Indian J Pure Appl Math (2024) 55:1234–1250
https://doi.org/10.1007/s13226-023-00430-1

ORIGINAL RESEARCH

Generalized Cassini identities via the generalized Fibonacci fundamental system. Applications

I. M. Craveiro · E. V. Pereira Spreafico · M. Rachidi

Received: 27 January 2022 / Accepted: 15 May 2023 / Published online: 23 May 2023

Abstract In this paper we explore the generalized Cassini identities for the weighted generalized Fibonacci sequences, through the associated generalized Fibonacci fundamental system. Some algebraic, combinatoric and analytic properties of these identities are established. Applications to generalized Fibonacci and Pell numbers are provided, and some special cases are studied.

Keywords Weighted generalized Fibonacci sequences · Generalized Cassini identities · Generalized Fibonacci Fundamental system · Combinatorial formulas · Analytical formulas · Generalized Fibonacci numbers · Generalized Pell numbers

Mathematics Subject Classification Primary 11B39 · 39A06 · 39A60 · Secondary 65Q10 · 65Q30

1 Introduction

Cassini's first identity is known to be one of the first properties of Fibonacci numbers, it is stated by the next formula,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n; \text{ for every } n \geq 1, \tag{1}$$

where $\{F_n\}_{n\geq 0}$ is the usual sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. This identity was established by the Italian-born French astronomer and mathematician Giovanni D. Cassini (1625-1712), and independently by the Scottish mathematician and landscape artist Robert Simson (1687-1768) (see, for instance, [9,11]). Since this period several extensions, proofs and formulations of the Cassini identity has been provided. Especially, Cassini identity (1) has been considered under the determinant of a special matrix in [14,16].

For the generalized Fibonacci and Pell numbers, the Cassini identity has been generalized for $r \geq 2$, using the concept of determinant, in [8,14] and [15]. In [16], Soykan generalizes the usual Cassini identity (1) to the

Communicated by Sanoli Gun.

I. M. Craveiro
Mathematics Department - UFGD, Dourados, MS, Brazil
E-mail: irenecraveiro@ufgd.edu.br

E. V. Pereira Spreafico (✉) · M. Rachidi
Institute of Mathematics, INMA - UFMS, Campo Grande, MS, Brazil
E-mail: elen.spreafico@ufms.br

M. Rachidi
E-mail: mustapha.rachidi@ufms.br

so-called generalized m-step Fibonacci numbers and gives some related formulas, using also the concept of determinant.

Recently, several generalizations of Cassini identity have been established under various aspects. Among these generalizations we can cite the identity of Catalan and the Vajda's Formula (see, for instance, [1,3,9,13], and references therein). On the other side, the Cassini identity has been extended to other sequences of known numbers such that Lucas and Pell numbers (see [15] and references therein).

On the other hand, the sequences defining the generalized Fibonacci numbers, the generalized Pell numbers or the Jacobsthal numbers, are special cases of the sequences defined by linear recurrence relations of Fibonacci. More precisely, they represent a special case of the so-called weighted r-generalized Fibonacci sequences. Such sequences, widely studied in the literature, are defined as follows. Let consider the weighted generalized Fibonacci sequences $\{v_n\}_{n\geq 0}$ of higher order $r \geq 2$, defined by giving two sequences $\{a_j\}_{0\leq j\leq r-1}$ and $\{\alpha_j\}_{0\leq j\leq r-1}$ ($r \geq 2$) of $\mathbb{K}$ (field of real or complex numbers) , such that

$$v_n = \sum_{i=0}^{r-1} a_i v_{n-i-1} \qquad \text{for} \qquad n \geq r, \tag{2}$$

$$v_n = \alpha_n \qquad \text{for} \qquad n = 0,\ 1,\ \ldots,\ r-1. \tag{3}$$

The sequences $\{a_j\}_{0\leq j\leq r-1}$ and $\{\alpha_j\}_{0\leq j\leq r-1}$ are known in the literature as the *coefficients* and the *initial conditions* (or *initial data*) of the sequence (2). For reasons of simplicity, in the sequel we will refer to these sequences by sequences (2). When considering only Expression (2), without taking into account the initial conditions (3), we are then dealing with a linear difference equation of constant coefficients.

In the present study, we focus on the generalization of the Cassini identity (1) to the weighted r-generalized Fibonacci sequences (2). Our generalization is based on the fundamental role of a specific set $\{\{v_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$, whose elements $\{v_n^{(s)}\}_{n\geq 0}$ represent solutions of Equation (2), considered as a linear difference equation. Currently, this set is known as the generalized Fibonacci fundamental system of Equation (2). More precisely, our approach is theoretically based on the techniques and properties of the linear difference equations with constant coefficients, and those of the weighted generalized Fibonacci sequences, as well as the properties of the determinants of matrices. To reach our goal, we would like to precise that the progress of our process is as follows. First, we explicitly describe the closed connection between the family of weighted generalized Fibonacci $\{v_n^{(s)}\}_{n\geq 0}$, for $1 \leq s \leq r-2$, and the fundamental weighted r-generalized Fibonacci numbers $\{v_n^{(r)}\}_{n\geq 0}$. Second, we utilize the matrix properties related to this generalized fundamental system for establishing the generalization of the Cassini identity associated to the sequence $\{v_n\}_{n\geq 0}$ defined by (2), throughout properties of the Casoratian matrix and the Casoratian, associated to the aforementioned generalized Fibonacci fundamental system $\{\{v_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$, considered as a fundamental system of solutions of Equation (2), conceived as a linear difference equation. Furthermore, the combinatorial and the analytical representations of this generalized Cassini identity are provided explicitly. As a consequence, we get various properties on the generalized Cassini identities, related to the generalized Fibonacci and Pell numbers. Moreover, some results of the literature are recovered, especially, those of [8,14,15]. Finally, significant illustrative examples and applications are furnished and some perspectives are proposed.

The content of this paper is structured as follows. In Section 2 we establish some results on the generalized Fibonacci fundamental system $\{\{v_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$, whose elements are conceived as solutions for the linear difference equation (2), and establish its closed connection with the Casoratian matrix and the Casoratian. Moreover, some identities related to the generalized Fibonacci fundamental system are derived. Section 3 is devoted to fundamental solution $\{v_n^{(r)}\}_{n\geq 0}$ of Equation (2), and its main role in the study of the generalized Cassini identity for sequences (2). Moreover, its combinatorial and analytical properties are explored, by providing its explicit formulas in Section 4. In Section 5 some applications of the results of Sections 2, 3 and 4 are established. More precisely, we are concerned with the generalized Cassini identities for generalized Fibonacci and Pell numbers. Furthermore, their related combinatorial and analytical expressions are furnished. In the last section, we give some final considerations and perspectives.

2 Generalized Fibonacci fundamental system

2.1 Casoratian matrix, Casoratian and generalized fundamental system

Let $w^{(1)}, \ldots, w^{(r)}$ be a set of r functions $w^{(j)} : \mathbb{N} \to \mathbb{K}$. For every $n \geq 0$, we put $w^{(j)}(n) = w_n^{(j)}$. Suppose that $\{w_n^{(1)}\}_{n\geq 0}, \ldots, \{w_n^{(r)}\}_{n\geq 0}$ are r solutions of Equation (2), viewed as a linear difference equation. We say that $w^{(1)}, \ldots, w^{(r)}$ are *linearly dependents* if there exists r scalars $\beta_1, \ldots, \beta_r$, with $(\beta_1, \ldots, \beta_r) \neq (0, \ldots, 0)$, such that

$$\beta_1 w^{(1)}(n) + \cdots + \beta_{r-1} w^{(r)}(n) = 0, \text{ for every } n \geq 0,$$

and $w^{(1)}, \ldots, w^{(r)}$ are said *linearly independents* if they are not linearly dependent. For every n, we consider the family of r vectors columns $\{\widehat{W}^{(j)}(n)\}_{1\leq j\leq r}$ defined by $\widehat{W}^{(j)}(n) = (w^{(j)}(n), w^{(j)}(n+1), \ldots, w^{(j)}(n+r-1))^t$, for $n \geq 0$ and $1 \leq j \leq r$. The Casoratian matrix $\widehat{C}(n)$, associated to the family $w^{(1)}, \ldots, w^{(r)}$, is defined $\widehat{C}(n) = (\widehat{W}^{(r)}(n), \ldots, \widehat{W}^{(1)}(n))$, namely,

$$\widehat{C}(n) = \begin{pmatrix} w_n^{(r)} & \cdots & w_n^{(j)} & w_n^{(1)} \\ \vdots & \ldots & \vdots & \vdots \\ w_{n+r-1}^{(r)} & \cdots & w_{n+r-1}^{(j)} & w_{n+r-1}^{(1)} \end{pmatrix}.$$

The Casoratian is defined as the determinant of the Casoratian matrix, namely, $C(n) = \det \widehat{C}(n)$, or equivalently,

$$C(n) = \det \widehat{C}(n) = det(\widehat{W}^{(r)}(n), \ldots, \widehat{W}^{(1)}(n)).$$

It is well known that the functions $w^{(1)}, \ldots, w^{(r)}$ are linearly independent, if and only if, $C(n) = \det \widehat{C}(n) \neq 0$, for every $n \geq 1$. When this condition is satisfied, we say that the set of functions $\{w^{(1)}, \ldots, w^{(r)}\}$ is *a fundamental system of solutions* of Equations (2).

2.2 Generalized Fibonacci fundamental system and Casoratian

Let consider $\mathcal{E}_{\mathbb{K}}^{(r)}(a_0, \ldots, a_{r-1})$, the $\mathbb{K}$-vector space of finite dimension r, of solutions of Equation (2). Let $\{v_n^{(s)}\}_{n\geq 0}$ be the family of sequences of $\mathcal{E}_{\mathbb{K}}^{(r)}(a_0, \ldots, a_{r-1})$, indexed by s $(1 \leq s \leq r)$ defined as follows,

$$\begin{cases} v_{n+1}^{(s)} = a_0 v_n^{(s)} + \cdots + a_{r-1} v_{n-r+1}^{(s)}, & \text{for } n \geq r-1, \\ v_n^{(s)} = \delta_{s-1,n} & \text{for } 0 \leq n \leq r-1. \end{cases} \tag{4}$$

The set of sequences $\{v_n^{(s)}\}_{n\geq 0}$, $1 \leq s \leq r$, represents r copies of sequences (2) with mutually different sets of initial values, viz. $v_j^{(s)} = \delta_{s-1,j}$ $(0 \leq j \leq r-1,\ 1 \leq s \leq r)$, where $\delta_{s-1,j}$ is the Kronecker symbol. For every$\{v_n\}_{n\geq 0}$ in $\mathcal{E}_{\mathbb{K}}^{(r)}(a_0, \ldots, a_{r-1})$ of initial data $\alpha_0, \cdots, \alpha_{r-1}$, we can verify that,

$$v_n = \sum_{s=1}^{r} \alpha_s v_n^{(s)}, \tag{5}$$

for every $n \geq 0$. Moreover, the set of functions $\{v^{(s)},\ s = 1, \ldots, r\}$, where $v^{(s)}(n) = v_n^{(s)}$, is a basis of the $\mathbb{K}$-vector space $\mathcal{E}_{\mathbb{K}}^{(r)}(a_0, \ldots, a_{r-1})$. In summary, we have the subsequent proposition.

Proposition 1 *The set $\{\{v_n^{(s)}\}_{n\geq 0}\ ,\ 1 \leq s \leq r\}$ is a fundamental system of solutions of Equation (2), called the generalized Fibonacci fundamental system.*

Consider the vector column $\mathbb{Y}_n = (v_n, \ \ldots, \ v_{n-r+1})^t$. The sequences (2) takes the equivalent matrix form,

$$\mathbb{Y}_{n+1} = \mathbb{A}\mathbb{Y}_n, \quad n \geq r-1, \tag{6}$$

where $\mathbb{A}$ is the companion matrix,

$$\mathbb{A} = \begin{pmatrix} a_0 & a_1 & & \cdots & a_{r-1} \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

Matrix formulation (6) of Equation (2), has been considered in various research papers of the literature (see for example [7,10], and references therein). The matrix Expression (6) was used in [4,10], for studying several properties of sequences (2).

To proceed with the work, consider the following questions. What is the closed connection between the Casoratian, the Casoratian matrix of (2) and the generalized Fibonacci fundamental system (4) of (2)? What is the closed connection between the generalized fundamental system (4) of (2) and the companion matrix $\mathbb{A}$ of (2)? What consequences on the generalized Fibonacci and Pell numbers? These questions will be addressed in the remainder of this work, with the aim of studying the concept of generalized Cassini identities.

First of all, the generalized Fibonacci fundamental system will play a central role in the computation of the powers $\mathbb{A}^n$ of the companion matrix (7), and also for studying its closed connection with the Casoratian matrix and the Casoratian. That is, the matrix equation (6), can be written under the form,

$$\mathbb{Y}_{n+r-1} = \mathbb{A}^n\mathbb{Y}_{r-1}, \quad n \geq r-1, \tag{8}$$

where $\mathbb{Y}_{r-1}$ is the vector column $\mathbb{Y}_{r-1} = (v_{r-1}, \ \ldots, \ v_0)^t$. By considering the generalized fundamental system $\{v^{(s)}, \ s = 1, \ldots, r\}$, each entries of the vector $\mathbb{Y}_{n+r-1} = (v_{n+r-1}, \ \ldots, \ v_n)^t$, can be written under the form $v_{n+j} = \sum_{s=1}^{r} \alpha_s v_{n+j}^{(s)}$, where $0 \leq j \leq r-1$. Thus, a direct computation shows that the vector $\mathbb{Y}_{n+r-1}$ can be written under the matrix form,

$$\mathbb{Y}_{n+r-1} = \mathbb{M}_n\mathbb{Y}_{r-1}, \tag{9}$$

where the entries $m_{ij}^{(n)}$ of the matrix $\mathbb{M}_n$ are stated, in terms of the elements of the generalized Fibonacci fundamental system, under the form $m_{ij}^{(n)} = v_{n+r-i}^{(r-j+1)}$. Therefore, the Casoratian matrix $\widehat{C}(n)$ and the matrix powers $\mathbb{A}^n$ of the companion matrix (7) are related to the matrix $\mathbb{M}_n$. More precisely, taking into account Formulas (8)-(9), and the fact that the initial conditions are arbitrary, we can formulate the following result.

Theorem 1 *Let* $\{\{v_n^{(s)}\}_{n\geq 0}\ , \ 1 \leq s \leq r\}$ *be the generalized Fibonacci fundamental system* (4)*. Then, for every* $n \geq 0$*, the entries of the powers* $\mathbb{A}^n = (a_{ij}^{(n)})_{1\leq i,j\leq r}$ *are imparted under the form,*

$$a_{ij}^{(n)} = v_{n+r-i}^{(r-j+1)}. \tag{10}$$

In other terms, we have

$$\mathbb{A}^n = \mathbb{M}_n = \begin{pmatrix} v_{n+r-1}^{(r)} & \cdots & v_{n+r-1}^{(j)} & v_{n+r-1}^{(1)} \\ \vdots & \ldots & \vdots & \vdots \\ v_n^{(r)} & \cdots & v_n^{(j)} & v_n^{(1)} \end{pmatrix}. \tag{11}$$

It is worth noting that Expressions (10)-(11) have been established by induction (see [4,6], and references therein). Since the Casoratian matrix of the generalized Fibonacci fundamental system (4), namely, $\{\{v_n^{(s)}\}_{n\geq 0}\ , \ 1 \leq s \leq r\}$ of (2), is defined by,

$$\widehat{C}(n) = (\widehat{V}^{(r)}(n), \ldots, \widehat{V}^{(1)}(n)) = \begin{pmatrix} v_n^{(r)} & \cdots & v_n^{(j)} & v_n^{(1)} \\ \vdots & \ldots & \vdots & \vdots \\ v_{n+r-1}^{(r)} & \cdots & v_{n+r-1}^{(j)} & v_{n+r-1}^{(1)} \end{pmatrix}, \tag{12}$$

where $\widehat{V}^{(j)}(n) = (v^{(j)}(n), \ldots, v^{(j)}(n+r-1))^t$ for $n \geq 0$ and $1 \leq j \leq r$; the next result allows us to connect the Casoratian matrix $\widehat{C}(n)$ of the Fibonacci fundamental system (4) and the powers $\mathbb{A}^n$ of the matrix (7).

Theorem 2 *The Casoratian matrix $\widehat{C}(n)$ of the Fibonacci fundamental system* (4) *and the powers $\mathbb{A}^n$ of the companion matrix* (7)*, are similar. More precisely, we have,*

$$\widehat{C}(n) = J\mathbb{A}^n J = (c_{ij}^{(n)})_{1 \leq i,\, j \leq r}, \tag{13}$$

where the entries $c_{ij}^{(n)}$ are given as in $c_{ij}^{(n)} = v_{n+i-1}^{(j)}$ $(1 \leq i, j \leq r)$ and $J = (b_{i,\, j})_{1 \leq i,\, j \leq r}$ is the anti-diagonal unit matrix, whose entries are defined by $b_{i,\, j} = 1$ for $i + j = r + 1$ and $b_{i,\, j} = 0$ otherwise.

Expression (13) will play a central role for formulating the general Cassini identity, in the general setting of sequences (2), as well as for its application to the special cases of generalized Fibonacci and Pell numbers.

2.3 Fundamental solution

In order to identify the combinatorial formulas of sequences (2), we consider the expression,

$$\rho(n, r) = \sum_{k_0 + 2k_1 + \cdots + rk_{r-1} = n-r+1} \frac{(k_0 + \cdots + k_{r-1})!}{k_0! k_1! \ldots k_{r-1}!} a_0^{k_0} a_1^{k_1} \ldots a_{r-1}^{k_{r-1}}, \tag{14}$$

with $\rho(r, r) = 1$ and $\rho(n, r) = 0$ for $n \leq r - 1$, where k_i is a natural number for all $0 \leq i \leq r - 1$. According to Theorem 2.5 and the Subsection 2.4 of [12], the general term v_n of the sequence (2), can be written under the form,

$$v_n = A_0 \rho(n, r) + A_1 \rho(n-1, r) + \cdots + A_{r-1} \rho(n - r + 1, r), \tag{15}$$

for every $n \geq r$, where $A_k = a_{r-1}\alpha_k + \ldots + a_k \alpha_{r-1}$ $(k = 0, \ldots, r-1)$ and the $\rho(n, r)$ are as in (14). Applying Expression (15) to each sequence $\{v_n^{(s)}\}_{n \geq 0}$ $(1 \leq s \leq r)$ of the generalized Fibonacci fundamental system (4), we acquire the hereafter result.

Theorem 3 *Let $\{\{v_n^{(s)}\}_{n \geq 0},\ 1 \leq s \leq r\}$ be the generalized Fibonacci fundamental system* (4)*. Then, for every $n \geq r$, we have*

$$v_n^{(r)} = \rho(n+1, r) \text{ and } v_n^{(1)} = a_{r-1} v_{n-1}^{(r)} = a_{r-1} \rho(n, r), \tag{16}$$

and for $2 \leq s \leq r - 1$, we have,

$$v_n^{(s)} = \sum_{j=1}^{s} a_{r-j} \rho(n + s - j, r), \quad \text{for every } n \geq r. \tag{17}$$

Expressions (17) shows that each element $\{v_n^{(s)}\}_{n \geq 0}$ of the generalized Fibonacci fundamental system (4) of Equation (2), is a linear combination of the terms $\rho(n + s - j + 1, r)$. Therefore, in view of Expression (16), we get the next result.

Corollary 1 *Let $\{\{v_n^{(s)}\}_{n \geq 0},\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system (4) of Equation (2). Then, for every $n \geq r$, we have*

$$v_n^{(s)} = \sum_{j=1}^{s} a_{r-j} v_{n+s-j-1}^{(r)}, \quad \text{for every } n \geq r, \tag{18}$$

where $v_n^{(r)}$ is imparted in (16).

Theorem 3 and Corollary 1, show that $\{v_n^{(r)}\}_{n \geq 0}$, the solution of Equation (2), represents a privileged element of the generalized fundamental Fibonacci system (4), which will play a key role in the sequel. For this reason we introduce the next definition.

Definition 1 The element $\{v_n^{(r)}\}_{n\geq 0}$ of the generalized Fibonacci fundamental system (4) given as in (16), is called the *fundamental solution* of Equation (2).

The fundamental solution $\{v_n^{(r)}\}_{n\geq 0}$ will perform a basic role, for exhibiting compact formulas of generalized Cassini identities and its applications to generalized Fibonacci and Pell numbers. More precisely, the Casoratian matrix expressed in terms of elements of the generalized Fibonacci system, will make it possible to establish combinatorial and analytical schemes explicit formulas of generalized Cassini identity of sequences (2).

3 Casoratian matrix and Generalized Cassini identity

As a consequence of Expressions (10)-(11) and (13), we show that

$$det[\widehat{C}(n)] = det[\mathbb{A}^n] = det[\mathbb{A}]^n.$$

Since $det[\mathbb{A}] = (-1)^{r-1}a_{r-1}$, we obtain the afterwards important property.

Proposition 2 *Under the data of Theorem* 2*, the Casoratian $C(n)$ of the Fibonacci fundamental system* (4)*, is determined by,*

$$C(n) = det[\widehat{C}(n)] = det[\mathbb{A}^n] = (-1)^{n(r-1)}a_{r-1}^n. \tag{19}$$

For $a_{r-1} = 1$ in Expression (19), we deduce that,

$$C(n) = det[\widehat{C}(n)] = (-1)^{n(r-1)}. \tag{20}$$

Expressions (19)–(20) represent a generalization of the usual Cassini identities such as the identity (1). That is, for $a_0 = \ldots = a_{r-1} = 1$, these formulas have been considered for defining the Cassini identity for generalized Fibonacci numbers in [14]. Furthermore, in order to illustrate here the scope of these two formulas, we provide the next two examples.

Example 1 Let $\{F_n\}_{n\geq 0}$ is the sequence of usual Fibonacci numbers defined by the recursive relation of order $r = 2$, and with arbitrary initial conditions $F_0 = \alpha_0$ and $F_1 = \alpha_1$. Let $\{\{F_n^{(1)}\}_{n\geq 0},\quad \{F_n^{(2)}\}_{n\geq 0}\}$ be the associated Fibonacci fundamental system (4). Since $F_0^{(1)} = 1$, $F_1^{(1)} = 0$ and $F_0^{(2)} = 0$, $F_2^{(1)} = 1$, we can verify easily that $F_n^{(1)} = F_{n-1}^{(2)}$, for every $n \geq 1$. If we set $F_n^{(2)} = F_n$, for every $n \geq 1$, then, the associated Casoratian is,

$$C(n) = \begin{vmatrix} F_{n+1}^{(2)} & F_{n+1}^{(1)} \\ F_n^{(2)} & F_n^{(1)} \end{vmatrix} = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Therefore, the Casoratian of the Fibonacci fundamental system related to the sequence of Fibonacci numbers, is nothing else but the well known Cassini Identity of the sequence of Fibonacci numbers.

Example 2 Let $\{P_n\}_{n\geq 0}$ is the sequence of usual Pell numbers defined by the recursive relation of order $r = 2$, and with arbitrary initial conditions $P_0 = \alpha_0$ and $P_1 = \alpha_1$. Let $\{\{P_n^{(1)}\}_{n\geq 0},\quad \{P_n^{(2)}\}_{n\geq 0}\}$ be the associated Fibonacci fundamental system (4). Since $P_0^{(1)} = 1$, $P_1^{(1)} = 0$ and $P_0^{(2)} = 0$, $P_2^{(1)} = 1$, we can verify easily that $P_n^{(1)} = P_{n-1}^{(2)}$, for every $n \geq 1$. If we set $P_n^{(2)} = P_n$, for every $n \geq 1$, then, the associated Casoratian is formulated under the form,

$$C(n) = \begin{vmatrix} P_{n+1}^{(2)} & P_{n+1}^{(1)} \\ P_n^{(2)} & P_n^{(1)} \end{vmatrix} = \begin{vmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{vmatrix} = P_{n+1}P_{n-1} - P_n^2 = (-1)^n.$$

Therefore, the Casoratian of the Fibonacci fundamental system related to the sequence of Pell numbers is nothing else but the well known Cassini Identity of the sequence of Pell numbers.

Motivated by results of [14] and the two previous examples we can then formulate the following general definition.

Definition 2 Let $\{\{v_n^{(s)}\}_{n\geq 0}\ ;\ 1 \leq s \leq r\}$ be the generalized Fibonacci fundamental system (4). The determinant of the associated Casoratian matrix $\widehat{C}(n)$ defined as in Expression (12), is called the *Generalized Cassini Identity* of the weighted generalized Fibonacci sequence $\{v_n\}_{n\geq 0}$, namely, the generalized Cassini identity of the sequence (2) is formulated by,

$$C(n) = \det(\widehat{C}(n)) = \det((\widehat{V}^{(r)}(n), \ldots, \widehat{V}^{(1)}(n))) = (-1)^{(r-1)n} a_{r-1}^n, \tag{21}$$

where $\widehat{V}^{(j)}(n)$ are the vector column $\widehat{V}^{(j)}(n) = (v^{(j)}(n), \ldots, v^{(j)}(n+r-1))^t$, for $n \geq 0$ and $1 \leq j \leq r$.

According to Expressions (11)-(12), the related generalized Cassini identity (21) is expressed in terms of the generalized Fibonacci fundamental system (4).

In fact, Expressions (10)-(11) and (13) reveal that the generalized Cassini identity (21) is determined via the formula $C(n) = \det(\mathbb{A}^n)$, where $\mathbb{A}^n = (a_{ij}^{(n)})_{1\leq i,\, j\leq r}$, is the powers of the companion matrix (7). Now, let recall the well known expression of the determinant of a square matrix $M = (a_{ij})_{1\leq i,\, j\leq r}$, defined by,

$$\det(M) = \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \ldots a_{\sigma(r),r},$$

where $\mathcal{S}_r$ is the group of permutations σ of r elements and $sgn(\sigma)$ is the signature of the permutation σ. Therefore, taking into account Expression (10), we show that the generalized Cassini identity, takes the subsequent form,

$$C(n) = \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) a_{\sigma(1),1}^{(n)} a_{\sigma(2),2}^{(n)} \cdots a_{\sigma(r),r}^{(n)} = (-1)^{(r-1)n} a_{r-1}^n. \tag{22}$$

However, for the matrix powers $\mathbb{A}^n = (a_{ij}^{(n)})_{1\leq i,\, j\leq r}$, the entries $a_{ij}^{(n)}$ $(1 \leq i,\, j \leq r)$ are given by Expression (10), namely, $a_{ij}^{(n)} = v_{n+r-i}^{(r-j+1)}$. Therefore, we come to have the result below.

Theorem 4 *Expression* (22) *of the generalized Cassini identity, related to the generalized Fibonacci fundamental system* (4)*, is given under the following form,*

$$C(n) = \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) v_{n+r-\sigma(1)}^{(r)} \cdots v_{n+r-\sigma(i)}^{(r-i+1)} \cdots v_{n+r-\sigma(r)}^{(1)} = (-1)^{(r-1)n} a_{r-1}^n. \tag{23}$$

On the other hand, the generalized Cassini identity can be stated with the aid of the fundamental solution $\{v_n^{(r)}\}_{n\geq 0}$ of Equation (2), considered as a generalized generalized Fibonacci sequence. That is, using Theorem 1, namely, Expression (18), and Expression (10), we derive that the entries $a_{ij}^{(n)}$ of the powers $\mathbb{A}^n$ of the companion matrix (7), can be expressed in terms of the fundamental solution of the Fibonacci fundamental system (4). More precisely, we have,

$$a_{ir}^{(n)} = v_{n+r-i}^{(1)} = a_{r-1} v_{n+r-i-1}^{(r)} \tag{24}$$

and, for $r \neq j$

$$a_{ij}^{(n)} = v_{n+r-i}^{(r-j+1)} = \sum_{k=1}^{r-j+1} a_{r-k} v_{n+2r-i-j-k}^{(r)}, \quad \text{for every } n \geq r. \tag{25}$$

Therefore, result of Theorem 4, namely Expression (23), shows that the Casoratian of the Fibonacci fundamental system (4), can be also stated in terms of the fundamental solution. That is, Expressions (10), (24) and (25), imply that

$$a_{\sigma(i),r}^{(n)} = v_{n+r-\sigma(i)}^{(1)} = a_{r-1} v_{n+r-\sigma(i)-1}^{(r)},$$
$$a_{\sigma(i),i}^{(n)} = v_{n+r-\sigma(i)}^{(r-i+1)} = \sum_{k=1}^{r-i+1} a_{r-k} v_{n+2r-\sigma(i)-i-k}^{(r)}, \quad \text{for every } n \geq r,$$

and substitution of this former formula in Expression (23) allows us to get,

$$
\begin{aligned}
C(n) &= \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) v^{(r)}_{n+r-\sigma(1)} v^{(r-1)}_{n+r-\sigma(2)} \cdots v^{(r-i+1)}_{n+r-\sigma(i)} \cdots v^{(1)}_{n+r-\sigma(r)} \\
&= \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) a_{r-1} v^{(r)}_{n+r-\sigma(1)} v^{(r)}_{n+r-\sigma(r)-1} \prod_{i=2}^{r-1} \left[\sum_{k=1}^{r-i+1} a_{r-k} v^{(r)}_{n+2r-\sigma(i)-i-k} \right].
\end{aligned}
$$

Therefore, the generalized Cassini identity of the Fibonacci fundamental system (4), can be also formulated in terms of the fundamental solution as shown in the next theorem.

Theorem 5 *Expression* (22) *of the generalized Cassini identity, related to the Fibonacci fundamental system* (4)*, is stated under the form,*

$$
C(n) = \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) a_{r-1} v^{(r)}_{n+r-\sigma(1)} v^{(r)}_{n+r-\sigma(r)-1} \Pi_\sigma(n) = (-1)^{(r-1)n} a^n_{r-1}, \tag{26}
$$

where

$$
\Pi_\sigma(n) = \prod_{i=2}^{r-1} \left[\sum_{k=1}^{r-i+1} a_{r-k} v^{(r)}_{n+2r-\sigma(i)-i-k} \right]. \tag{27}
$$

From Expression (24)–(25), we can show that, for every $n \geq r$, we have,

$$
a^{(n)}_{\sigma(1),1} = v^{(r)}_{n+r-\sigma(1)} = \sum_{k=1}^{r} a_{r-k} v^{(r)}_{n+2r-\sigma(1)-k-1} = a_0 v^{(r)}_{n+r-\sigma(1)-1} + \sum_{k=1}^{r-1} a_{r-k} v^{(r)}_{n+2r-\sigma(1)-k-1},
$$

and

$$
a^{(n)}_{\sigma(r),r} = v^{(1)}_{n+r-\sigma(r)} = a_{r-1} v^{(r)}_{n+r-\sigma(r)-1}.
$$

Therefore, Expression (26) of the generalized Cassini identity, can be written under the form,

$$
C(n) = \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) a_{r-1} v^{(r)}_{n+r-\sigma(r)-1} \Delta_\sigma(n) \Pi_\sigma(n) = (-1)^{(r-1)n} a^n_{r-1}, \tag{28}
$$

where $\Pi_\sigma(n)$ is given by Expression (27) and

$$
\Delta_\sigma(n) = a_0 v^{(r)}_{n+r-\sigma(1)-1} + \sum_{k=1}^{r-1} a_{r-k} v^{(r)}_{n+2r-\sigma(1)-k-1}, \tag{29}
$$

Among the special cases, which will interest us, within the framework of the applications of our results, is the one for which $a_0 = a$, $a_1 = \ldots = a_{r-2} = 0$ and $a_{r-1} = h \neq 0$. In this case the previous formulas (27)–(28) will take the upcoming expression,

$$
C(n) = \sum_{\sigma\in\mathcal{S}_r} sgn(\sigma) h v^{(r)}_{n+r-\sigma(r)} \Delta'_\sigma(n) \Pi'_\sigma(n) = (-1)^{(r-1)n} h^n, \tag{30}
$$

where

$$
\Delta'_\sigma(n) = a v^{(r)}_{n+r-\sigma(1)-1} + h v^{(r)}_{n+2r-\sigma(1)-2} \text{ and } \Pi'_\sigma(n) = h^{r-2} \prod_{i=2}^{r-1} v^{(r)}_{n+2r-\sigma(i)-i-1}. \tag{31}
$$

In order to better visualize the results of Theorems 4 and 5 , we treat the special case of $r = 3$. Thus, we can state the next corollary.

Corollary 2 *For $r=3$, is valid the formula of the generalized Cassini identities below,*

$$C(n)=\sum_{\sigma\in\mathcal{S}_3} sgn(\sigma)v^{(3)}_{n+3-\sigma(1)}v^{(2)}_{n+3-\sigma(2)}v^{(1)}_{n+3-\sigma(3)}=(-1)^{2n}a_2^n,$$
$$C(n)=\sum_{\sigma\in\mathcal{S}_3} sgn(\sigma)v^{(3)}_{n+3-\sigma(1)}v^{(3)}_{n+4-\sigma(2)}v^{(3)}_{n+2-\sigma(3)}=a_2^{n-2}.$$

In summary, we have defined the generalized Cassini identity, in terms of the determinant of the Casoratian matrix associated with the Fibonacci fundamental system. Thanks to the properties of the determinant and the results of Section 2, we could express the generalized Cassini identity in terms of the fundamental solution $\{v_n^{(r)}\}_{n\geq 0}$ of the generalized Fibonacci fundamental system (4). As the fundamental solution (16), namely, $\{v_n^{(r)}\}_{n\geq 0}$, has combinatorial and analytical properties, we will exploit these properties to develop combinatorial and analytical formulas for the generalized Cassini identity of the linear recursive sequences of Fibonacci type (2).

Finally, we would like to point out that, Expressions (28)–(29), will be very useful in the next sections concerning their applications to generalized Cassini identities for generalized Fibonacci and Pell numbers.

4 Combinatorial and analytical aspects of the generalized Cassini identities for sequences (2)

4.1 Combinatorial aspect of the generalized Cassini identity

Expression (26) shows that the generalized Cassini identity is formulated in terms the sequence $\{v_n^{(r)}\}_{n\geq 0}$, namely, the fundamental solution (16). On the other side, the fundamental solution has been started under its combinatorial form as follows $v_n^{(r)}=\rho(n+1,r)$, where $\rho(n+1,r)$ is conveyed by the formula (14). Therefore, we can provide the combinatorial expression of the generalized Cassini identity. Indeed, by substituting the expression of $v_n^{(r)}$ by $\rho(n+1,r)$ in Expression (26), we acquire the forthcoming.

Theorem 6 *The combinatorial expression of the generalized Cassini identity, related to the Fibonacci fundamental system* (4)*, is imparted under the following form,*

$$C(n)=\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)\prod_{i=1}^{r}\left[\sum_{k=1}^{r-i+1}a_{r-k}\rho(n+2r-\sigma(i)-i-k,r)\right]=(-1)^{(r-1)n}a_{r-1}^n. \quad (32)$$

Furthermore, we can show that $a^{(n)}_{\sigma(1),1}=v^{(r)}_{n+r-\sigma(1)}=\rho(n+r-\sigma(1)+1,r)$, for every $n\geq r$, and we deduce that $a^{(n)}_{\sigma(1),1}=\sum_{k=1}^{r}a_{r-k}\rho(n+2r-\sigma(1)-k,r)$. Therefore, we obtain $a^{(n)}_{\sigma(1),1}=a_0\rho(n+r-\sigma(1),r)+\sum_{k=1}^{r-1}a_{r-k}\rho(n+2r-\sigma(1)-k,r)$ and $a^{(n)}_{\sigma(r),r}=a_{r-1}\rho(n+r-\sigma(r)-1,r)$. Hence, it ensues that the combinatorial Expression (32) of the generalized Cassini identity, can be written under the form,

$$C(n)=\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)a_{r-1}\rho(n+r-\sigma(r),r)\Delta_\sigma(n)\Pi_\sigma(n)=(-1)^{(r-1)n}a_{r-1}^n, \quad (33)$$

where

$$\Delta_\sigma(n)=a_0\rho(n+r-\sigma(1),r)+\sum_{k=1}^{r-1}a_{r-k}\rho(n+2r-\sigma(1)-k,r), \quad (34)$$
$$\Pi_\sigma(n)=\prod_{i=2}^{r-1}\left[\sum_{k=1}^{r-i+1}a_{r-k}\rho(n+2r-\sigma(i)-i-k+1,r)\right]. \quad (35)$$

Expressions (33)–(35), will be very useful in the next sections concerning their applications to generalized Cassini identities for generalized Fibonacci and Pell numbers. Before developing these two interesting applications, we first provide a special case that illustrates Theorem 6, by substituting $r=3$ in Expression (32).

Corollary 3 *For $r = 3$ is valid the identity*

$$C(n) = \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma)\rho(n+4-\sigma(1), 3)\rho(n+4-\sigma(2), 3)\rho(n+3-\sigma(3), 3) = a_2^{n-2}.$$

4.2 Analytical aspect of the generalized Cassini identity

Since the fundamental solution $\{v_n^{(r)}\}_{n\geq 0}$ is linear recursive sequence satisfying Equation (2) (see also [4,5, 12]). Therefore, this sequence owns an analytical formulation, which is expressed in terms of the roots of the characteristic polynomial of Equation (2), namely, the roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-1}$. More generally, the analytical formula (or the so-called generalized Binet formula) for sequences (2) is obtained from the roots of the characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-1}$ and the initial conditions (3). More precisely, for the sequences (2), the analytic formula of its general term v_n is conveyed as follows,

$$v_n = \sum_{j=1}^{s} \sum_{i=0}^{m_j - 1} \beta_{i,j} n^i \lambda_j^n, \text{ for every } n \geq 0, \tag{36}$$

where the λ_j $(1 \leq j \leq s)$ are the roots of $P(z)$, of multiplicities m_j $(1 \leq j \leq s)$ respectively. On the other hand, the scalars $\beta_{i,j}$ $(1 \leq i \leq s$ and $0 \leq i \leq m_j - 1)$ are solutions of the generalized Vandermonde system,

$$\sum_{j=1}^{s} \sum_{i=0}^{m_j - 1} \beta_{i,j} n^i \lambda_j^n = \alpha_n, \text{ for } n = 0, 1, \ldots, r-1.$$

If the roots $\{\lambda_i\}_{1\leq i\leq r}$ of characteristic polynomial are simple, i.e., with multiplicity $m_i = 1$, we have $v_n = \sum_{i=1}^{r} \beta_i \lambda_i^n$, $(n \geq 0)$. Expressions established in [2,4,5] gives us an analytic expression of $\rho(n,r)$ in case of simple roots.

Lemma 1 (Rachidi *et al.*) *Suppose that the roots $\lambda_1, \ldots, \lambda_r$ of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-2} z - a_{r-1}$ $(a_{r-1} \neq 0)$, are simple, namely, $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, we have,*

$$\rho(n,r) = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)}, \textit{ for every } n \geq r, \tag{37}$$

with $\rho(r,r) = 1$, $\rho(i,r) = 0$ para $i \leq r-1$, where $P'(z) = \frac{dP}{dz}(z)$.

Formula (37) shows that, the analytic expression of the fundamental solution is presented under the form,

$$v_n^{(r)} = \rho(n+1, r) = \sum_{i=1}^{r} \frac{\lambda_i^n}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^n}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \text{ for every } n \geq r.$$

In order to apply our results in the next section we are interested in the case when the characteristic polynomial owns a simple roots.

With reference to Formula (32), the generalized Cassini identity is expressed in terms of of the combinatorial expression $\rho(n + 2r - \sigma(i) - i - k, r)$. Suppose that the roots $\lambda_1, \ldots, \lambda_r$ of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-2} z - a_{r-1}$ $(a_{r-1} \neq 0)$, are simple, namely, $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, applying Formula (37) to each $\rho(n + 2r - \sigma(i) - i - k + 2, r)$, we get,

$$\rho(n + 2r - \sigma(i) - i - k, r) = \sum_{i=1}^{r} \frac{\lambda_i^{n+2r-\sigma(i)-i-k-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n+2r-\sigma(i)-i-k-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)},$$

for every $n + 2r - \sigma(i) - i - k - 1 \geq r$. Then, the generalized Cassini identity (32), can be expressed under an analytical form. More precisely, we arrive to have the result below.

Theorem 7 *Let $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-1}$ be the characteristic polynomial of the sequence* (2). *Suppose that the roots $\lambda_1, \cdots, \lambda_r$ of $P(z)$ are simple. Then, the analytical expression of the generalized Cassini identity, associated to the Fibonacci fundamental system* (4)*, is given by the following form,*

$$\sum_{\sigma \in \mathcal{S}_r} sgn(\sigma) \prod_{i=1}^{r} \left[\sum_{k=1}^{r-i+1} a_{r-k} \Gamma_{r,\sigma(i)}(n; \lambda_1, \ldots, \lambda_r) \right] = (-1)^{(r-1)n} a_{r-1}^{n}, \tag{38}$$

where

$$\Gamma_{r,\sigma(i),k}(n; \lambda_1, \ldots, \lambda_r) = \sum_{i=1}^{r} \frac{\lambda_i^{n+2r-\sigma(i)-i-k-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n+2r-\sigma(i)-i-k-1}}{\prod_{k \neq i} (\lambda_i - \lambda_k)}. \tag{39}$$

As part of the application of Theorem 7, namely, Expressions (38)–(39), we can also formulate Expressions (33)–(35) under an analytical form. Indeed, the generalized Cassini identity, can be written under the analytic form,

$$\sum_{\sigma \in \mathcal{S}_r} sgn(\sigma) a_{r-1} \Gamma_{r,\sigma(r),r}(n; \lambda_1, \ldots, \lambda_r) \Delta_\sigma(n) \Pi_\sigma(n) = (-1)^{(r-1)n} a_{r-1}^{n}, \tag{40}$$

where $\Gamma_{r,\sigma(i),k}(n; \lambda_1, \ldots, \lambda_r)$ is given by Expression (39) and,

$$\Delta_\sigma(n) = a_0 \Gamma_{r,\sigma(1),1}(n; \lambda_1, \ldots, \lambda_r) + \sum_{k=1}^{r-1} a_{r-k} \Gamma_{r,\sigma(1),k}(n; \lambda_1, \ldots, \lambda_r), \tag{41}$$

$$\Pi_\sigma(n) = \prod_{i=2}^{r-1} \left[\sum_{k=1}^{r-i+1} a_{r-k} \Gamma_{r,\sigma(i),k}(n; \lambda_1, \ldots, \lambda_r) \right]. \tag{42}$$

Expressions (40)–(42), will be very useful in the next sections concerning their applications to generalized Cassini identities for generalized Fibonacci and Pell numbers.

5 Applications

Results of Sections 2, 3 and 4, can be applied to some well known sequences of usual numbers. In this section we are interested in some identities and the generalized Cassini identities related to generalized Fibonacci and Pell numbers. Thereby, we recover results of [8,14,15] established for these sequences of numbers.

5.1 Identities and generalized Cassini identities for generalized Fibonacci numbers

5.1.1 Linear and combinatorial aspect

For $a_0 = \ldots = a_{r-1} = 1$ in Expression (2), we get the sequence of generalized Fibonacci numbers, namely, we have,

$$F_n = \sum_{i=0}^{r-1} F_{n-i-1} \qquad \text{for} \qquad n \geq r, \tag{43}$$

with given integer initial conditions $F_n = \alpha_n$, for $n = 0, 1, \ldots, r-1$. The associated fundamental solution is characterized by the initial conditions,

$$F_{r-1} = 1 \text{ and } F_n = 0, \text{ for } n = 0, 1, \ldots, r-2.$$

And, as claimed by Expression (14), the combinatorial formula for F_n, the generalized Fibonacci numbers (43), is shaped in the form below,

$$F_n = \rho(n+1, r) = \sum_{\mathcal{T}_{n,r}} \frac{(k_0 + \cdots + k_{r-1})!}{k_0! k_1! \ldots k_{r-1}!},$$

with $\rho(r,r)=1$ and $\rho(n,r)=0$ for $n\leq r-1$, where $\mathcal{T}_{n,r}=\{(k_0,k_1,\ldots,k_{r-1})\in\mathbb{N}^r,\ k_0+2k_1+\cdots+rk_{r-1}=n-r+1\}$ (for more details, see also [14]).

The related generalized Cassini identity established in [14], is started under the form,

$$\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)F_{n-\sigma(1)+1}\ldots F_{n-\sigma(r)+r}=\varepsilon(\sigma_r)C(n)=(-1)^{(n+1)(r-1)},$$

$$\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)\left[\prod_{i=1}^{r}\rho(n-\sigma(i)+i+1,r)\right]=(-1)^{(n+1)(r-1)},$$

where $\mathcal{S}_r$ is the group of permutations of $\{1,\ 2,\ \ldots,\ r\}$, $sgn(\sigma)$ is the signature of $\sigma\in\mathcal{S}_r$. However, Expressions (26)–(32), allow us to get a new formula for the generalized Cassini identity for the generalized Fibonacci numbers (43). So, we hold the following new result,

$$C_F(n)=\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)\prod_{i=1}^{r}\left[\sum_{k=1}^{r-i+1}F^{(r)}_{n+2r-\sigma(i)-i-k-1}\right]=(-1)^{(r-1)n} \tag{44}$$

whose combinatorial form is determined by,

$$C_F(n)=\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)\prod_{i=1}^{r}\left[\sum_{k=1}^{r-i+1}\rho(n+2r-\sigma(i)-i-k,r)\right]=(-1)^{(r-1)n}. \tag{45}$$

In the best of our knowledge the identities (44)–(45) of the generalized Cassini identities, for generalized Fibonacci numbers, are not known in the literature.

5.1.2 Analytical aspect

The analytical formula of the generalized Fibonacci numbers (43) is established in terms of the roots of the associated characteristic polynomial, namely, $P(z)=z^r-z^{r-1}-\ldots-1$. Lemma 4.11 of [8] shows that the roots $\lambda_1,\ldots,\lambda_r$ of the characteristic polynomial $P(z)$ are simple. Hence, applying Expression (37), the analytical formula of the fundamental solution is started under the form,

$$F_n=\rho(n+1,r)=\sum_{i=1}^{r}\frac{\lambda_i^n}{P'(\lambda_i)}=\sum_{i=1}^{r}\frac{\lambda_i^n}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\ \text{for every}\ \ n\geq r.$$

Therefore, with reference to the combinatorial formula (45), we arrive to the subsequent analytic expression of the generalized Cassini identity,

$$C_F(n)=\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)\prod_{i=1}^{r}\left[\sum_{k=1}^{r-i+1}\sum_{i=1}^{r}\frac{\lambda_i^{n+2r-\sigma(i)-i-k-1}}{P'(\lambda_i)}\right]=(-1)^{(r-1)n}, \tag{46}$$

or equivalently,

$$C_F(n)=\sum_{\sigma\in\mathcal{S}_r} sgn(\sigma)\prod_{i=1}^{r}\left[\sum_{k=1}^{r-i+1}\sum_{i=1}^{r}\frac{\lambda_i^{n+2r-\sigma(i)-i-k-1}}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\right]=(-1)^{(r-1)n}. \tag{47}$$

It is significant to point out that the analytical formulas (46)–(47), concerning the generalized Cassini identity for generalized Fibonacci numbers, are not common in the literature.

5.2 Generalized Cassini identity for the generalized Pell numbers

5.2.1 Linear and combinatorial aspect

Let study some identities and the generalized Cassini identities, for the model of generalized Pell numbers introduced in [8]. The elements of such model are defined as follows. Let $d \geq 0$, i $(0 \leq i \leq r-2)$ and $h \geq 1$, be integer parameters, and consider the linear difference equation of order $r \geq 2$,

$$P_{d,i,h,n+1} = 2^d P_{d,i,h,n} + P_{d,i,h,n-i-1} + \cdots + hP_{d,i,h,n-r+1}, \qquad \text{for} \qquad n \geq r, \tag{48}$$

where the initial conditions $P_{d,i,h,0} = \alpha_0, \ldots, P_{d,i,h,r-1} = \alpha_{r-1}$ are chosen adequately. We also consider the two special cases related to $i = 0$, $h = 1$ and $i = r-2$, which defined by the equations,

$$P_{d,0,1,n+1} = 2^d P_{d,0,1,n} + P_{d,0,1,n-1} + \cdots + P_{d,0,1,n-r+1}, \qquad \text{for} \qquad n \geq r, \tag{49}$$

$$P_{d,r-2,h,n+1} = 2^d P_{d,r-2,h,n} + hP_{d,r-2,h,n-r+1}, \qquad \text{for} \qquad n \geq r, \tag{50}$$

with $d \geq 0$ and $h \geq 1$. The former case (49) has been studied in [15]. The two cases (48) and (50) have been largely explored in [8].

For reason of simplicity, we denote by $P_n = P_{d,i,h,n}$ the fundamental solution of Equation (48), defined by the initial data $P_0 = \ldots = P_{r-2} = 0$ and $P_{r-1} = 1$. And we denote by $R_n = P_{d,r-1,h,n}$ the fundamental solution of Equation (50) defined by the initial data $R_0 = \ldots = R_{r-2} = 0$ and $R_{r-1} = 1$. In both cases, when considering special cases, we will specify the values of the parameters $d \geq 0$, i $(0 \leq i \leq r-2)$ and $h \geq 1$. In accordance with Expression (14), the combinatorial formula of the fundamental solution $P_n = P_{d,i,h,n}$ is given by,

$$P_n = \rho_1(n+1, r) = \sum_{\mathcal{T}_{i,n}} \frac{(k_0 + k_{i+1} + \cdots + k_{r-1})}{k_0! k_{i+1}! \ldots k_{r-1}!} 2^{dk_0} h^{k_{r-1}}, \tag{51}$$

for every positive integer i and $n \geq r$, where $P_j = P_{d,j,h,j} = 0$ for $j \leq r-1$ and $P_{r-1} = P_{d,i,h,r-1} = 1$, and $\mathcal{T}_{i,n} = \{(k_0, k_{i+1}, \ldots, k_{r-1}), k_0 + (i+2)k_{i+1} + \cdots + rk_{r-1} = n-r+1\}$. For $i = 0$ we have,

$$P_n = P_{d,0,h,n} = \sum_{k_0 + 2k_1 + \cdots + rk_{r-1} = n-r+1} \frac{(k_0 + \cdots + k_{r-1})!}{k_0! \ldots k_{r-1}!} 2^{dk_0} h^{k_{r-1}},$$

where $P_j = P_{d,0,h,j} = 0$ for $j \leq r-1$ and $P_{r-1} = P_{d,0,h,r-1} = 1$ (for more details, see also [8]). Furthermore, concerning the fundamental solution of Eq. (50), namely, $R_n = P_{d,r-1,h,n}$ we have

$$R_n = \rho_2(n+1, r) = \sum_{\mathcal{T}_n} \frac{(k_0 + k_{r-1})!}{k_0! k_{r-1}!} 2^{dk_0} h^{k_{r-1}}, \tag{52}$$

for every $n \geq r$, where $R_j = P_{d,r-1,h,j} = 0$ for $j \leq r-1$, $R_{r-1} = P_{d,r-1,h,r-1} = 1$ and $\mathcal{T}_n = \{(k_0, k_{r-1}), k_0 + rk_{r-1} = n-r+1\}$.

According to Expression (26), the generalized Cassini identity can be written under the form (28) and (29). For the generalized Pell numbers (48), the coefficients are $a_0 = 2^d$, $a_j = 1$, for $1 \leq j \leq r-2$ and $a_{r-1} = h$, the fundamental solutions is given by $v_n^{(r)} = P_n = P_{d,i,h,n}$. Therefore, we obtain the linear formula for the generalized Cassini identity below,

$$C_P(n) = \sum_{\sigma \in \mathcal{S}_r} sgn(\sigma) h P_{n+r-\sigma(r)-1} \Delta_\sigma(n) \Pi_\sigma(n) = (-1)^{(r-1)n} h^n, \tag{53}$$

where

$$\Delta_\sigma(n) = 2^d P_{n+r-\sigma(1)-1} + \sum_{k=2}^{r-1} P_{n+2r-\sigma(1)-k-1} + hP_{n+2r-\sigma(1)-2},$$

and

$$\Pi_\sigma(n)=\prod_{i=2}^{r-1}\left[\sum_{k=1}^{r-i+1}a_{r-k}\rho(n+2r-\sigma(i)-i-k+1,r)\right].$$

Expression (53) is valid for every $i\in\mathbb{N}$ and $h>0$, especially, it is also verified for (49), namely, when $i=0$. However, for Expression (50), we show that $a_0=2^d$, $a_1=\ldots=a_{r-2}=0$ and $a_{r-1}=h>0$. Therefore, using Expressions (30)–(31) we derive that the generalized Cassini identity for (50), takes the form,

$$C_R(n)=\sum_{\sigma\in\mathcal{S}_r}sgn(\sigma)hR_{n+r-\sigma(r)-1}\Delta_\sigma(n)\Pi_\sigma(n)=(-1)^{(r-1)n}h^n, \tag{54}$$

such that $\Delta_\sigma(n)=2^dR_{n+r-\sigma(1)-1}+hR_{n+2r-\sigma(1)-2}$ and $\Pi_\sigma(n)=h^{r-2}\prod_{i=2}^{r-1}R_{n+2r-\sigma(i)-i}$, where $R_n=P_{d,r-1,h,n}$ is the fundamental solution of Equation (50).

It seems for us that the identities (53)–(54), of the generalized Cassini identities, for the generalized Pell numbers are not current in the literature.

Furthermore, the combinatorial form of the fundamental solutions of Eqs. (48) and (50), are furnished by formulas (51) and (52), namely,

$$P_n=\sum_{\mathcal{T}_{i,n}}\binom{k_0+k_{i+1}+\cdots+k_{r-1}}{k_0\ k_{i+1}\ \ldots\ k_{r-1}}2^{dk_0}h^{k_{r-1}}\text{ and }R_n=\sum_{\mathcal{T}_n}\binom{k_0+k_{r-1}}{k_0\ k_{r-1}}2^{dk_0}h^{k_{r-1}},$$

for $n\geq r$, where $\binom{m_1+\cdots+m_s}{m_1\ \ldots\ m_s}=\frac{(m_1+\cdots+m_s)!}{m_1!\ldots m_s!}$, $\mathcal{T}_{i,n}=\{(k_0,k_{i+1},\ldots,k_{r-1}),k_0+(i+2)k_{i+1}+\cdots+rk_{r-1}=n-r+1\}$ and $\mathcal{T}_n=\{(k_0,k_{r-1}),k_0+rk_{r-1}=n-r+1\}$. Therefore, the combinatorial form of the generalized Cassini identity, for the generalized Pell numbers (48), are determined by the formula,

$$C_P(n)=\sum_{\sigma\in\mathcal{S}_s}sgn(\sigma)\left[\sum_{\mathcal{T}_{i,n+r-\sigma(r)}}\binom{m_{i,r}}{k_0\ k_{i+1}\ \ldots\ k_{r-1}}2^{dk_0}h^{k_{r-1}}\right]\Delta_r(n)\Phi_r(n)=(-1)^{(r-1)n}, \tag{55}$$

where $m_{i,r}=k_0+k_{i+1}+\cdots+k_{r-1}$, and

$$\Phi_r(n)=2^d\sum_{\mathcal{D}_{i,n,\sigma(1)}}\binom{m_{i,r}}{k_0\ k_{i+1}\ \ldots\ k_{r-1}}2^{dk_0}h^{k_{r-1}}+\sum_{k=1}^{r-1}\sum_{\mathcal{D}_{i,n,k}}\binom{m_{i,r}}{k_0\ k_{i+1}\ \ldots\ k_{r-1}}2^{dk_0}h^{k_{r-1}},$$

with $\mathcal{D}_{i,n,\sigma(1)}=\mathcal{T}_{i,n+r-\sigma(1)}$, and $\mathcal{D}_{i,n,k}=\mathcal{T}_{i,n+2r-\sigma(1)-k}$, and

$$\Delta_r(n)=\prod_{s=2}^{r-1}\left[\sum_{k=1}^{r-s+1}\sum_{\mathcal{D}_{i,n,s,k}}\binom{m_{i,r}}{k_0\ k_{i+1}\ \ldots\ k_{r-1}}2^{dk_0}h^{k_{r-1}}\right],$$

where $m_{i,r}=k_0+k_{i+1}+\cdots+k_{r-1}$ and $\mathcal{D}_{i,n,s,k}=\mathcal{T}_{i,n+2r-\sigma(s)-s-k+1}$.

For the generalized Pell numbers (50), the combinatorial formula of their associated generalized Cassini identity is specified by the next form,

$$C_R(n)=\sum_{\sigma\in\mathcal{S}_r}sgn(\sigma)h\sum_{\mathcal{T}_{n+r-\sigma(r)}}\binom{k_0+k_{r-1}}{k_0\ k_{r-1}}2^{dk_0}h^{k_{r-1}}\Delta_r(n)\Phi_r(n)=(-1)^{(r-1)n}h^n, \tag{56}$$

such that

$$\Phi_r(n)=2^d\sum_{\mathcal{S}_{n+r-\sigma(1)}}\binom{k_0+k_{r-1}}{k_0\ k_{r-1}}2^{dk_0}h^{k_{r-1}}+h\sum_{\mathcal{T}_{n+2r-\sigma(1)-1}}\binom{k_0+k_{r-1}}{k_0\ k_{r-1}}2^{dk_0}h^{k_{r-1}},$$

$$\Delta_r(n)=h^{r-2}\prod_{i=2}^{r-1}\sum_{\mathcal{T}_{n+2r-\sigma(i)-i}}\binom{k_0+k_{r-1}}{k_0\ k_{r-1}}2^{dk_0}h^{k_{r-1}}.$$

5.2.2 Analytical aspect

Formulas (53)–(54) show that the analytical aspect of the generalized Cassini identities can be exhibited from those of the fundamental solutions P_n and R_n, given respectively by (48), (49) and (50). For these former solutions, the analytical formulas are obtained from the roots of their associated characteristic polynomials, namely,

$$P(z) = z^r - 2^d z^{r-1} - z^{r-2} - \ldots - z - h \text{ and } R(z) = z^r - 2^d z^{r-1} - h.$$

For reason of simplicity, we consider the special case (49) of (48), namely, $i = 0$ and $h = 1$.

Lemma 2 *(see [8, Lemmas 4.1, 4.3]) The roots of each polynomials*

$$P(z) = z^r - 2^d z^{r-1} - z^{r-2} - \ldots - z - 1 \textit{ and } R(z) = z^r - 2^d z^{r-1} - h.$$

are simple.

We denote by $\lambda_1, \ldots, \lambda_r$ the roots of the polynomial $P(z)$ and $\gamma_1, \ldots, \gamma_r$ those of the polynomial $R(z)$. Then, with the aid of the Lemma 1, the analytic formula of the fundamental solutions P_n and R_n, are imparted under the following form,

$$P_n = \rho_1(n+1, r) = \sum_{i=1}^{r} \frac{\lambda_i^n}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} \text{ and } R_n = \rho_2(n+1, r) = \sum_{i=1}^{r} \frac{\gamma_i^n}{\prod\limits_{k\neq i}(\gamma_i - \gamma_k)}, \tag{57}$$

for every $n \geq r$. By substituting the analytical expressions of P_n and R_n, which are yielded as in (57), we then establish the explicit analytical formulas of the generalized Cassini identities associated with the two families of generalized Pell numbers (49) and (50). More precisely, the analytic expression of the generalized Cassini identity for the generalized Pell numbers (49), is presented under the form,

$$C_P(n) = \sum_{\sigma \in \mathcal{S}_r} sgn(\sigma) \left[\sum_{i=1}^{r} \frac{\lambda_i^{n+r-\sigma(r)}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} \right] \Delta_r(n) \Phi_r(n) = (-1)^{(r-1)n}, \tag{58}$$

where

$$\Phi_r(n) = 2^d \sum_{i=1}^{r} \frac{\lambda_i^{n+r-\sigma(1)}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} + \sum_{k=1}^{r-1} \sum_{i=1}^{r} \frac{\lambda_i^{n+2r-\sigma(1)-k}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)},$$

$$\Delta_r(n) = \prod_{s=2}^{r-1} \left[\sum_{k=1}^{r-s+1} \sum_{i=1}^{r} \frac{\lambda_i^{n+2r-\sigma(s)-s-k+1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} \right].$$

Through the analytic expression of the generalized Cassini identity for the generalized Pell numbers (50), we hold the formula,

$$C_R(n) = \sum_{\sigma \in \mathcal{S}_r} sgn(\sigma) h \sum_{i=1}^{r} \frac{\gamma_i^{n+r-\sigma(r)}}{\prod\limits_{k\neq i}(\gamma_i - \gamma_k)} \Delta_r(n) \Phi_r(n) = (-1)^{(r-1)n} h^n, \tag{59}$$

such that

$$\Phi_r(n) = 2^d \sum_{i=1}^{r} \frac{\gamma_i^{n+r-\sigma(1)}}{\prod\limits_{k\neq i}(\gamma_i - \gamma_k)} + h \sum_{i=1}^{r} \frac{\gamma_i^{n+2r-\sigma(1)-1}}{\prod\limits_{k\neq i}(\gamma_i - \gamma_k)}, \; \Delta_r(n) = h^{r-2} \prod_{i=2}^{r-1} \left[\sum_{i=1}^{r} \frac{\gamma_i^{n+2r-\sigma(i)-i}}{\prod\limits_{k\neq i}(\gamma_i - \gamma_k)} \right].$$

Note that the analytical formulas (58)–(59), of the generalized Cassini identity for the generalized Pell numbers (49)- (50), seem new to us.

For the general setting, the characteristic polynomial of generalized Pell numbers (48), where d, i $(0 \leq i \leq r-2)$ and $h \geq 1$ are arbitrary integer, is,

$$P(z) = z^r - 2^d z^{r-1} - z^{r-i-2} - \ldots - z - h.$$

In view of the difficulties in characterizing the simplicity of the roots of the preceding polynomial, some significant cases have been studied in [8]. More precisely, the analytic aspect of the following three special cases of (48) were studied,

1. $r = 4$ and $i = 1$, whose characteristic polynomial is $S_1(z) = z^4 - 2^d z^3 - z - h$,
2. $r = 5$ and $i = 1$, whose characteristic polynomial is $S_2(z) = z^5 - 2^d z^4 - z^2 - z - h$,
3. $r = 5$ and $i = 2$, whose characteristic polynomial is $S_3(z) = z^5 - 2^d z^4 - z - h$.

Regarding these three special cases, it was established in [8] that the roots of each one of the polynomials $S_1(z)$, $S_2(z)$ and $S_3(z)$ are simple (see Lemmas 5.2, 5.6 and Proposition 5.10 of [8]). Therefore, analogous analytic formulas of the generalized Cassini identity, related to the associated generalized Pell numbers, can be exhibited. For example, in third case where $r = 5$ and $i = 1$, according to the Lemma 5.6 of [8], the characteristic polynomial $S_2(z) = z^5 - 2^d z^4 - z^2 - z - h$, of the related generalized Pell numbers (48), owns 5 distinct roots λ_j, $1 \leq j \leq 5$. Therefore, the analytic form of fundamental solution takes the form,

$$P_n = \rho_1(n+1, 5) = \sum_{j=1}^{5} \frac{\lambda_j^n}{S_2'(\lambda_j)} = \sum_{j=1}^{5} \frac{\lambda_j^n}{\prod_{k \neq j} (\lambda_j - \lambda_k)}, \text{ for } n \geq 5,$$

Thus, the analytic formula for the associated generalized Cassini identity, for the generalized Pell numbers (48), is started under the form,

$$C_S(n) = \sum_{\sigma \in \mathcal{S}_5} sgn(\sigma) \sum_{j=1}^{5} \frac{\lambda_j^{n+5-\sigma(5)}}{\prod_{k \neq j} (\lambda_j - \lambda_k)} \Delta_5(n) \Phi_5(n) = (-1)^{(5-1)n} = 1,$$

where

$$\Phi_5(n) = 2^d \sum_{j=1}^{5} \frac{\lambda_j^{n+5-\sigma(1)}}{\prod_{k \neq j} (\lambda_j - \lambda_k)} + \sum_{k=1}^{4} \sum_{j=1}^{5} \frac{\lambda_j^{n+10-\sigma(1)-k}}{\prod_{k \neq j} (\lambda_j - \lambda_k)}, \Delta_5(n) = \prod_{s=2}^{4} \left[\sum_{k=1}^{6-s} \sum_{j=1}^{5} \frac{\lambda_j^{n+11-\sigma(s)-s-k}}{\prod_{k \neq j} (\lambda_j - \lambda_k)} \right].$$

Through the same process, we can establish the analytical form of the generalized Cassini identity for the other two special cases, namely, case $r = 4$, $i = 1$ and case $i = 5$, $i = 1$.

6 Conclusion and perspectives

This study presents new results regarding the generalized Cassini identity as continuation of [14,15]. We do not claim that we have given a complete presentation of all methods for approaching generalized Cassini identities. However, we would like to clarify that the material presented here is a reflection of our approach concerning generalized identities of Cassini, throughout the fundamental system related to the associated to difference equations. Indeed, as shown in the development, our approach is based on the techniques and properties of the linear difference equations with constant coefficients, and those of the weighted generalized Fibonacci sequences, as well as the properties of the determinants of matrices. In the best of our knowledge, our results on this topic are not current in the literature.

In addition, our approach can also be applied to Cassini identities, related to other sequences of usual numbers such as those of Lucas, Jacobsthal, Morgan-Voyce, Chebyshev and Dickson polynomial.

Acknowledgements The authors express their sincere thanks to the Universidade Federal da Grande Dourados-UFGD/FACET/MEC - Brazil and to the Universidade Federal de Mato Grosso do Sul - UFMS/MEC - Brazil for their valuable support. The third author is supported by the Profmat and PPGEdumat programs of the INMA-UFMS. He expresses his sincere thanks to the INMA and the UFMS for their valuable support and encouragements.

References

1. S. Abramovich and G. A. Leonov, *Fibonacci-like polynomials : computational experiments, proofs and conjectures,* International Journal of Pure and Applied Mathematics, **53.4** (2009), 489–496.
2. D. Aiat Hadj Ahmed, A. Bentaleb, M. Rachidi and F. Zitan,*Powers of matrices by density and divided differences*, International Journal of Algebra **3.9** (2009), 407–422.
3. F. R. V. Alves and R. R. de Oliveira, *Sobre o modelo de Fibonacci na variável complexa: identidades generalizadas.* C.Q.D.-Revista Eletrônica Paulista de Matemática, Bauru, **11** (2017), 116–135.
4. R. Ben Taher and M. Rachidi,*On the matrix powers and exponential by r-generalized Fibonacci sequences methods: the companion matrix case*, Linear Algebra and Its Applications **370** (2003), 341–353.
5. R. Ben Taher and M. Rachidi, *Solving some generalized Vandermonde systems and inverse of their associate matrices via new approaches for the Binet formula.* Applied Math. and Computation, **290** (2016), 267–280.
6. B. El Wahbi, M. Mouline and M. Rachidi, *Solving nonhomogeneous recurrences relations by matrix method*, Fibonacci Quarterly, **40.2** (2002), 109–117.
7. M. C. Er, *Sums of Fibonacci Numbers by Matrix Methods,* Fibonacci Quarterly, **Vol 22. 3** (1984), 2004–207.
8. I. M. Craveiro , E. V. Pereira Spreafico and M. Rachidi, *On a model of generalized Pell numbers,* International Journal of Advanced Engineering Research and Science (IJAERS), **8.9** (2021), 527–550.
9. T. Guardia and D. Jiménez, *Fiboquadratic sequences and extensions of the Cassini identity raised from the study of Rithmomachia,* arXiv:1509.03177v3 [math.HO] (2016).
10. M. Mouline and M. Rachidi, *Suites de Fibonacci généralisées et chaines de Markov,* Real Acad. Cienc. Exact. Fís. Natur. Madrid **89 (1-2)** (1995), 61–77.
11. R. Grimaldi, *Fibonacci and Catalan Numbers: an Introduction,* Wiley, (2012).
12. M. Mouline and M. Rachidi, *Application of Markov chains properties to r-generalized Fibonacci sequences*, Fibonacci Quarterly **37** (1999), 34–38.
13. Y. K. Panwar, M. Singh, *Certain Properties of Generalized Fibonacci Sequence,* Turkish Journal of Analysis and Number Theory, **2.1** (2014), 6–8.
14. E.V. Pereira-Spreafico and M. Rachidi, *Fibonacci Fundamental System and Generalized Cassini Identity*, Fibonacci Quarterly, **57. 2** (2019), 155–157.
15. E.V. Pereira-Spreafico and M. Rachidi, *On Generalized Pell Numbers of Order $r \geq 2$*, Trends in Computational and Applied Mathematics , **22.1** (2021), 125–138.
16. Y. Soykan, *Simson identity of generalized m-step Fibonacci numbers,* Int. J. Adv. Appl. Math. and Mech. **7** (2019), 45–56.
17. H. Belbachir, E.V. Pereira-Spreafico and M. Rachidi, *Linear Recursive Relations for Bernoulli numbers and applications*. Journal of Ramanujan Society Mathematics and Mathematical Sciences, **8** (2020), 07-30