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ORIGINAL RESEARCH

On path-factor critical uniform graphs

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Abstract A graph G is $P_{\geq k}$-factor uniform if for arbitrary two distinct edges e_1 and e_2 of G, G contains a $P_{\geq k}$-factor including e_1 and excluding e_2. In this paper, we study the relationship between some graphic parameters and the existence of path factor with specific properties. Our main contributions are three-fold: firstly, we define the new concept of $(P_{\geq k}, n)$-critical uniform graph, namely, a graph G is $(P_{\geq k}, n)$-critical uniform if the graph $G - V'$ is $P_{\geq k}$-factor uniform for any $V' \subseteq V(G)$ with $|V'| = n$. Secondly, the relationship between binding number and $(P_{\geq k}, n)$-critical uniform graphs ($k = 2, 3$) is studied. Thirdly, we demonstrate the sharpness of the main results in this paper by constructing several graph classes. Furthermore, the relationship between other graph parameters and $(P_{\geq k}, n)$-critical uniform graphs can be studied further.

Keywords Binding number · $P_{\geq 2}$-factor · $P_{\geq 3}$-factor · $(P_{\geq 2}, n)$-critical uniform graph · $(P_{\geq 3}, n)$-critical uniform graph

Mathematics Subject Classification 05C70 · 05C38

1 Introduction

From the perspective of network security, researchers must consider the robustness and vulnerability of the network. It is well known that the denser the network, the higher its robustness and the stronger its resistance to attacks; on the contrary, the sparse network is more vulnerable to attacks by comparison. We look upon the whole network as a graph. The positions match the vertices, and the channels match the edges of the graph. In the communication networks, large data packets are sent to different destinations through channels. Scientists model the feasible data allocation problem and divide the large data packets into small data packets, known as the network flow problem in graph theory.

In communication transmission networks, the data transmission between two positions passes through a path between two corresponding vertices. The feasibility of data transmission corresponds to the existence of path factor in the corresponding network. The existence of a path-factor critical uniform graph is also of vital importance in data transmission for network security. The path-factor critical uniform graph requires the presence of the factor under the combined frame of deleting vertices and edges, which corresponds to data transmissibility under the situation where multiple stations and transmission channels are destroyed in the network. In addition, the existence of path-factor critical uniform graph, is equivalent to the possibility of data transmission in a communication network in a specific setting when some nonadjacent nodes with each other are damaged and

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a special channel is assigned. Graphic parameters (such as: toughness, binding number, neighborhood union, neighbor set and minimum degree) are often used to measure the ruggedness and vulnerability of network attacks. In this paper, we investigate the relationship between graphic parameter binding number and the existence of a path-factor critical uniform graph. The study on the existence of path-factor critical uniform graph can provide theoretical guidance for network security, which help scientists to design and construct networks with high data transmission.

In this paper, we deal with only finite undirected graphs without loops or multiple edges. Let $N_G[x]$ denote $N_G(x) \cup \{x\}$. Denote by $i(G)$ and $\omega(G)$ the number of isolated vertices and the number of connected components in G, respectively. For a subset $X \subseteq V(G)$, we use $N_G(X)$ to denote the union of $N_G(x)$ for every $x \in X$, and $G[X]$ to denote the subgraph of G induced by X, respectively. For two given graphs G_1 and G_2, we define $G_1 \cup G_2$ by the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and $G_1 \vee G_2$ by the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{e = xy : x \in V(G_1), y \in V(G_2)\}$. K_n and P_n denote the complete graph and the path of order n, respectively. We use $\kappa(G)$ to denote the vertex connectivity of G. Woodall [18] introduced the concept of binding number. For a connected graph G, its binding number, was defined by

$$bind(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

A path-factor is a spanning subgraph F of G such that each component of F is a path of order at least two. Assume $k \geq 2$ be an integer. A $P_{\geq k}$-factor is a spanning subgraph of G whose components are paths of order at least k. A graph G is $P_{\geq k}$-factor covered if for each edge e of G, G admits a $P_{\geq k}$-factor containing e. If for each edge e of G, G admits a $P_{\geq k}$-factor excluding e, then graph G is called $P_{\geq k}$-factor deleted. $P_{\geq k}$-factor uniform graphs satisfy both $P_{\geq k}$-factor covered and $P_{\geq k}$-factor deleted. Namely, a graph G is $P_{\geq k}$-factor uniform if for any two distinct edges e_1 and e_2 of G, G contains a $P_{\geq k}$-factor including e_1 and excluding e_2.

A graph G is said to be $(P_{\geq k}, n)$-factor critical covered (deleted) if for any $V' \subseteq V(G)$ with $|V'| = n$, the graph $G - V'$ is $P_{\geq k}$-factor covered (deleted). We define the new concept of $(P_{\geq k}, n)$-critical uniform graph, namely, a graph G is $(P_{\geq k}, n)$-critical uniform if the graph $G - V'$ is $P_{\geq k}$-factor uniform for any $V' \subseteq V(G)$ with $|V'| = n$. Obviously, a $(P_{\geq k}, 0)$-critical uniform graph is equivalent to a $P_{\geq k}$-factor uniform graph.

Kaneko [10] introduced the concept of a sun so as to characterize a graph possessing a $P_{\geq 3}$-factor. A spanning subgraph F of G is called a 1-factor (namely, a perfect matching) if $d_F(v) = 1$ for any $v \in V(G)$. If a graph H satisfying $H - v$ admits a 1-factor for any $v \in V(H)$, then we call H a factor-critical graph. Let H be a factor-critical graph such that $V(H) = \{v_1, v_2, \cdots, v_n\}$. By adding new vertices $\{u_1, u_2, \cdots, u_n\}$ together with new edges $\{v_i u_i : 1 \leq i \leq n\}$ to H, the resulting new graph is called a sun. We also look upon K_1 and K_2 as suns. A sun other than K_1 and K_2 is called a big sun. It is obvious that the order of a big sun is at least 6. We call a component of G a sun component if it is isomorphic to a sun. Denote by $sun(G) = |Sun(G)|$ the number of sun components in G.

There has been some research on path-factors in graphs by Egawa, Furuya and Ozeki [4], Johansson [9], and Kelmans [13]. Kano, Lu and Yu [12] demonstrated that a graph G admits a $P_{\geq 3}$-factor if $i(G-X) \leq \frac{2}{3}|X|$ for any $X \subseteq V(G)$. Bazgan, Benhamdine, Li, and Woźniak [2] justified that a 1-tough graph G of order at least 3 has a $P_{\geq 3}$-factor. Zhou, Bian and Pan [26] put forward some sufficient conditions on $P_{\geq 3}$-factor critical deleted graphs. Chen and Dai [3] improved Zhou, Bian and Pan's result. Zhou and Sun [29] posed a binding number condition for a graph being a $P_{\geq 3}$-factor uniform graph. Gao and Wang [6] improved Zhou and Sun's result on $P_{\geq 3}$-factor uniform graphs. Wang and Zhang [15] presented a minimum degree and independence number condition for the existences of $P_{\geq 3}$-factor critical deleted graph. Liu [14] posed some sufficient conditions for the existence of path-factor uniform graphs. Gao, Wang and Chen [7] derived some tight bounds for the existence of path factors in graphs. Zhou, Sun and Liu [30] showed toughness and isolated toughness conditions for $P_{\geq 3}$-factor deleted graphs.

Zhou and Bian [25], Zhou, Sun and Liu [31], Zhou, Wu and Xu [33], Gao, Chen and Wang [5], Wang and Zhang [16] obtained some results on $P_{\geq 3}$-factors of graphs with specific properties. Zhou [23] showed a degree condition for graphs to be $P_{\geq 3}$-factor deleted graphs. Zhou, Liu and Xu [28], Zhou [21], Zhou, Wu and Bian [32], Wang and Zhang [17] established some relationships between minimum degree and graph factors. Wu [19] discussed the relationships between independence number and graph factors. Some other research on graph factors can see [8,22,24,27].

Akiyama, Avis and Era [1] acquired a necessary and sufficient condition for a graph to contain a $P_{\geq 2}$-factor.

Theorem 1.1 *([1]) A graph G has a $P_{\geq 2}$-factor if and only if $i(G-X) \leq 2|X|$ for all $X \subseteq V(G)$, where $i(G-X)$ denotes the number of isolated vertices of $G-X$.*

The following criterion for a graph possessing a $P_{\geq 3}$-factor is due to Kaneko [10], for which Kano, Katona and Kiraly [11] put forward a simpler proof.

Theorem 1.2 *([10,11]) A graph G permits a $P_{\geq 3}$-factor if and only if $sun(G-X) \leq 2|X|$ for every $X \subseteq V(G)$.*

Zhang and Zhou [20] put forward two characterizations for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs.

Theorem 1.3 *([20]) A connected graph G is a $P_{\geq 2}$-factor covered graph if and only if $i(G-X) \leq 2|X| - \varepsilon_1(X)$ for any $X \subseteq V(G)$, in which $\varepsilon_1(X)$ is represented by*

$$\varepsilon_1(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set\ and\ there\ exists\ a\ nontrivial \\ & component\ of\ G-X; \\ 0, & otherwise. \end{cases}$$

Theorem 1.4 *([20]) A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if $sun(G-X) \leq 2|X| - \varepsilon_2(X)$ for any $X \subseteq V(G)$, in which $\varepsilon_2(X)$ is represented by*

$$\varepsilon_2(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set\ and\ there\ exists\ a\ non-sun \\ & component\ of\ G-X; \\ 0, & otherwise. \end{cases}$$

In this article, we investigate path-factor critical uniform graphs, and verify two binding number conditions for graphs to be $(P_{\geq 2}, n)$-critical uniform and $(P_{\geq 3}, n)$-critical uniform graphs, which are shown in Sections 2 and 3. Both Theorem 1.3 and Theorem 1.4 will be applied in the proof of our main results.

2 Binding number and $(P_{\geq 2}, n)$-critical uniform graphs

Theorem 2.1 *Let $n \geq 1$ be an integer, and let G be a graph with $\kappa(G) \geq n+2$. Suppose that G satisfies $bind(G) > \frac{2+n}{3}$. Then G is a $(P_{\geq 2}, n)$-critical uniform graph.*

Proof of Theorem2.1 We verify the theorem by contradiction. Let $G' = G - V'$ for any $V' \subseteq V(G)$ with $|V'| = n$. Suppose that $H = G' - e$ is not a $P_{\geq 2}$-factor covered graph for some $e = xy \in E(G')$. Then by Theorem 1.3,

$$i(H-X) \geq 2|X| - \varepsilon_1(X) + 1 \tag{1}$$

for some $X \subseteq V(H)$. □

Claim 1. $X \neq \emptyset$.

Proof Let $X = \emptyset$. Then $\varepsilon_1(X) = 0$. In view of $\kappa(G) \geq n+2$, H is connected. By (1), we obtain $i(H) \geq 1$. This contradicts that H is connected. Claim 1 is verified. □

In what follows, we shall consider two cases and derive a contradiction in each case.

Case 1. $|X| = 1$.

We write $Y = \{x : x \in V(H) \setminus X, d_{H-X}(x) = 0\}$.

Subcase 1.1 $H - X$ does that possess a nontrivial component.

Note that $\varepsilon_1(X) = 0$. In light of (1),

$$i(H-X) \geq 2|X| - \varepsilon_1(X) + 1 = 2|X| + 1 = 3.$$

It follows that $|Y| = i(H-X) \geq 2|X|+1 = 3$. Apparently, $Y \neq \emptyset$ and $|N_G(Y)| \leq |V' \cup X| = |V'|+|X| = n+1$. According to the definition of binding number, we admit

$$bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{1+n}{3}.$$

Combining this inequality with the assumption of Theorem 2.1, we have

$$\frac{1+n}{3} \geq bind(G) > \frac{2+n}{3},$$

which is a contradiction.

Subcase 1.2 $H - X$ possesses a nontrivial component Q.

Clearly, $\varepsilon_1(X) = 1$. By (1), we derive

$$i(H-X) \geq 2|X| - \varepsilon_1(X) + 1 = 2|X| = 2.$$

Obviously, $|Y| = i(H-X) \geq 2$ and $|V(Q)| \geq 2$. We see that $|N_G(Y \cup V(Q))| \leq |V'| + |X| + |V(Q)| = n + 1 + |V(Q)|$. Thus, we get

$$\begin{aligned} bind(G) &\leq \frac{|N_G(Y \cup V(Q))|}{|Y \cup V(Q)|} \leq \frac{n+1+|V(Q)||}{|Y|+|V(Q)|} \leq \frac{n+1+|V(Q)|}{2+|V(Q)|} \\ &= 1 + \frac{n-1}{2+|V(Q)|} \leq 1 + \frac{n-1}{4} = \frac{3+n}{4}, \end{aligned}$$

Combining this inequality with the assumption of Theorem 2.1, we have

$$\frac{3+n}{4} \geq bind(G) > \frac{2+n}{3},$$

which is a contradiction.

Case 2. $|X| \geq 2$.

In this case, $\varepsilon_1(X) \leq 2$. According to (1), we obtain

$$i(H-X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 3. \tag{2}$$

It follows that $|Y| = i(H-X) \geq 2|X| - 1 \geq 3$. Thus $Y \neq \emptyset$ and $|N_G(Y)| \leq |V' \cup X| = |V'| + |X| = n + |X|$. It follows that

$$\begin{aligned} bind(G) &\leq \frac{|N_G(Y)|}{|Y|} = \frac{|N_G(Y)|}{i(H-X)} \leq \frac{n+|X|}{2|X|-1} \\ &= \frac{1}{2}\left(\frac{2n+1}{2|X|-1} + 1\right) \leq \frac{1}{2}\left(\frac{2n+1}{3} + 1\right) = \frac{2+n}{3} \end{aligned}$$

by (2) and $|X| \geq 2$. This contradicts that $bind(G) > \frac{2+n}{3}$. We complete the proof of Theorem 2.1. □

Remark 2.1 In what follows, we claim that $bind(G) > \frac{2+n}{3}$ in Theorem 2.1 is sharp, that is, it cannot be replaced by $bind(G) \geq \frac{2+n}{3}$.

We construct a graph $G = K_{n+2} \vee (3K_1)$, where $n \geq 1$ is an integer. Note that $bind(G) = \frac{2+n}{3}$ and $\kappa(G) = n+2$. Let $G' = G - V'$ for any $V' \subseteq V(G)$ with $|V'| = n$. We choose $e = uv$, where $u \in (K_{n+2} \setminus V')$, $v \in (3K_1)$. Let $H = G' - e$. We select $X = V(K_{n+2} \setminus V') \subseteq V(H)$, and so $|X| = 2$. Then X is not an independent set and $\varepsilon_1(X) = 2$. Thus,

$$i(H-X) = 3 > 2 = 2|X| - \varepsilon_1(X).$$

It follows from Theorem 1.3 that H is not a $P_{\geq 2}$-factor covered graph, which implies that G' is not a $P_{\geq 2}$-factor uniform graph. Hence, G is not a $(P_{\geq 2}, n)$-critical uniform graph.

Remark 2.2 Next, we claim that the condition $\kappa(G) \geq n+2$ is sharp, that is, it cannot be replaced by $\kappa(G) \geq n+1$.

We construct a graph $G = K_{n+1} \vee (2K_1)$, where $n > 1$ is an integer. Obviously, $\kappa(G) \geq n+1$ and $bind(G) = \frac{n+1}{2} > \frac{2+n}{3}$. We choose $V' \subseteq V(K_{n+1})$ with $|V'| = n$, and let $G' = G - V'$. Clearly, G' is not a $P_{\geq 2}$-factor uniform graph, and so G is not a $(P_{\geq 2}, n)$-critical uniform graph. □

3 Binding number and $(P_{\geq 3}, n)$-critical uniform graphs

Theorem 3.1 *Let $n \geq 1$ be an integer, and let G be a graph with $\kappa(G) \geq n+2$ and $|V(G)| \geq n+3$. Then G is a $(P_{\geq 3}, n)$-critical uniform graph if $bind(G) \geq \frac{4+n}{3}$.*

Proof of Theorem3.1 For any $V' \subseteq V(G)$ with $|V'| = n$, we write $G' = G - V'$. Let $H = G' - e$ for any $e = xy \in E(G')$. Obviously, H is connected. In order to prove Theorem 3.1, it suffices to present that H is $P_{\geq 3}$-factor covered. Assume, to the contrary, that H is not $P_{\geq 3}$-factor covered. Then applying Theorem 1.4, there exists some subset $X \subseteq V(H)$ such that

$$sun(H - X) \geq 2|X| - \varepsilon_2(X) + 1. \tag{3}$$

□

Claim 2. $|X| \geq 1$.

Proof Suppose $X = \emptyset$. Then $\varepsilon_2(X) = 0$. By (3), we have $sun(H) \geq 1$. Note that H is connected. Thus, $1 \leq sun(H) \leq \omega(H) = 1$, which implies $sun(H) = \omega(H) = 1$. Hence H is a big sun. (In fact, suppose that H is not a big sun. Then $H = K_1$ or K_2. Thus $|V(H)| \leq 2$ and $|V(G)| = |V(H)| + |V'| \leq 2+n$, which contradicts that $|V(G)| \geq n+3$.) We denote by R the factor-critical subgraph of H, and thus $|V(H) \setminus V(R)| = |V(R)| \geq 3$. Hence, we admit

$$\begin{aligned} bind(G) &\leq \frac{|N_G(V(H) \setminus V(R))|}{|V(H) \setminus V(R)|} = \frac{|N_G(V(G) \setminus (V' \cup V(R)))|}{|V(R)|} \leq \frac{|V' \cup V(R)|}{|V(R)|} \\ &= \frac{|V'| + |V(R)|}{|V(R)|} = 1 + \frac{n}{|V(R)|} \leq 1 + \frac{n}{3}, \end{aligned}$$

which contradicts that $bind(G) \geq \frac{4+n}{3}$. This completes the proof of Claim 2. □

Suppose that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, in $H - X$. By (3),

$$a + b + c = sun(H - X) \geq 2|X| - \varepsilon_2(X) + 1. \tag{4}$$

In the following, we consider two cases and derive a contradiction in each case.

Case 1. $|X| = 1$.

Subcase 1.1 $H - X$ has a non-sun component Y.

In this case, $\varepsilon_2(X) = 1$ and $|V(Y)| \geq 3$. Otherwise, $Y = K_1$ or K_2, which contradicts that Y is a non-sun component. Hence, by (4), we get

$$a + b + c = sun(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 2|X| = 2. \tag{5}$$

Subcase 1.1.1 $a = 0$.

By (5), we have $b + c \geq 2$. Let $W = Y \cup (bK_2) \cup H_1 \cup \cdots \cup H_c$. Then there exist $x, y \in V(W)$ with $d_W(x) = 1$ and $xy \in E(W)$. Hence, we have

$$\begin{aligned} |N_G(V(W) \setminus \{y\})| &\leq |V'| + |X| + |V(Y)| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1 \\ &= n + 1 + |V(Y)| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1 \\ &= n + |V(Y)| + 2b + \sum_{i=1}^{c} |V(H_i)| \end{aligned}$$

and

$$|V(W) \setminus \{y\}| = |V(Y)| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1 \geq 3 + 2b + 6c - 1 = 2 + 2b + 6c.$$

Note that $b + c \geq 2$ and $bind(G) \geq \frac{4+n}{3}$, we obtain

$$\begin{aligned}\frac{4+n}{3} \leq bind(G) &\leq \frac{|N_G(V(W)\setminus\{y\})|}{|V(W)\setminus\{y\}|} \leq \frac{n+|V(Y)|+2b+\sum_{i=1}^{c}|V(H_i)|}{|V(Y)|+2b+\sum_{i=1}^{c}|V(H_i)|-1}\\ &= 1+\frac{n+1}{|V(Y)|+2b+\sum_{i=1}^{c}|V(H_i)|-1} \leq 1+\frac{n+1}{2+2b+6c}\\ &\leq 1+\frac{n+1}{2+2b+2c} \leq 1+\frac{n+1}{6},\end{aligned}$$

which is a contradiction.

Subcase 1.1.2 $a \geq 1$.

Let $U = V(Y) \cup V(aK_1) \cup V(bK_2) \cup V(H_1) \cup \cdots \cup V(H_c)$. Then

$$\begin{aligned}|N_G(U)| &\leq |V'| + |X| + |V(Y)| + 2b + \sum_{i=1}^{c}|V(H_i)|\\ &= n + 1 + |V(Y)| + 2b + \sum_{i=1}^{c}|V(H_i)|\end{aligned}$$

and

$$|U| = |V(Y)| + a + 2b + \sum_{i=1}^{c}|V(H_i)| \geq 3 + a + 2b + 6c.$$

Thus,

$$\begin{aligned}bind(G) &\leq \frac{|N_G(U)|}{|U|} \leq \frac{n+1+|V(Y)|+2b+\sum_{i=1}^{c}|V(H_i)|}{|V(Y)|+a+2b+\sum_{i=1}^{c}|V(H_i)|}\\ &= 1+\frac{n+1-a}{|V(Y)|+a+2b+\sum_{i=1}^{c}|V(H_i)|} \leq 1+\frac{n}{3+a+2b+6c}\\ &\leq 1+\frac{n}{3+a+b+c} \leq 1+\frac{n}{5}\end{aligned}$$

by (5) and the definition of $bind(G)$, which contradicts that $bind(G) \geq \frac{4+n}{3}$.

Subcase 1.2 $H - X$ does not have a non-sun component.

In this case, $\varepsilon_2(X) = 0$. Hence,

$$a + b + c = sun(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 2|X| + 1 = 3. \tag{6}$$

by (4).

Subcase 1.2.1 $a = 0$.

Let $Z = (bK_2) \cup H_1 \cup \cdots \cup H_c$. Then there exist $u, v \in V(Z)$ with $d_Z(u) = 1$ and $uv \in E(Z)$. In view of $bind(G) \geq \frac{4+n}{3}$, we obtain

$$\begin{aligned} \frac{4+n}{3} &\leq bind(G) \leq \frac{|N_G(V(Z) \setminus \{v\})|}{|V(Z) \setminus \{v\}|} \leq \frac{|V'| + |X| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1}{2b + \sum_{i=1}^{c} |V(H_i)| - 1} \\ &\leq \frac{n + 2b + \sum_{i=1}^{c} |V(H_i)|}{2b + \sum_{i=1}^{c} |V(H_i)| - 1} = 1 + \frac{n+1}{2b + \sum_{i=1}^{c} |V(H_i)| - 1}. \end{aligned}$$

Combining this with (6), $a = 0$ and $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, we derive

$$\frac{4+n}{3} \leq 1 + \frac{n+1}{2b+6c-1} \leq 1 + \frac{n+1}{2a+2b+2c-1} \leq 1 + \frac{n+1}{6-1} = 1 + \frac{n+1}{5},$$

which is a contradiction.

Subcase 1.2.2 $a \geq 1$.

Let $T = (aK_1) \cup (bK_2) \cup (H_1) \cup \cdots \cup (H_c)$. We have

$$\begin{aligned} \frac{4+n}{3} &\leq bind(G) \leq \frac{|N_G(V(T))|}{|V(T)|} \leq \frac{|V'| + |X| + 2b + \sum_{i=1}^{c} |V(H_i)|}{a + 2b + \sum_{i=1}^{c} |V(H_i)|} \\ &= \frac{n + 1 + 2b + \sum_{i=1}^{c} |V(H_i)|}{a + 2b + \sum_{i=1}^{c} |V(H_i)|} \leq \frac{n + a + 2b + \sum_{i=1}^{c} |V(H_i)|}{a + 2b + \sum_{i=1}^{c} |V(H_i)|} \\ &= 1 + \frac{n}{a + 2b + \sum_{i=1}^{c} |V(H_i)|}. \end{aligned}$$

Combining this with (6) and $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, we admit

$$\frac{4+n}{3} \leq bind(G) \leq 1 + \frac{n}{a+2b+6c} \leq 1 + \frac{n}{a+b+c} \leq 1 + \frac{n}{3} = \frac{n+3}{3},$$

which is a contradiction.

Case 2. $|X| \geq 2$.

Subcase 2.1 $b + c = 0$.

In this case, $\varepsilon_2(X) \leq 2$. By (4), we have

$$a + b + c = sun(H - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \geq 3. \tag{7}$$

By (7), we have $a \geq 3$. Hence, in view of (7) and $bind(G) \geq \frac{4+n}{3}$,

$$\begin{aligned} \frac{4+n}{3} &\leq bind(G) \leq \frac{|N_G(V(aK_1))|}{|V(aK_1)|} \leq \frac{|V'| + |X|}{a} = \frac{n + |X|}{a} \\ &\leq \frac{n + \frac{a+b+c+1}{2}}{a} \leq \frac{n + \frac{a+1}{2}}{a} = \frac{2n + a + 1}{2a} \\ &= \frac{1}{2} + \frac{2n+1}{2a} \leq \frac{1}{2} + \frac{2n+1}{6} = \frac{n+2}{3}, \end{aligned}$$

which is a contradiction.

Subcase 2.2 $b + c \geq 1$.

We write $M = (bK_2) \cup H_1 \cup \cdots \cup H_c$. Thus there exist $x, y \in V(M)$ with $d_M(x) = 1$ and $xy \in E(M)$. By $bind(G) \geq \frac{4+n}{3}$ and $|V(H_i)| \geq 6$ for $1 \leq i \leq c$,

$$\frac{4+n}{3} \leq bind(G) \leq \frac{|N_G(V(aK_1) \cup (V(M) \setminus \{y\}))|}{|V(aK_1) \cup (V(M) \setminus \{y\})|} \leq \frac{|V'| + |X| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1}{a + 2b + \sum_{i=1}^{c} |V(H_i)| - 1}$$

$$\leq \frac{n + |X| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1}{a + 2b + \sum_{i=1}^{c} |V(H_i)| - 1} = 1 + \frac{n + |X| - a}{a + 2b + \sum_{i=1}^{c} |V(H_i)| - 1} \leq 1 + \frac{n + |X| - a}{a + 2b + 6c - 1},$$

Combining this with (7), $|X| \geq 2$, $b + c \geq 1$ and $n \geq 1$, we obtain

$$\frac{4+n}{3} \leq 1 + \frac{n + |X| - a}{a + 2b + 6c - 1} \leq 1 + \frac{n + \frac{a+b+c+1}{2} - a}{a + 2b + 6c - 1} = \frac{a + 5b + 13c + 2n - 1}{2a + 4b + 12c - 2},$$

which implies $5a + b + 9c - 5 + (2a + 4b + 12c - 8)n \leq 0$. This is a contradiction since

$$\begin{aligned} 5a + b + 9c - 5 + (2a + 4b + 12c - 8)n &= 7a + 5b + 21c - 13 + (2a + 4b + 12c - 8)(n - 1) \\ &\geq 5(a + b + c) - 13 + (2a + 4b + 12c - 5)(n - 1) \\ &\geq 2 + [2(a + b + c) - 5](n - 1) \\ &\geq 2 + n - 1 = n + 1 > 0 \end{aligned}$$

by $a + b + c \geq 3$ and $n \geq 1$. We complete the proof of Theorem 3.1. □

Remark 3.1 The condition $bind(G) \geq \frac{4+n}{3}$ in Theorem 3.1 is tight, it cannot be substitute for $bind(G) \geq \frac{4+n}{4}$.

We construct a graph $G = K_{n+2} \vee (2K_1 \cup K_2)$, where $n \geq 1$ is an integer. Note that $bind(G) = \frac{4+n}{4} < \frac{4+n}{3}$ and $\kappa(G) = n + 2$. Let $G' = G - V'$ for any $V' \subseteq V(K_{n+2})$ with $|V'| = n$. We select $e = uv$, where $u \in (K_{n+2} \setminus V')$, $v \in (2K_1)$. Let $H = G' - e$. We choose $X = V(K_{n+2} \setminus V') \subseteq V(H)$, and so $|X| = 2$. Then X is not an independent set and $\varepsilon_2(X) = 2$. Thus,

$$sun(H - X) = 3 > 2 = 2|X| - \varepsilon_2(X).$$

It follows from Theorem 1.4 that H is not a $P_{\geq 3}$-factor covered graph, which implies that G' is not a $P_{\geq 3}$-factor uniform graph. Hence, G is not a $(P_{\geq 3}, n)$-critical uniform graph.

Remark 3.2 The condition $\kappa(G) \geq n + 2$ is tight, it cannot be substitute for $\kappa(G) \geq n + 1$.

We construct a graph $G = K_{n+1} \vee (2K_1)$, where $n \geq 5$ is an integer. Obviously, $\kappa(G) \geq n + 1$ and $bind(G) = \frac{n+1}{2} \geq \frac{4+n}{3}$. For $V' \subseteq V(K_{n+1})$ with $|V'| = n$, let $G' = G - V'$. Apparently, G' is not a $P_{\geq 3}$-factor uniform graph, and so G is not a $(P_{\geq 3}, n)$-critical uniform graph.

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