Indian J Pure Appl Math (2024) 55:1210–1221
https://doi.org/10.1007/s13226-023-00421-2

ORIGINAL RESEARCH

On automorphisms and fixing number of co-normal product of graphs

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Received: 26 June 2022 / Accepted: 26 April 2023 / Published online: 12 May 2023

Abstract An automorphism of a graph describes its structural symmetry and the concept of fixing number of a graph is used for breaking its symmetries (except the trivial one). In this paper, we evaluate automorphisms of the co-normal product graph $G_1 * G_2$ of two simple graphs G_1 and G_2 and give sharp bounds on the order of its automorphism group. We study the fixing number of $G_1 * G_2$ and prove sharp bounds on it. Moreover, we compute the fixing number of the co-normal product of some families of graphs.

Keywords Fixing set · Automorphism · Co-normal product of graphs

Mathematics Subject Classification 05C25 · 05C76

1 Introduction

In examining a given simple graph symmetries are used to describe its structure. Each symmetry of a simple graph is known as automorphism. Mathematically, a bijective mapping γ from the vertex set of a simple graph H to itself which satisfies the property that $\gamma(a)$ and $\gamma(b)$ are neighbors if and only if a and b are neighbors in H, is known as *automorphism* of H. The notations $V(H)$ and $E(H)$ stands for the vertex set and the edge set of a simple graph H. The set $Aut(H)$ comprises on all automorphisms of a graph H forms a group under the operation of composition of mappings. The notation $S_{|V(H)|}$ stands for the group of all permutations of $V(H)$. Clearly, $Aut(H)$ is a subgroup of $S_{|V(H)|}$. The concept of automorphisms has its roots in graph drawing [1], graph coloring [2], model checking [3], programming [4], etc.

An automorphism $\gamma \in Aut(H)$ is said to fix a vertex $\acute{a} \in V(H)$, if $\gamma(\acute{a}) = \acute{a}$. The *stabilizer* of a vertex $\acute{a} \in V(H)$ is a set which contain automorphisms of H under which $\acute{a}$ is fixed, denoted by $Stab(\acute{a})$. A set of automorphisms of H under which each vertex of a set $\mathcal{F} \subseteq V(H)$ remains fixed is called the stabilizer of $\mathcal{F}$ (i.e.,

Communicated by Shariefuddin Pirzada.

This research is supported by the UPAR Grants of United Arab Emirates University, Al Ain, UAE via Grant No. G00002590 and G00003271

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$Stab(\mathcal{F}) = \bigcap_{w\in\mathcal{F}} Stab(w))$. A vertex $\acute{a} \in V(H)$ is called a *fixed vertex* of H, if $\gamma(\acute{a}) = \acute{a}$ for all $\gamma \in Aut(H)$. For a vertex $\acute{a} \in V(H)$, the set $\{\gamma(\acute{a}) : \gamma \in Aut(H)\}$ is called the *orbit* of $\acute{a}$ under $Aut(H)$, denoted by $\mathcal{O}(\acute{a})$. Two vertices in the same orbit are called *similar* vertices. Recently, Dehmer et al. [5] used the cardinality of vertex orbits to find zeros of graph polynomials. For basic concepts and terminology which are not given in this paper, please see [6,7].

Since each automorphism of a graph H describes a structural symmetry of H so the fixing number of H is a measure of its structural complexity. Harary [8] presented several methods of fixing a graph. Erwin and Harary [9] coined the term fixing set, and for the same idea Boutin [10] used the term determining set. This notion has its application in manipulating objects [11]. For more results on fixing number and its related parameters of graphs, see [10,12–21]. A set $\mathcal{F} \subseteq V(H)$ with the property $Stab(\mathcal{F}) = \{id_H\}$ is called a *fixing set* of H, where id_H is the identity automorphism of H. The *fixing number* of H [9], denoted by $fix(H)$ defined as the cardinality of possible minimum fixing set of H. A graph H for which $fix(H) = 0$ (i.e., $Aut(H)$ is trivial) is named as *rigid graph* while H is a *non-rigid* graph if $fix(H) > 0$ (i.e., $Aut(H)$ is non-trivial). Results on the fixing number of well known families of graphs are given in [12]. All graphs considered in this paper are non-trivial, simple, finite and non-rigid, unless otherwise stated.

Until now, several products have been defined, see [22,23] for brief overview of product graphs. Ore [7] presented a product graph named it *Cartesian sum* of graphs that is also called the co-normal product of graphs [24]. For a graph G_1 with vertex set $\{g_1^1, g_2^1, \ldots, g_m^1\}$ and a graph G_2 with vertex set $\{g_1^2, g_2^2, \ldots, g_n^2\}$, the *co-normal product* of G_1 and G_2 (the terminology we use) is a graph having the vertex set $V(G_1) \times V(G_2) = \{(g_i^1, g_j^2) : g_i^1 \in V(G_1), g_j^2 \in V(G_2)\}$ and two distinct vertices (g_i^1, g_j^2) and (g_k^1, g_l^2) are adjacent if and only if g_i^1 is adjacent to g_k^1 in G_1 or g_j^2 is adjacent to g_l^2 in G_2, denoted by $G_1 * G_2$. The study of product graphs is to get information about a product graph by using the knowledge of its component graphs.

A graph can be fixed by coloring a subset of its vertex set [8] as well as by using the distance between the vertices of the graph [9]. Several properties and results are discussed in [24–29] about coloring of the co-normal product of graphs. The metric dimension and strong metric dimension of $G_1 * G_2$ were studied by Javaid et al. [30] and Kuziak et al. [31], respectively. As a function of the order of the graph, Garijo et al. [32] analyzed the difference between its metric dimension and its fixing number. In this paper, we study the automorphism group and the fixing number of $G_1 * G_2$.

We evaluate automorphisms of $G_1 * G_2$ in the next section and give sharp bounds on the cardinality of its automorphism group. We also show that $G_1 * G_2$ is a rigid graph (i.e., $Aut(G_1 * G_2)$ is trivial) if and only if both G_1 and G_2 are indeed rigid and non-isomorphic ones. We depict conditions on G_1 and G_2 which yields that $Aut(G_1 * G_2) = S_2$ (symmetric group). In Sect. 3, we study the fixing number of $G_1 * G_2$ and prove that $max\{fix(G_1), fix(G_2)\} \leq fix(G_1 * G_2)$ for any two graphs G_1 and G_2. We also identify conditions on G_1 and G_2 under which these bounds can be achieved. Further, we give formulae for the fixing number of the co-normal product of two path graphs, two star graphs, two complete multipartite graphs and a path with a null graph.

2 Automorphisms of Co-normal Product

For two graphs G_1 and G_2, $G_1 * G_2 \cong G_2 * G_1$. In [24], Kuziak et al. characterized $G_1 * G_2$ by using its connectivity. In [31], the authors characterized $G_1 * G_2$ by using its diameter. For each $(g_i^1, g_j^2) \in V(G_1 * G_2)$ $deg(g_i^1, g_j^2) = |V(G_2)|deg(g_i^1) + (|V(G_1)| - deg(g_i^1))deg(g_j^2)$ and $N(g_i^1, g_j^2) = N(g_i^1) \times V(G_2) \cup (N(g_i^1))^c \times N(g_j^2)$, where $(N(g_i^1))^c = V(G_1) \setminus N(g_i^1)$, as given in [30]. If $deg(h) = |V(H)| - 1$ for $h \in V(H)$, then h is called a dominating vertex in H.

Lemma 2.1 *[30] A vertex $(g_i^1, g_j^2) \in V(G_1 * G_2)$ is a dominating vertex if and only if g_i^1 and g_j^2 are dominating in G_1 and G_2, respectively.*

We have the following observation from the definition of a rigid graph.

Observation 2.2 *A rigid graph has at most one dominating vertex.*

In [30], Javaid et al. proved the following result.

Theorem 2.3 *[30] For any two distinct vertices $(g_i^1, g_j^2), (g_k^1, g_l^2) \in V(G_1 * G_2)$, $N((g_i^1, g_j^2)) = N((g_k^1, g_l^2))$ if and only if $N(g_i^1) = N(g_k^1)$ in G_1 and $N(g_j^2) = N(g_l^2)$ in G_2.*

In the next proposition, we provide restrictions on G_1 and G_2 whereby two different vertices of $G_1 * G_2$ are true twins.

Proposition 2.4 *Two distinct vertices $(g_i^1, g_j^2), (g_k^1, g_l^2)$ of $G_1 * G_2$ are true twins, if and only if one of the following holds:*

(1) For $g_i^1 \neq g_k^2$ and $g_j^2 \neq g_l^2$, the vertices g_i^1, g_k^1 and g_j^2, g_l^2 are dominating vertices of G_1 and G_2, respectively.
(2) For $g_i^1 = g_k^1$ and $g_j^2 \neq g_l^2$, vertex g_i^1 is a dominating vertex and $N[g_j^2] = N[g_l^2]$.
(3) For $g_i^1 \neq g_k^1$ and $g_j^2 = g_l^2$, vertex g_j^2 is a dominating vertex and $N[g_i^1] = N[g_k^1]$.

Proof (1) Suppose $(g_i^1, g_j^2), (g_k^1, g_l^2)$ are true twins in $G_1 * G_2$ and $g_i^1 \neq g_k^1$, $g_j^2 \neq g_l^2$. In order to prove that g_i^1, g_k^1 are dominating in G_1 and g_j^2, g_l^2 are dominating in G_2. Suppose on contrary that at least one of these vertices is not a dominating vertex. In particular, suppose g_i^1 is the only vertex which is not a dominating vertex in G_1, then Lemma 2.1 gives that (g_i^1, g_j^2) is not a dominating vertex, a contradiction.

Converse arguments follows from Lemma 2.1.

(2) Suppose $(g_i^1, g_j^2), (g_k^1, g_l^2)$ are true twins in $G_1 * G_2$ and $g_i^1 = g_k^1$, $g_j^2 \neq g_l^2$. In order to prove that g_i^1 is a dominating vertex in G_1 and g_j^2, g_l^2 are true twins in G_2 suppose on contrary that g_i^1 is not a dominating vertex in G_1 and $N[g_j^2] = N[g_l^2]$, then there exist at least one vertex say $g_r^1 \in V(G_1)\backslash\{g_i^1\}$ such that $g_r^1 \notin N(g_i^1)$ and $(g_r^1, g_j^2) \notin N(g_i^1, g_j^2)$ but $(g_r^1, g_j^2) \in N(g_i^1, g_l^2)$, hence $N[(g_i^1, g_j^2)] \neq N[(g_i^1, g_l^2)]$, a contradiction. Now suppose that g_i^1 is a dominating vertex of G_1 and $N[g_j^2] \neq N[g_l^2]$, then by the definition of the co-normal product $N[(g_i^1, g_j^2)] \neq N[(g_i^1, g_l^2)]$, a contradiction. Hence, g_i^1 is a dominating vertex and g_j^2, g_l^2 are true twins in G_2.

Converse arguments follows from the definition of the co-normal product of graphs.

(3) The proof directly follows from the commutative property of the co-normal product of two graphs. □

Automorphisms of a graph provide basic knowledge about its symmetrical structure. For every $\gamma \in Aut(G_1)$ and $\eta \in Aut(G_2)$, mappings λ_{G_1} and λ_{G_2} on $G_1 * G_2$ defined as $\lambda_{G_1}(g_i^1, g_j^2) = (\gamma(g_i^1), g_j^2)$ and $\lambda_{G_2}(g_i^1, g_j^2) = (g_i^1, \eta(g_j^2))$ are automorphisms of $G_1 * G_2$. The next theorem gives automorphisms of $G_1 * G_2$.

Theorem 2.5 *Let G_1 and G_2 be two graphs and $\lambda : V(G_1 * G_2) \to V(G_1 * G_2)$ be a mapping, then the following assertions holds:*

(1) For every automorphism α of G_1 and β of G_2, the mapping λ defined as $\lambda(g_i^1, g_j^2) = (\alpha(g_i^1), \beta(g_j^2))$ for all $(g_i^1, g_j^2) \in V(G_1 * G_2)$, is an automorphism of $G_1 * G_2$.
(2) If $G_1 \cong G_2$, then for every isomorphism α of G_1 to G_2 and β of G_2 to G_1, the mapping λ defined as $\lambda(g_i^1, g_j^2) = (\beta(g_j^2), \alpha(g_i^1))$ for all $(g_i^1, g_j^2) \in V(G_1 * G_2)$, is an automorphism of $G_1 * G_2$.

Proof (1) Consider two distinct vertices $(g_i^1, g_j^2), (g_k^1, g_l^2) \in V(G_1 * G_2)$ and suppose (g_i^1, g_j^2) is adjacent to (g_k^1, g_l^2), then g_i^1 is adjacent to g_k^1 in G_1 or g_j^2 is adjacent to g_l^2 in G_2. Since α and β are automorphisms of G_1 and G_2, respectively, so $\alpha(g_i^1)$ is adjacent to $\alpha(g_k^1)$ or $\beta(g_j^2)$ is adjacent to $\beta(g_l^2)$. Thus $\lambda(g_i^1, g_j^2)$ is adjacent to $\lambda(g_k^1, g_l^2)$. Now, suppose (g_i^1, g_j^2) is not adjacent to (g_k^1, g_l^2), then g_i^1 is not adjacent to g_k^1 in G_1 and g_j^2 is not adjacent to g_l^2 in G_2. Clearly, $\lambda(g_i^1, g_j^2)$ is not adjacent to $\lambda(g_k^1, g_l^2)$, which gives that $\lambda(g_i^1, g_j^2) = (\alpha(g_i^1), \beta(g_j^2))$, is a homomorphism of $G_1 * G_2$. Also, λ is a bijective mapping because α and β are bijective mappings. Hence, $\lambda(g_i^1, g_j^2) = (\alpha(g_i^1), \beta(g_j^2))$ is an automorphism of G_1.

(2) The proof follows from the fact that every isomorphism is an automorphism when the labels are removed from G_1 and G_2. □

For an automorphism $\alpha \in Stab(g_i^1)$ and $\beta \in Stab(g_j^2)$, Theorem 2.5 (1) gives that mapping $\lambda(g_i^1, g_j^2) = (\alpha(g_i^1), \beta(g_j^2))$ is an automorphism of $G_1 * G_2$ and $\lambda \in Stab(g_i^1, g_j^2)$. Hence, we have the following corollary:

Corollary 2.6 *For any two graphs G_1 and G_2, $Stab(g_i^1) \times Stab(g_j^2) \subseteq Stab(g_i^1, g_j^2)$ for each $(g_i^1, g_j^2) \in V(G_1 * G_2)$.*

For a positive integer $k \geq 2$, we define a product graph obtained from $G_1, G_2, \ldots, G_k$ with $|V(G_i)| = m_i \geq 2$ for each i. The graph having the vertex set $V(G_1) \times V(G_2) \times \ldots \times V(G_k)$, where $(g_1, g_2, \ldots, g_k)$ is adjacent

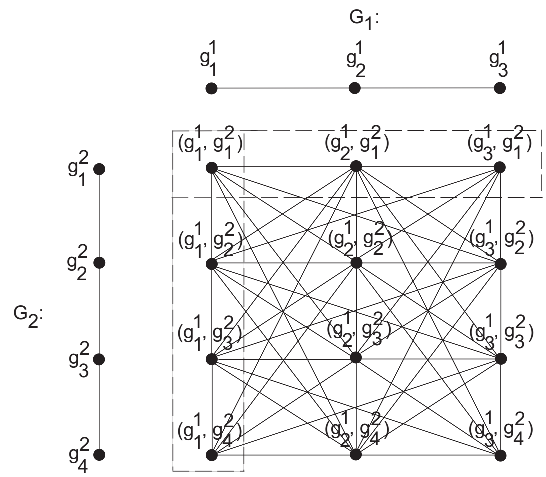

Fig. 1 The co-normal product graph of $G_1 = P_3$ and $G_2 = P_4$

to $(\acute{g_1}, \acute{g_2}, \ldots, \acute{g_k})$ whenever g_i is adjacent to $\acute{g_i}$ in G_i for some $1 \leq i \leq k$, is called the *generalized co-normal product graph* of $G_1, G_2, \ldots, G_k$, denoted by $G_1 * G_2 * \ldots * G_k$. Let $\mathcal{G} = G_1 * G_2 * \ldots * G_k$, where $|V(G_i)| \geq 2$ for each i. For any vertex $x \in V(G_i)$ for some $1 \leq i \leq k$, we define a vertex set $\mathcal{G}(x) = \{(g^1, g^2, \ldots, g^{i-1}, x, g^{i+1}, \ldots, g^k) | g^j \in V(G_j), j \neq i\}$. Note that, $\langle \mathcal{G}(x) \rangle \cong G_1 * G_2 * \ldots * G_{i-1} * G_{i+1} * \ldots * G_k$ for each $x \in V(G_i)$. In particular for $k = 2$, $\mathcal{G} = G_1 * G_2$ is the usual co-normal product graph and for $k > 2$, $\mathcal{G}$ is a natural generalization of usual co-normal product, therefore its properties too. Theorem 2.5 can be generalized to the generalized co-normal product of graphs as:

Theorem 2.7 *Let $\pi \in S_k$ be a permutation such that $\psi_i : V(G_i) \to V(G_{\pi(i)})$ is an isomorphism for each $1 \leq i \leq k$. The mapping $\lambda : V(\mathcal{G}) \to V(\mathcal{G})$, defined as $\lambda(g^1, g^2, \ldots, g^k) = (\psi_{\pi^{-1}(1)}(g^{\pi^{-1}(1)}), \psi_{\pi^{-1}(2)}(g^{\pi^{-1}(2)}), \ldots, \psi_{\pi^{-1}(k)}(g^{\pi^{-1}(k)})$ is an automorphism of $\mathcal{G}$.*

Corollary 2.8 *Let $\mathcal{G}$ be the co-normal product graph of $k \geq 2$ graphs. For every vertex $(g^1, g^2, \ldots, g^k) \in V(\mathcal{G})$, we have $Stab(g^1) \times Stab(g^2) \times \ldots \times Stab(g^k) \subseteq Stab(g^1, g^2, \ldots, g^k)$.*

For the co-normal product graph $\mathcal{G} = G_1 * G_2$, we define the vertex sets $\mathcal{G}(g_j^2) = \{(g_i^1, g_j^2) | g_i^1 \in V(G_1)\}$ and $\mathcal{G}(g_i^1) = \{(g_i^1, g_j^2) | g_j^2 \in V(G_2)\}$ for each $g_j^2 \in V(G_2)$ and $g_i^1 \in V(G_1)$. Moreover, $\langle \mathcal{G}(g_j^2) \rangle \cong G_1$ and $\langle \mathcal{G}(g_i^1) \rangle \cong G_2$ for each $g_j^2 \in V(G_2)$ and $g_i^1 \in V(G_1)$. Remark 2.9, immediately follows from the fact that every automorphism is an isometry [33], i.e., for every $\alpha \in Aut(G_1)$ and $x, y \in V(G_1)$, $d(x, y) = d(\alpha(x), \alpha(y))$.

Remark 2.9 For every automorphism $\alpha \in Aut(G)$ and every subgraph H of G, $\langle \alpha(V(H)) \rangle \cong H$.

Let id_{G_1} and id_{G_2} denotes the identity automorphism of G_1 and G_2, respectively. The automorphism $\lambda(g_i^1, g_j^2) = (\alpha(g_i^1), \beta(g_j^2))$ of $G_1 * G_2$ as defined in Theorem 2.5(1) is called a $rotation$ about G_1 (or G_2) if $\alpha \neq id_{G_1}$ and $\beta = id_{G_2}$ or $\alpha = id_{G_1}$ and $\beta \neq id_{G_2}$. For $G_1 \cong G_2$, automorphism $\lambda(g_i^1, g_j^2) = (\beta(g_j^2), \alpha(g_i^1))$ as defined in Theorem 2.5(2) exists and is called a $flip$ of $G_1 * G_2$. For two distinct vertices $u, v \in V(H)$, an automorphism $\alpha \in Aut(H)$ such that $\alpha(u) = v$, $\alpha(v) = u$ and $\alpha(x) = x$ for all $x \in V(H) \setminus \{u, v\}$ is called a $(u, v)-interchange$. All results given in this paper for $G_1 * G_2$ are also hold for $G_2 * G_1$ due to the commutative property of co-normal product and the results can be generalized to the generalized co-normal product graph of $k > 2$ graphs. For a fixed vertex $g_k^1 \in V(G_1)$ and any $\psi \in Aut(G_2)$, we define a mapping λ on $V(G_1 * G_2)$ as:

$$\lambda(g_i^1, g_j^2) \in V(G_1 * G_2) \to \begin{cases} (g_k^1, \psi(g_j^2)) & if \quad i = k, \\ (g_i^1, g_j^2) & otherwise. \end{cases} \tag{1}$$

Note that if $\psi = id_{G_2}$, then λ is a trivial automorphism of $G_1 * G_2$. Also, mapping λ as defined in (1) is a non-trivial automorphism for $P_3 * P_4$ as shown in Fig. 1 for $k = 2$ and $id_{G_2} \neq \psi \in Aut(P_4)$.

The proof of next proposition is straightforward.

Proposition 2.10 *For each dominating vertex g_k^1 of G_1 and for every non-trivial automorphism ψ of G_2, mapping λ as defined in (1) built from ψ is a non-trivial automorphism of $G_1 * G_2$.*

Note that λ as defined in (1) is not an automorphism for graph $P_4 * P_4$ for any k as shown in Fig. 2. In next theorem, we give conditions on G_1 and G_2 so that mapping λ as defined in (1) is an automorphism of $G_1 * G_2$ when g_k^1 is not a dominating vertex.

Theorem 2.11 *Let $g_k^1 \in V(G_1)$ is not a dominating vertex and $\psi \in Aut(G_2)$ is a non-trivial automorphism of G_2, then λ as defined in (1) built from ψ is a non-trivial automorphism of $G_1 * G_2$ if and only if $N(g_j^2) = N(\psi(g_j^2))$ in G_2 for each $g_j^2 \in V(G_2)$.*

Proof Let $g_k^1 \in V(G_1)$ and $\psi \in Aut(G_2)$ is a non-trivial automorphism and λ as defined in (1) built from ψ is a non-trivial automorphism of $G_1 * G_2$. Since ψ is non-trivial, therefore there exist $g_j^2 \neq g_l^2 \in V(G_2)$ such that $\psi(g_j^2) = g_l^2$. Suppose $N(g_j^2) \neq N(\psi(g_j^2))$ in G_2, then there exist at least one vertex $g_s^2 \in V(G_2)$ such that $g_s^2 \in N(g_j^2)$ and $g_s^2 \notin N(\psi(g_j^2))$ or $g_s^2 \notin N(g_j^2)$ and $g_s^2 \in N(\psi(g_j^2))$. Also, $g_k^1 \in V(G_1)$ is not a dominating vertex so there exist at least one vertex g_i^1 such that $g_i^1 \notin N(g_k^1)$. The definition of λ gives that $\lambda((g_i^1, g_s^2)) = (g_i^1, g_s^2)$ and $\lambda((g_k^1, g_j^2)) = (g_k^1, g_l^2)$, moreover, $d(\lambda((g_i^1, g_l^2)), \lambda((g_k^1, g_j^2))) > 1$ and $d((g_i^1, g_s^2), (g_k^1, g_j^2)) = 1$, which gives that λ is not an isometry, a contradiction. Hence, $N(g_j^2) = N(\psi(g_j^2))$ for each $g_j^2 \in V(G_2)$.

Conversely, suppose $N(g_j^2) = N(\psi(g_j^2))$ for each $g_j^2 \in V(G_2)$. Theorem 2.3 gives that for each $g_i^1 \in V(G_1)$, vertices (g_i^1, g_j^2) and $(g_i^1, \psi(g_j^2))$ are also false twins in $G_1 * G_2$. Hence, λ is a non-trivial automorphism of $G_1 * G_2$. □

Using the structure of co-normal product of two graphs and Theorem 2.11, we have the following corollary.

Corollary 2.12 *For any two graphs G_1 and G_2, $Aut(G_1 * G_2)$ has a $((g_i^1, g_j^2), (g_k^1, g_l^2)) - interchange$ if and only if one of the following holds:*

(1) $g_i^1 = g_k^1$, $g_j^2 \neq g_l^2$ *and* $N(g_j^2) = N(g_l^2)$.
(2) $g_i^1 \neq g_k^1$, $g_j^2 = g_l^2$ *and* $N(g_i^1) = N(g_k^1)$.
(3) $g_i^1 = g_k^1$, $g_j^2 \neq g_l^2$ *with* g_i^1 *dominating in* G_1 *and* $N[g_j^2] = N[g_l^2]$.
(4) $g_i^1 \neq g_k^1$, $g_j^2 = g_l^2$ *with* g_j^2 *dominating in* G_2 *and* $N[g_i^1] = N[g_k^1]$.
(5) $g_i^1 \neq g_k^1$, $g_j^2 \neq g_l^2$ *with* $N(g_i^1) = N(g_k^1)$ *and* $N(g_j^2) = N(g_l^2)$.
(6) $g_i^1 \neq g_k^1$, $g_j^2 \neq g_l^2$ *with* g_i^1, g_k^1 *are dominating in* G_1 *and* g_j^2, g_l^2 *are dominating in* G_2.

The definition of co-normal product of graphs gives that $Aut(G_1 * G_2)$ is isomorphic to $S_{|V(G_1 * G_2)|}$ (symmetric group) if and only if both G_1 and G_2 are complete or null graphs. Moreover, Theorem 2.5 (1) gives that $Aut(G_1) \times Aut(G_2) \subseteq Aut(G_1 * G_2)$ for any two graphs G_1 and G_2. Hence, we have the following bounds:

$$|Aut(G_1)||Aut(G_2)| \leq |Aut(G_1 * G_2)| \leq (|V(G_1)||V(G_2)|)!$$

In the next theorem, we give conditions on G_1 and G_2 so that $G_1 * G_2$ is a rigid graph and the lower bound is attained.

Theorem 2.13 *For any two graphs G_1 and G_2, graph $G_1 * G_2$ is a rigid graph if and only if G_1 and G_2 are non-isomorphic rigid graphs.*

Proof Suppose $G_1 * G_2$ is a rigid graph. In order to prove that G_1 and G_2 are both rigid graphs, suppose on contrary that G_1 is non-rigid and $\alpha \in Aut(G_1)$ is a non-trivial automorphism, then $\bar{\lambda}$ built from α and $\beta = id_{G_2}$ described in Theorem 2.5 (1) is a non-trivial automorphism of $G_1 * G_2$, a contradiction. A similar argument holds, when G_2 is a non-rigid graph. Hence, G_1 and G_2 must be rigid graphs. Now, suppose $G_1 \cong G_2$ and α is an isomorphism from G_1 to G_2 and β is an isomorphism from G_2 to G_1, then there exists a non-trivial automorphism $\bar{\lambda} \in Aut(G_1 * G_2)$ built from α and β as described in Theorem 2.5 (2), a contradiction. Hence G_1 and G_2 are non-isomorphic graphs.

Conversely, suppose that G_1 and G_2 are non-isomorphic rigid graphs and $\lambda \in Aut(G_1 * G_2)$ such that $\lambda(g^1, g^2) = (g_i^1, g_j^2)$ for some $(g^1, g^2) \in V(G_1 * G_2)$. Since $G_1 \ncong G_2$ so λ is not a flip and λ is not an interchange because G_1 and G_2 has no dominating vertices. Now to prove that λ is trivial, we study the following cases:

Case 1) Suppose $g^1 = g_i^1$ and $g^2 \neq g_j^2$. Since λ is not a flip and $\langle \lambda(\mathcal{G}(g^1)) \rangle \cong G_2$ so there exists a non-trivial automorphism β of G_2 such that $\beta(g^2) = g_j^2$ which contradicts that G_2 is a rigid graph.
Case 2) Suppose $g^1 \neq g_i^1$ and $g^2 = g_j^2$. Since λ is not a flip and $\langle \lambda(\mathcal{G}(g^1)) \rangle \cong G_2$ so there exists a non-trivial automorphism α of G_1 such that $\alpha(g^1) = g_i^1$ which contradicts that G_1 is a rigid graph.
Case 3) Suppose $g^1 \neq g_i^1$ and $g^2 \neq g_j^2$. Since λ is not a flip and $\langle \lambda(\mathcal{G}(g^2)) \rangle \cong G_1$ and $\langle \lambda(\mathcal{G}(g^1)) \rangle \cong G_2$ so there exists a non-trivial automorphism α of G_1 such that $\alpha(g^1) = g_i^1$ and a non-trivial automorphism β of G_2 such that $\beta(g^2) = g_j^2$ which contradicts that G_1 and G_2 are rigid graphs.

Concluding all above cases, we have $\lambda(g^1, g^2) = (g^1, g^2)$ for all $(g^1, g^2) \in V(G_1 * G_2)$ i.e., λ is trivial which completes the proof. □

In the next two theorems, we give conditions on G_1 and G_2 so that $Aut(G_1 * G_2) = Aut(G_1) \times Aut(G_2)$ when at least one of G_1 and G_2 is a non-rigid graph.

Theorem 2.14 *For a rigid graph G_1 and a non-rigid graph G_2, $Aut(G_1 * G_2) = Aut(G_1) \times Aut(G_2)$ if and only if G_1 has no dominating vertex and G_2 has no false twins.*

Proof Suppose $Aut(G_1 * G_2) = Aut(G_1) \times Aut(G_2)$. In order to prove that G_1 has no dominating vertex suppose on contrary that G_1 has a dominating vertex $g_k^1 \in V(G_1)$ and $\bar{\psi} \in Aut(G_2)$ is non-trivial, then mapping $\bar{\lambda} \in Aut(G_1 * G_2)$ built from $\bar{\psi}$ as described in Proposition 2.10 is a non-trivial automorphism of $G_1 * G_2$ such that $\bar{\lambda} \notin Aut(G_1) \times Aut(G_2)$, a contradiction. Hence, G_1 has no dominating vertex. Now, suppose $g_j^2 \neq g_l^2 \in V(G_2)$ are false twins in G_2, then Corollary 2.12 (1) gives that for each $g_i^1 \in V(G_1)$ there exists a $((g_i^1, g_j^2), (g_i^1, g_l^2)) - interchange$ in $Aut(G_1 * G_2)$ which does not belong to $Aut(G_1) \times Aut(G_2)$, a contradiction. Hence, G_2 has no false twins.

Conversely, suppose G_1 is a rigid graph without dominating vertex and G_2 is without false twins. Theorem 2.5 (1) gives that $Aut(G_1) \times Aut(G_2) \subseteq Aut(G_1 * G_2)$. Consider $\lambda \in Aut(G_1 * G_2)$ since G_1 and G_2 are non-isomorphic graphs so λ is not a flip and Corollary 2.12 gives that λ is not an interchange because G_1 and G_2 has no false twins. Further, G_1 has no dominating vertex so for every non-trivial automorphism $\psi \in Aut(G_2)$, λ does not have the form described in Proposition 2.10 built from ψ. Suppose $\lambda(g_i^1, g_j^2) = (g_k^1, g_l^2)$ such that $g_i^1 \neq g_k^1$, then there exists a non-trivial automorphism α of G_1 for which $\alpha(g_i^1) = g_k^1$, a contradiction to the fact that G_1 is a rigid graph. Hence $\lambda(g_i^1, g_j^2) = (g_i^1, g_l^2)$ i.e., λ is constant on the first coordinate. Now suppose $\lambda \notin Aut(G_1) \times Aut(G_2)$ and g_j^2 is similar to g_l^2 in G_2, then there exists a vertex $(g_i^1, g_r^2) \in \mathcal{G}(g_i^1)$ such that $\lambda(g_i^1, g_r^2) = (g_i^1, g_t^2)$ and G_2 has no automorphism β for which $\beta(g_j^2) = g_l^2$ and $\beta(g_r^2) = g_t^2$ which yields that $\langle \lambda(\mathcal{G}(g_i^1)) \rangle \ncong G_2$ but $\langle \mathcal{G}(g_i^1) \rangle \cong G_2$, a contradiction to the fact that λ is an automorphism by Remark 2.9. Now suppose that g_j^2 is not similar to g_l^2 in G_2, then $\langle \lambda(\mathcal{G}(g_i^1)) \rangle \ncong G_2$ because λ is a bijective mapping, a contradiction to the fact that λ is an automorphism by Remark 2.9. □

Theorem 2.15 *For any two non-rigid, non-isomorphic graphs G_1 and G_2, $Aut(G_1 * G_2) = Aut(G_1) \times Aut(G_2)$ if and only if neither G_1 nor G_2 has dominating vertex and false twins.*

Proof Let $Aut(G_1 * G_2) = Aut(G_1) \times Aut(G_2)$, we will prove that neither G_1 nor G_2 has dominating vertex or false twins. First, suppose that G_1 has a dominating vertex say g_k^1, then for every non-trivial automorphism $\bar{\psi} \in Aut(G_2)$, mapping $\bar{\lambda}$ built from $\bar{\psi}$ described in Proposition 2.10 is a non-trivial automorphism of $G_1 * G_2$ such that $\lambda \notin Aut(G_1) \times Aut(G_2)$. Now, suppose G_1 or G_2 has false twins. In particular, suppose $g_j^2 \neq g_l^2 \in V(G_2)$ are false twins, then Corollary 2.12 (1) gives that for each $g_i^1 \in V(G_1)$ there exists a $((g_i^1, g_j^2), (g_i^1, g_l^2)) - interchange$ in $Aut(G_1 * G_2)$ which does not belong to $Aut(G_1) \times Aut(G_2)$, a contradiction. Hence, neither G_1 nor G_2 has false twins and dominating vertices.

Conversely, suppose G_1 and G_2 are non-isomorphic, non-rigid graphs without false twins and dominating vertices. Theorem 2.5 (1) gives that $Aut(G_1) \times Aut(G_2) \subseteq Aut(G_1 * G_2)$. Now to prove that $Aut(G_1 * G_2) \subseteq Aut(G_1) \times Aut(G_2)$ assume contrary that there exists $\lambda \in Aut(G_1 * G_2)$ such that $\lambda \notin Aut(G_1) \times Aut(G_2)$. Now G_1 and G_2 are non-isomorphic graphs so λ is not a flip and Corollary 2.12 gives that λ is not an interchange because $G_1 * G_2$ has no false twins. Further, for each non-trivial automorphism $\psi \in Aut(G_2)$, λ does not have the form described in Proposition 2.10 built from ψ. A similar argument holds for each non-trivial automorphism $\phi \in Aut(G_1)$ due to the commutative property of co-normal product of G_1 and G_2. Suppose $\lambda(g_i^1, g_j^2) = (g_k^1, g_l^2)$ then we study two cases as follows:

Case 1) Suppose g_i^1 is similar to g_k^1 in G_1 and g_j^2 is similar to g_l^2 in G_2. We have two possibilities:
(1) Since $\lambda \notin Aut(G_1) \times Aut(G_2)$ so there exists a vertex $(g_i^1, g_r^2) \in \mathcal{G}(g_i^1)$ such that $\lambda(g_i^1, g_r^2) = (g_k^1, g_t^2)$ and G_2 has no automorphism β for which $\beta(g_j^2) = g_l^2$ and $\beta(g_r^2) = g_t^2$ which yields that $\langle \lambda(\mathcal{G}(g_i^1)) \rangle \ncong G_2$ but $\langle \mathcal{G}(g_i^1) \rangle \cong G_2$, a contradiction to the fact that λ is an automorphism by Remark 2.9.
(2) Since $\lambda \notin Aut(G_1) \times Aut(G_2)$ so there exists a vertex $(g_r^1, g_j^2) \in \mathcal{G}(g_j^2)$ such that $\lambda(g_r^1, g_j^2) = (g_t^1, g_l^2)$ and G_1 has no automorphism α for which $\alpha(g_i^1) = g_k^1$ and $\alpha(g_r^1) = g_t^1$ which yields that $\langle \lambda(\mathcal{G}(g_j^2)) \rangle \ncong G_1$ but $\langle \mathcal{G}(g_j^2) \rangle \cong G_1$, a contradiction to the fact that λ is an automorphism by Remark 2.9.
Case 2) Suppose $\lambda(g_i^1, g_j^2) = (g_k^1, g_l^2)$ such that g_i^1 is not similar to g_k^1 in G_1 then $\langle \lambda(\mathcal{G}(g_i^1)) \rangle \ncong G_2$ because λ is a bijective mapping, a contradiction to the fact that λ is an automorphism by Remark 2.9. Now suppose g_j^2 is not similar to g_l^2 in G_2 then $\langle \lambda(\mathcal{G}(g_j^2)) \rangle \ncong G_1$ because λ is a bijective mapping, a contradiction to the fact that λ is an automorphism by Remark 2.10. □

A path graph with three vertices has false twins and each path graph with at least four vertices has no false twins. Hence, we have the following corollary from Theorem 2.14.

Corollary 2.16 *For a rigid graph G_1 with no dominating vertex and $G_2 \cong P_n$; $n \geq 4$, $Aut(G_1 * G_2) \cong S_2$.*

The next corollary is a direct consequence of Theorem 2.15.

Corollary 2.17 *Let G_1 and G_2 be two non-isomorphic, non-rigid graphs. For each $(g_i^1, g_j^2) \in V(G_1 * G_2)$, $Stab(g_i^1, g_j^2) = Stab(g_i^1) \times Stab(g_j^2)$ if and only if neither G_1 nor G_2 has dominating vertex and false twins.*

In the following theorem, we give the automorphism group of $G_1 * G_2$, when G_1 and G_2 are isomorphic rigid graphs.

Theorem 2.18 *For any two rigid isomorphic graphs G_1 and G_2 with no dominating vertices, $Aut(G_1 * G_2) \cong S_2$.*

Proof Since $G_1 \cong G_2$ so there exists isomorphisms ϕ and ψ from G_1 to G_2 and G_2 to G_1, respectively, such that mapping λ defined in Theorem 2.5 (2) built from ϕ and ψ is a non-trivial automorphism of $G_1 * G_2$. Given that G_1 and G_2 are rigid graphs which gives that ϕ and ψ are unique isomorphisms. Hence, $G_1 * G_2$ has a unique non-trivial automorphism. □

3 Fixing Number of Co-normal Product

This section is devoted towards the study of the fixing number of the co-normal product of two graphs. For a rigid graph H, $fix(H) = 0$ and Theorem 2.13 gives that $fix(G_1 * G_2) = 0$ if and only if G_1 and G_2 are non-isomorphic rigid graphs. Also, Theorem 2.18 gives that for two isomorphic rigid graphs G_1 and G_2, $fix(G_1 * G_2) = 1$. The graph $G_1 + G_2$ obtained from two graphs G_1 and G_2, has $V(G_1) \cup V(G_2)$ as its vertex set and $E(G_1) \cup E(G_2) \cup \{g_i^1 \sim g_j^2 | g_i^1 \in V(G_1), g_j^2 \in V(G_2)\}$ as its edge set, is named as join graph. The next observation and Proposition 3.2 will help in proving our next results.

Observation 3.1 *If G_1 has a dominating vertex g_i^1, then $G_1 * G_2 = \langle V(G_1) \backslash \{g_i^1\} \rangle * G_2 + \langle \mathcal{G}(g_i^1) \rangle$.*

Proposition 3.2 *For a rigid graph G with a dominating vertex v, the induced subgraph $G' = \langle V(G) \setminus \{v\} \rangle$ of G, is also a rigid graph.*

Proof Let ψ be an automorphism of G'. The mapping $\psi_v : V(G) \to V(G)$, defined as $\psi_v(v) = v$ and $\psi_v(u) = \psi(u)$ for $u \in V(G) \backslash \{v\}$, is a non-trivial automorphism of G if and only if ψ is a non-trivial automorphism of G'. Hence, G' is rigid. □

For a graph H, a graph H^c with $V(H)$ as its vertex set and two vertices are adjacent whenever they are not adjacent in H is named as *complement graph* of H. The *strong product* of two graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$ and the adjacency is defined as: $(g_i^1, g_j^2) \sim (g_k^1, g_l^2)$ if and only if $g_i^1 = g_k^1$ and $g_j^2 \sim g_l^2$, $g_i^1 \sim g_k^1$ and $g_j^2 = g_l^2$ or $g_i^1 \sim g_k^1$ and $g_j^2 \sim g_l^2$, denoted as $G_1 \boxtimes G_2$. In [31], Kuziak et al. proved that $(G_1 * G_2)^c = G_1^c \boxtimes G_2^c$, where $G_1^c \boxtimes G_2^c$ is the strong product of G_1^c and G_2^c. Moreover, $fix(G_1) = fix(G_1^c)$ [12]. This yields the following.

Theorem 3.3 *For any two graphs G_1 and G_2, $fix(G_1 * G_2) = fix(G_1^c \boxtimes G_2^c)$.*

The following observation follows from the definition of co-normal product of graphs.

Observation 3.4 *If G_1 has r components $G_1^1, G_2^1, \ldots, G_r^1$ and G_2 has s components $G_1^2, G_2^2, \ldots, G_s^2$, then for each $x \in V(G_i^1) \times V(G_j^2)$ and $y, z \in V(G_k^1) \times V(G_l^2)$, we have $d(x, y) = d(x, z)$ for all $i \neq k$ and $j \neq l$ in $G_1 * G_2$.*

The next lemma follows from Observation 3.4.

Lemma 3.5 *If g_i^1 is a dominating vertex in G_1 then vertices of the class $\mathcal{G}(g_i^1)$ are not fixed by any vertex from $V(G_1 * G_2) \setminus \mathcal{G}(g_i^1)$.*

In the next theorem, we give sharp bounds on the fixing number of the co-normal product of two graphs.

Theorem 3.6 *For any two graphs G_1 and G_2,*

$$max\{fix(G_1), fix(G_2)\} \leq fix(G_1 * G_2) \leq |V(G_1)||V(G_2)| - 1.$$

Proof Let $\mathcal{F} \subseteq V(G_1 * G_2)$, be an arbitrary set such that $|\mathcal{F}| < max\{fix(G_1), fix(G_2)\}$. Without loss of generality, suppose that $max\{fix(G_1), fix(G_2)\} = fix(G_2)$ and define $\mathcal{F}_1 = \{g_i^1 \in V(G_1) | (g_i^1, g_j^2) \in \mathcal{F}$ for some $g_j^2 \in V(G_2)\} \subseteq V(G_1)$ and $\mathcal{F}_2 = \{g_j^2 \in V(G_2) | (g_i^1, g_j^2) \in \mathcal{F}$ for some $g_i^1 \in V(G_1)\} \subseteq V(G_2)$. Since $|\mathcal{F}| < fix(G_2)$ so $\cap_{g_j^2 \in \mathcal{F}_2} Stab(g_j^2) \neq \{id_{G_2}\}$. Let $id_{G_2} \neq \psi \in \cap_{g_j^2 \in \mathcal{F}_2} Stab(g_j^2)$, then mapping $\lambda = (id_{G_1}, \psi)$ is a non-trivial automorphism of $G_1 * G_2$. Hence, $\mathcal{F}$ is not a fixing set for $G_1 * G_2$ and $max\{fix(G_1), fix(G_2)\} \leq fix(G_1 * G_2)$. The upper bound follows directly from the definition of $G_1 * G_2$ and the fixing number of a graph. □

The upper bound given in Theorem 3.6 is attained, when both G_1 and G_2 are complete or null graphs. The next lemma follows from Theorem 2.14.

Lemma 3.7 *For a rigid graph G_1 without dominating vertex and a non-rigid graph G_2 without false twins, $Stab(g_i^1, g_j^2) = Stab(g_i^1) \times Stab(g_j^2)$ for each $(g_i^1, g_j^2) \in V(G_1 * G_2)$.*

In next theorem, we give some conditions on G_1 and G_2 under which the lower bound given in Theorem 3.6 is attained.

Theorem 3.8 *For any two non-isomorphic graphs G_1 and G_2, $fix(G_1 * G_2) = max\{fix(G_1), fix(G_2)\}$, if one of the following holds:*

(1) G_1 and G_2 are rigid graphs.
(2) G_1 is a rigid graph without dominating vertex and G_2 is a non-rigid graph without false twins.
(3) G_1 and G_2 are non-rigid graphs without dominating vertices and false twins.

Proof (1) Proof follows from Theorem 2.13.

(2) Let $F_2 = \{g_1^2, g_2^2, \ldots, g_l^2\} \subseteq V(G_2)$ be a minimum fixing set for G_2. For any vertex $g_i^1 \in V(G_1)$, consider set $F(g_i^1) = \{(g_i^1, g_j^2) | g_j^2 \in F_2\} \subseteq V(G_1 * G_2)$. First, we prove that $F(g_i^1)$ is a fixing set for $G_1 * G_2$. As G_1 and G_2 satisfies the hypothesis of Lemma 3.7 yields that $Stab(g_i^1, g_j^2) = Stab(g_i^1) \times Stab(g_j^2)$ for all $(g_i^1, g_j^2) \in V(G_1 * G_2)$. Moreover, G_1 is a rigid graph and $\cap_{g_j^2 \in F_2} Stab(g_j^2) = \{id_{G_2}\}$ implies that $\cap_{(g_i^1, g_j^2) \in F(g_i^1)} Stab(g_i^1, g_j^2) = \{id_{G_1 * G_2}\}$. Hence, $F(g_i^1)$ is a fixing set for $G_1 * G_2$. In order to prove that $F(g_i^1)$ is a minimum fixing set consider any subset $\mathcal{F}$ of $V(G_1 * G_2)$ such that $|\mathcal{F}| < l$ and define $\mathcal{F}_1 = \{g_i^1 \in V(G_1) | (g_i^1, g_j^2) \in \mathcal{F}$ for some $g_j^2 \in V(G_2)\}$ and $\mathcal{F}_2 = \{g_j^2 \in V(G_2) | (g_i^1, g_j^2) \in \mathcal{F}$ for some $g_i^1 \in V(G_1)\}$. Since, $|\mathcal{F}| < l$ so $|\mathcal{F}_2| < l$, moreover, $fix(G_2) = l$ yields that $\cap_{g_j^2 \in \mathcal{F}_2} Stab(g_j^2) \neq \{id_{G_2}\}$ and $\cap_{(g_i^1, g_j^2) \in \mathcal{F}} Stab(g_i^1, g_j^2) \neq \{id_{G_1 * G_2}\}$ which gives that $\mathcal{F}$ is not a fixing set for $G_1 * G_2$. Hence, $F(g_i^1)$ is a minimum fixing set for $G_1 * G_2$.

(3) Let $F_1 = \{g_1^1, g_2^1, \ldots, g_r^1\} \subseteq V(G_1)$ and $F_2 = \{g_1^2, g_2^2, \ldots, g_l^2\} \subseteq V(G_2)$ are minimum fixing sets for G_1 and G_2, respectively. We define $F = \{(g_1^1, g_1^2), (g_2^1, g_2^2), \ldots, (g_r^1, g_r^2), (g_r^1, g_{r+1}^2), \ldots, (g_r^1, g_l^2)\} \subseteq V(G_1 * G_2)$ such that $|F| = l$. Since F_1 and F_2 are fixing sets so $\cap_{g_i^1 \in F_1} Stab(g_i^1) = \{id_{G_1}\}$ and $\cap_{g_j^2 \in F_2} Stab(g_j^2) = \{id_{G_2}\}$,

further, G_1 and G_2 satisfies the conditions of Corollary 2.17 yields that $Stab(g_i^1, g_i^2) = Stab(g_i^1) \times Stab(g_i^2)$ for each $(g_i^1, g_i^2) \in V(G_1 * G_2)$. Hence, $\cap_{(g_i^1, g_j^2) \in F} Stab(g_i^1, g_j^2) = \{id_{G_1 * G_2}\}$ and F is a fixing set for $G_1 * G_2$. In order to prove that F is a minimum fixing set, consider any subset $\mathcal{F} \subseteq V(G_1 * G_2)$ such that $|\mathcal{F}| < l$ and define $\mathcal{F}_1 = \{g_i^1 \in V(G_1) | (g_i^1, g_j^2) \in \mathcal{F}$ for some $g_j^2 \in V(G_2)\}$ and $\mathcal{F}_2 = \{g_j^2 \in V(G_2) | (g_i^1, g_j^2) \in \mathcal{F}$ for some $g_i^1 \in V(G_1)\}$. Also, $|\mathcal{F}| < l$ gives that $|\mathcal{F}_2| < l$ which yields that $\mathcal{F}_2$ is not a fixing set for G_2, i.e., $\cap_{g_j^2 \in \mathcal{F}_2} Stab(g_j^2) \neq \{id_{G_2}\}$. Hence, $\cap_{(g_i^1, g_j^2) \in \mathcal{F}} Stab(g_i^1, g_j^2) \neq \{id_{G_1 * G_2}\}$ which yields that $\mathcal{F}$ is not a fixing set for $G_1 * G_2$ and we conclude that F is a minimum fixing set for $G_1 * G_2$. □

In the next theorem, we give the fixing number of $G_1 * G_2$, when exactly one of G_1 or G_2 is a rigid graph.

Theorem 3.9 *If G_1 is a rigid graph with a dominating vertex and G_2 is a non-rigid graph without false twins, then $fix(G_1 * G_2) = 2fix(G_2)$.*

Proof Let g_k^1 be dominating vertex of G_1 and $G' = \langle V(G_1) \setminus \{g_k^1\} \rangle$ be induced subgraph of G_1. Observation 3.4 yields that $G_1 * G_2 = G' * G_2 + \langle \mathcal{G}(g_k^1) \rangle$ and using Lemma 3.5 we have $fix(G_1 * G_2) = fix(G' * G_2) + fix(\langle \mathcal{G}(g_k^1) \rangle$. Further, $\langle \mathcal{G}(g_k^1) \rangle \cong G_2$ and Proposition 3.2 gives that G' is also a rigid graph, moreover, Theorem 3.8 (2) gives that $fix(G' * G_2) = fix(G_2)$. Hence, $fix(G_1 * G_2) = 2fix(G_2)$. □

Example 3.10 Let G_1 be a rigid graph with a dominating vertex. If $G_2 \cong P_n$ with $n \geq 4$, then Theorem 3.9 gives that $fix(G_1 * G_2) = 2$, because $fix(P_n) = 1$. If $G_2 \cong C_n$ with $n \geq 5$, then Theorem 3.9 gives that $fix(G_1 * G_2) = 4$, because $fix(C_n) = 2$.

For a vertex x in a graph G_1, set $C(x) = \{u \in V(G_1) : N(u) = N(x)\}$ is called the *equivalence class* of false twins of x in G_1. The next theorem, follows from Theorem 2.3 and the definition of co-normal product of graphs.

Theorem 3.11 *Let G_1 be a connected graph and G_2 be a graph such that $G_1^2, G_2^2, \ldots, G_s^2$ are distinct equivalence classes of false twins and forms a partition of $V(G_2)$, with $|G_j^2| \geq 2$, for each j. If G_1 has no false twins, then $fix(G_1 * G_2) = |V(G_1)| fix(G_2)$.*

Proof Since G_1 has no false twins, therefore, Theorem 2.3 yields that $(g_i^1, g_j^2), (g_k^1, g_l^2) \in V(G_1 * G_2)$ are false twins if and only if $g_i^1 = g_k^1$ and $g_j^2, g_l^2 \in G_r^2$ for some $1 \leq r \leq s$. We define set $\mathcal{G}_{ij} = \{g_i^1\} \times G_j^2$ for each $1 \leq j \leq s$ and $g_i^1 \in V(G_1)$ which is an equivalence class of false twins of $G_1 * G_2$ such that $|\mathcal{G}_{ij}| \geq 2$ and $G_1 * G_2$ has $|V(G_1)| \cdot s$ distinct such classes. Moreover, the collection $\mathcal{G}_{ij}$ *'s* forms a partition of $V(G_1 * G_2)$ because G_i^2 *'s* gives a partition of G_2. Hence, $fix(G_1 * G_2) = \sum_{g_i^1 \in V(G_1)} \sum_{j=1}^{s} |\mathcal{G}_{ij}| - |V(G_1)| \cdot s = |V(G_1)||V(G_2)| - |V(G_1)| \cdot s$ implies that $fix(G_1 * G_2) = |V(G_1)| fix(G_2)$. □

Example 3.12 If G_1 is a path graph of order m and G_2 be a null graph of order n, then Theorem 3.11 gives that $fix(G_1 * G_2) = m(n-1)$, because $fix(N_n) = n - 1$.

In the next two theorems, we give formulae for the fixing number of $G_1 * G_2$, under certain conditions.

Theorem 3.13 *In a graph G_1, let D be the set of all dominating vertices with $|D| \geq 1$ and $G_1^1, G_2^1, \ldots, G_r^1$ with $r \geq 1$ be the distinct equivalence classes of false twins with $|G_i^1| \geq 2$ for each $1 \leq i \leq r$. If the collection $D, G_1^1, G_2^1, \ldots, G_r^1$ gives a partition of $V(G_1)$ and G_2 has no false twins, then $fix(G_1 * G_2) = |V(G_2)|(\sum_{i=1}^{r} |G_i^1| - r) + |D| fix(G_2)$.*

Proof Let $G' = \langle V(G_1) \setminus D \rangle$, then $G_1^1, G_2^1, \ldots, G_r^1$ give a partition of $V(G')$ and are distinct equivalence classes of false twins in G'. Now for each $g_j^2 \in V(G_2)$ and for each G_i^1, the set $\mathcal{G}_{ij} = G_i^1 \times \{g_j^2\}$ is an equivalence class of false twins and there are $|V(G_2)| \cdot r$ such classes in $G' * G_2$. Moreover, the collection of $\mathcal{G}_{ij}$ *'s* yields a partition of $V(G' * G_2)$ implies that $fix(G' * G_2) = |V(G_2)|(\sum_{i=1}^{r} |G_i^1| - r)$. Now, for each $g_i^1 \in D$, Lemma 3.5 gives that the vertices of $\mathcal{G}(g_i^1)$ are not fixed by any $x \in V(G_1 * G_2) \setminus \mathcal{G}(g_i^1)$ and $\mathcal{G}(g_i^1) \cong G_2$ gives that $fix(\langle D \times V(G_2) \rangle) = |D| fix(G_2)$. Observation 3.4 extended to a set of dominating vertices gives that $G_1 * G_2 = \langle (V(G_1) \setminus D) \times V(G_2) \rangle + \langle D \times V(G_2) \rangle = G' * G_2 + \langle D \times V(G_2) \rangle$. Hence, $fix(G_1 * G_2) = |V(G_2)|(\sum_{i=1}^{r} |G_i^1| - r) + |D| fix(G_2)$. □

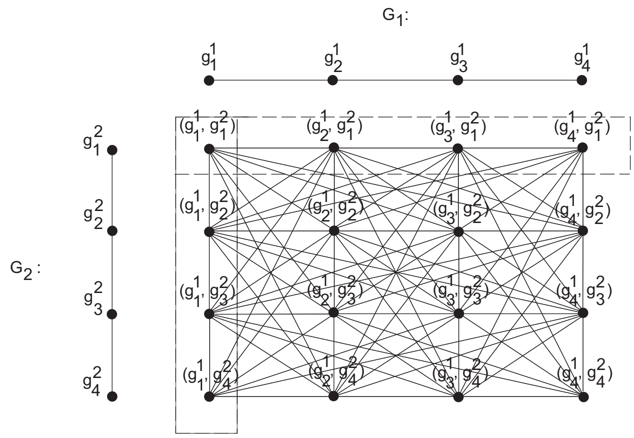

Fig. 2 The co-normal product graph of $G_1 = P_4$ and $G_2 = P_4$

Theorem 3.14 *Let G_1 and G_2 be two graphs such that G_1 has r distinct equivalence classes $G_1^1, G_2^1, \ldots, G_r^1$ of false twins and G_2 has s distinct equivalence classes $G_1^2, G_2^2, \ldots, G_s^2$ of false twins. If $\cup_{i=1}^{r} G_i^1 = V(G)$, $\cup_{j=1}^{s} G_j^2 = V(G_2)$ and $|G_i^1|, |G_j^2| \geq 2$ for each i and j, then $fix(G_1 * G_2) = |V(G_1)||V(G_2)| - rs$.*

Proof Theorem 2.3 gives that for each i and j, the set $G_i^1 \times G_j^2$ is an equivalence class of false twins in $G_1 * G_2$, moreover, the collection $G_i^1 \times G_j^2, 1 \leq i \leq r, 1 \leq j \leq s$ gives a partition of $V(G_1 * G_2)$ such that $|G_i^1 \times G_j^2| \geq 4$. Hence, $fix(G_1 * G_2) = |V(G_1)||V(G_2)| - rs$. □

The next corollary deals with the fixing number of the co-normal product of two multipartite graphs.

Corollary 3.15 *For $G_1 \cong K_{k_1,k_2,\ldots,k_r}$ and $G_2 \cong K_{l_1,l_2,\ldots,l_s}$, $fix(G_1 * G_2) = |V(G_1)|\,|V(G_2)| - rs$.*

In [9], Erwin and Harary gave the following observation.

Observation 3.16 *[9] For a graph G, $fix(G) = 1$ if and only if G has an orbit of cardinality $|Aut(G)|$.*

Note that the vertices (g_1^1, g_1^2), (g_1^1, g_4^2), (g_4^1, g_1^2) and (g_4^1, g_4^2) for the co-normal product graph $P_4 * P_4$ shown in Fig. 2, are similar. Also, set $\{(g_1^1, g_2^2)\}$ is a fixing set implies that $fix(P_4 * P_4) = 1$. In the next theorem, we give the fixing number of the co-normal product of two path graphs.

Theorem 3.17 *Let P_n be a path graph with n vertices, then for any two integers $s, t \geq 2$,*

$$fix(P_s * P_t) = \begin{cases} 1 & if \;\; s, t \geq 4, \\ 2 & if \;\; s = 2, t \geq 4, \\ 3 & if \;\; s = t = 2 or s = 2, t = 3, \\ 5 & if \;\; s = t = 3, \\ t+1 & if \;\; s = 3, t \geq 4. \end{cases}$$

Proof For $s, t \geq 4$, we discuss two cases:
Case 1) Suppose $s \neq t$, then $Aut(P_s * P_t) \cong S_2 \oplus S_2$ and the four nodes of degree $t + s - 1$ are similar so by Observation 3.16, $fix(P_t * P_s)$=1.
Case 2) Suppose $s = t$, then $Aut(P_s * P_t)$ is isomorphic to the dihedral group of order 8 and the eight nodes of degree $3s - 2$ are similar which are adjacent to the four nodes of degree $2s - 1$ (the corner nodes), so by Observation 3.16, $fix(P_s * P_t) = 1$.
Suppose $s = 2$ and $t \geq 4$, then P_s has two dominating vertices and Lemma 3.5 gives that $fix(P_s * P_t) = 2$.
Suppose $s = t = 2$, then the order of $P_s * P_t$ is 4 and it is a complete graph. Hence, $fix(P_s * P_t) = 3$.
Suppose $s = 2$ and $t = 3$, then $P_s * P_t$ has two equivalence classes of false twins with cardinality 2 and one class of true twins because P_t also has a dominating vertex. Hence, $fix(P_s * P_t) = 3$.
Suppose $s = t = 3$, then $P_s * P_t$ has an equivalence class of false twins with cardinality 4, two classes of false twins with cardinality 2 and one dominating vertex. Hence, $fix(P_s * P_t) = 5$.
Suppose $s = 3$ and $t \geq 4$, then $P_s * P_t$ has t equivalence of cardinality 2. Since P_s has a dominating vertex so Lemma 3.5 gives that $fix(P_s * P_t) = t + 1$. □

In the next theorem, we give formulae for the fixing number of the co-normal product of two graphs, when G_1 is a graph without false twins and G_2 is a star graph.

Theorem 3.18 *If G_1 is a graph without false twins and $G_2 \cong S_{n+1}$, then $fix(G_1 * G_2) = |V(G_1)|fix(G_2) + fix(G_1)$.*

Proof Let $V(G_1) = \{g_1^1, g_2^1, \ldots, g_m^1\}$ and $V(G_2) = \{g_1^2, g_2^2, \ldots, g_n^2, g_{n+1}^2\}$ and g_{n+1}^2 be the dominating vertex in G_2. Let F be a minimum fixing set for $G_1 * G_2$. Theorem 2.3 gives that for each $g_i^1 \in V(G_1)$, the set $\mathcal{G}_i = \{g_i^1\} \times (V(G_2) \setminus \{g_{n+1}^2\})$ is an equivalence class of false twins in $G_1 * G_2$ which gives that $|F \cap \mathcal{G}_i| \geq n-1$ for each $1 \leq i \leq m$. Lemma 3.5 gives that the vertices of the set $\mathcal{G}(g_{n+1}^2)$ are not fixed by any vertex in $V(G_1 * G_2) \backslash \mathcal{G}(g_{n+1}^2)$ and $\mathcal{G}(g_{n+1}^2) \cong G_1$ yields that $|F \cap \mathcal{G}(g_{n+1}^2)| \geq fix(G_1)$. Also, $\mathcal{G}(g_{n+1}^2) \cap \mathcal{G}_i = \emptyset$ for each i implies that $fix(G_1 * G_2) \geq |V(G_1)|fix(G_2) + fix(G_1)$. Let F_1 be a minimum fixing set for G_1 and F_2 be a minimum fixing set for G_2. Consider the set $F = \cup_{i=1}^{m}(\{g_i^1\} \times F_2) \cup (F_1 \times \{g_{n+1}^2\})$ such that $Stab(F) = Stab(F_1) \times Stab(F_2) = \{id_{G_1 * G_2}\}$ yields that F is a fixing set for $G_1 * G_2$ and $fix(G_1 * G_2) \leq |V(G_1)|fix(G_2) + fix(G_1)$. Hence, $fix(G_1 * G_2) = |V(G_1)|fix(G_2) + fix(G_1)$. □

Example 3.19 Let $G_1 \cong S_{m+1}$ and $G_2 \cong S_{n+1}$ whereas $V(G_1) = \{g_1^1, g_2^1, \ldots, g_m^1, g_{m+1}^1\}$ and $V(G_2) = \{g_1^2, g_2^2, \ldots, g_n^2, g_{n+1}^2\}$ such that g_{m+1}^1 is dominating vertex in G_1 and g_{n+1}^2 is dominating in G_2. The equivalence classes of false twins in G_1 are $G_1^1 = \{g_1, g_2, \ldots, g_m\}$, $G_2^1 = \{g_{m+1}\}$ and $G_1^2 = \{g_1^2, g_2^2, \ldots, g_n^2\}$, $G_2^2 = \{g_{n+1}^2\}$ are equivalence classes of false twins in G_2. The classes $G_1^1 \times G_1^2$, $G_1^1 \times G_2^2$, $G_2^1 \times G_1^2$ and $G_2^1 \times G_2^2$ are equivalence classes of false twins such that $|G_2^1 \times G_2^2| = 1$. Also, $G_1^1 \times G_2^2 = \mathcal{G}(g_{n+1}^2) \backslash \{(g_{m+1}^1, g_{n+1}^2)\}$, $G_2^1 \times G_1^2 = \mathcal{G}(g_{m+1}^1) \backslash \{(g_{m+1}^1, g_{n+1}^2)\}$ and (g_{m+1}^1, g_{n+1}^2) is the dominating vertex in $G_1 * G_2$. Using Lemma 3.5, we have $fix(G_1 * G_2) = mn + fix(G_1) + fix(G_2) - 1$.

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