial l^2} \asymp N^{\gamma} L^{-2} \tag{68}$$

and

$$\frac{\partial^3 f(d,l)}{\partial l^3} \asymp N^{\gamma} L^{-3}. \tag{69}$$

Using (58), (59), (61), (63), (67), (68), (69) and Lemma 2 with $k = 1$ we obtain

$$\begin{aligned} S_I &\ll N^{\varepsilon} \sum_{D<d\le D_1} \Big( h^{\frac{1}{6}} N^{\frac{\gamma}{6}} L^{\frac{1}{2}} + h^{-1} N^{-\gamma} L \Big) \\ &\ll N^{\varepsilon} \Big( D h^{\frac{1}{6}} N^{\frac{\gamma}{6}} L^{\frac{1}{2}} + h^{-1} N^{-\gamma} D L \Big) \\ &\ll N^{\varepsilon} \Big( D^{\frac{1}{2}} h^{\frac{1}{6}} N^{\frac{1}{2}+\frac{\gamma}{6}} + h^{-1} N^{1-\gamma} \Big) \\ &\ll N^{\varepsilon} \Big( h^{\frac{1}{6}} N^{\frac{3}{4}+\frac{\gamma}{6}} + h^{-1} N^{1-\gamma} \Big) \\ &\ll h^{\frac{1}{6}} N^{\frac{2\gamma+9}{12}+\varepsilon}. \end{aligned}$$

To estimate the sum defined by (57) we apply Abel's summation formula and proceed in the same way to deduce

$$S_I' \ll h^{\frac{1}{6}} N^{\frac{2\gamma+9}{12}+\varepsilon}.$$

This proves the lemma. □

**Lemma 6** *Set*

$$S_{II} = \sum_{\substack{D<d\le D_1}} a(d) \sum_{\substack{L<l\le L_1 \\ N<dl\le N_1}} b(l) e\big(h m'_{dl}\big), \tag{70}$$

*where*

$$\begin{aligned} &2^3 \le L \le 2^7 N^{\frac{1}{3}}, \quad D_1 \le 2D, \quad L_1 \le 2L, \quad N_1 \le 2N, \\ &a(d) \ll \tau_5(d) \log N, \quad b(l) \ll \tau_5(l) \log N. \end{aligned} \tag{71}$$

*Then*

$$S_{II} \ll h^{\frac{1}{4}} N^{\frac{3\gamma+8}{12}+\varepsilon}.$$

*Proof* First we notice that (70) and (71) give us

$$DL \asymp N. \tag{72}$$

From (3), (60), (70), (71), (72), Cauchy's inequality and Lemma 1 with $Q = N^{\frac{1}{2}}$ it follows

$$\begin{aligned}
S_{II} &\ll \Bigg(\sum_{D<d\le D_1}|a(d)|^2\Bigg)^{\frac{1}{2}}\Bigg(\sum_{D<d\le D_1}\Bigg|\sum_{\substack{L<l\le L_1\\ N<dl\le N_1}} b(l)e\big(hf(d,l)\big)\Bigg|^2\Bigg)^{\frac{1}{2}}\\
&\ll D^{\frac{1}{2}+\varepsilon}\Bigg(\sum_{D<d\le D_1}\frac{L}{Q}\sum_{|q|<Q}\Big(1-\frac{q}{Q}\Big)\sum_{\substack{L<l,\, l+q\le L_1\\ N<dl\le N_1\\ N<d(l+q)\le N_1}} b(l+q)\overline{b(l)}e\Big(hf(d,l+q)-hf(d,l)\Big)\Bigg)^{\frac{1}{2}}\\
&\ll D^{\frac{1}{2}+\varepsilon}\Bigg(\frac{L}{Q}\sum_{D<d\le D_1}\Bigg(L^{1+\varepsilon}\\
&\qquad+\sum_{1\le|q|<Q}\Big(1-\frac{q}{Q}\Big)\sum_{\substack{L<l,\, l+q\le L_1\\ N<dl\le N_1\\ N<d(l+q)\le N_1}} b(l+q)\overline{b(l)}e\Big(hf(d,l+q)-hf(d,l)\Big)\Bigg)\Bigg)^{\frac{1}{2}}\\
&\ll N^{\varepsilon}\Bigg(\frac{N^2}{Q}+\frac{N}{Q}\sum_{1\le|q|\le Q}\ \sum_{L<l,\, l+q\le L_1}\Bigg|\sum_{D'<d\le D_1'} e\big(hf(d,l,q)\big)\Bigg|\Bigg)^{\frac{1}{2}},
\end{aligned} \tag{73}$$

where

$$D' = \max\left\{D, \frac{N}{l}, \frac{N}{l+q}\right\}, \quad D_1' = \min\left\{D_1, \frac{N_1}{l}, \frac{N_1}{l+q}\right\} \tag{74}$$

and

$$f(d,l,q) = m'_{d(l+q)} - m'_{dl}. \tag{75}$$

From (71) and (74) for the sum in (73) we have

$$\left|\begin{array}{l} L < l,\ l+q \le L_1\\ D' < d \le D_1'\\ N < dl \le N_1\\ N < d(l+q) \le N_1\\ (D', D_1'] \subseteq (D, 2D] \end{array}\right. . \tag{76}$$

According to (10), (11), (28), (71), (75) and (76) the function $f(d,l,q)$ is the difference of two implicit functions of $d$ and $l$ defined by

$$y^c \tan^{\theta}(\log y) = d(l+q) \quad \text{for} \quad d(l+q) \in (N, N_1] \tag{77}$$

and

$$y^c \tan^{\theta}(\log y) = dl \quad \text{for} \quad dl \in (N, N_1] \tag{78}$$

and for both functions

$$y \subset [\Delta_1, \Delta_2). \tag{79}$$

We have

$$\frac{\partial f(d,l,q)}{\partial d} = \frac{(l+q)y^{1-c}}{\big(c\tan(\log y)+\theta\sec^2(\log y)\big)\tan^{\theta-1}(\log y)} - \frac{l y^{1-c}}{\big(c\tan(\log y)+\theta\sec^2(\log y)\big)\tan^{\theta-1}(\log y)} \tag{80}$$

Now (3), (4), (28), (72), (76), (79) and (80) imply

$$\frac{\partial f(d,l,q)}{\partial d} \asymp N^{\gamma} D^{-1}. \tag{81}$$

Arguing in the same way we get

$$\frac{\partial^2 f(d,l,q)}{\partial d^2} \asymp N^{\gamma} D^{-2}. \tag{82}$$

Now (71), (72), (73), (76), (81), (82) and Lemma 2 with $k=0$ give us

$$\begin{aligned} S_{II} &\ll N^{\varepsilon}\bigg(\frac{N^2}{Q} + \frac{N}{Q}\sum_{1\le q\le Q}\sum_{L<l\le L_1}\Big(h^{\frac{1}{2}}N^{\frac{\gamma}{2}} + h^{-1}N^{-\gamma}D\Big)\bigg)^{\frac{1}{2}} \\ &\ll N^{\varepsilon}\Big(N^2Q^{-1} + h^{\frac{1}{2}}LN^{1+\frac{\gamma}{2}} + h^{-1}N^{2-\gamma}\Big)^{\frac{1}{2}} \\ &\ll N^{\varepsilon}\Big(NQ^{-\frac{1}{2}} + h^{\frac{1}{4}}L^{\frac{1}{2}}N^{\frac{\gamma+2}{4}} + h^{-\frac{1}{2}}N^{1-\frac{\gamma}{2}}\Big) \\ &\ll N^{\varepsilon}\Big(N^{\frac{3}{4}} + h^{\frac{1}{4}}N^{\frac{3\gamma+8}{12}} + h^{-\frac{1}{2}}N^{1-\frac{\gamma}{2}}\Big) \\ &\ll h^{\frac{1}{4}}N^{\frac{3\gamma+8}{12}+\varepsilon}. \end{aligned}$$

This proves the lemma. □

**Lemma 7** *For the exponential sum denoted by (50) we have*

$$\Sigma_4(N,N_1) \ll N^{\varepsilon}\Big(h^{\frac{1}{6}}N^{\frac{2\gamma+9}{12}} + h^{\frac{1}{4}}N^{\frac{3\gamma+8}{12}}\Big).$$

*Proof* Take

$$U = 2^3, \quad V = 2^7N^{\frac{1}{3}}, \quad Z = 2^{-10}N^{\frac{1}{2}}. \tag{83}$$

According to Lemma 3, the sum $\Sigma_4(N,N_1)$ can be decomposed into $O\Big(\log^6 N\Big)$ sums, each of which is either of Type I

$$S_I = \sum_{\substack{D<d\le D_1 \\ N<dl\le N_1}} a(d)\sum_{L<l\le L_1} e\big(hm'_{dl}\big)$$

and

$$S'_I = \sum_{\substack{D<d\le D_1 \\ N<dl\le N_1}} a(d)\sum_{L<l\le L_1} e\big(hm'_{dl}\big)\log l,$$

where

$$L\ge Z, \quad D_1\le 2D, \quad L_1\le 2L, \quad a(d)\ll \tau_5(d)\log N$$

or of Type II

$$S_{II} = \sum_{\substack{D<d\le D_1 \\ N<dl\le N_1}} a(d) \sum_{L<l\le L_1} b(l) e\big(hm'_{dl}\big),$$

where

$$U \le L \le V, \quad D_1 \le 2D, \quad L_1 \le 2L, \quad a(d) \ll \tau_5(d)\log N, \quad b(l) \ll \tau_5(l)\log N.$$

Bearing in mind (83), Lemma 5 and Lemma 6 we establish the statement in the lemma. □

We are now in a good position to estimate the sum $\Sigma_5(N, N_1)$ defined by (55). Using (42), (55) and Lemma 7 we write

$$\begin{aligned}\Sigma_5(N, N_1) &\ll N^{\frac{2\gamma+9}{12}+\varepsilon} \sum_{1\le h\le M} h^{\frac{1}{6}} + N^{\frac{3\gamma+8}{12}+\varepsilon} \sum_{1\le h\le M} h^{\frac{1}{4}} \\ &\ll N^{\frac{2\gamma+9}{12}+\varepsilon} M^{\frac{7}{6}} + N^{\frac{3\gamma+8}{12}+\varepsilon} M^{\frac{5}{4}} \\ &\ll N^{\frac{23-12\gamma}{12}+\varepsilon} .\end{aligned} \tag{84}$$

We will now take the final step in estimating the sum $\Sigma$. By (28), (54) and (84) we deduce

$$\begin{aligned}\Sigma &\ll (\log x) \max_{\Delta_1^c \tan^\theta(\log\Delta_1)-1\le N\le \frac{1}{2}\Delta_2^c \tan^\theta(\log\Delta_2)} \left(N^{\gamma-1} N^{\frac{23-12\gamma}{12}+\varepsilon}\right) + \frac{x}{\log^2 x} \\ &\ll (\log x) \max_{\Delta_1^c \tan^\theta(\log\Delta_1)-1\le N\le \frac{1}{2}\Delta_2^c \tan^\theta(\log\Delta_2)} \left(N^{\frac{11}{12}+\varepsilon}\right) + \frac{x}{\log^2 x} \\ &\ll \frac{x}{\log^2 x} .\end{aligned} \tag{85}$$

### 4.3 The end of the proof

Bearing in mind (9), (19) and (85) we establish the lower bound (6).

This completes the proof of Theorem 1.

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