Indian J Pure Appl Math (2024) 55:1251–1260
https://doi.org/10.1007/s13226-023-00431-0

ORIGINAL RESEARCH

A new proof of quadratic series of Au-Yeung and explicit evaluation of its alternating sum

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Received: 5 August 2022 / Accepted: 15 May 2023 / Published online: 25 May 2023

Abstract We revisit the quadratic series of Au-Yeung $\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2$, which is quite well-known in the mathematical literature, and we consider its alternating sum $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{H_n}{n}\right)^2$. The central notion of this paper is to address a new approach to the famous quadratic series of Au-Yeung via the construction of a few classes of logarithmic integrals with the tails of the dilogarithm functions, which on computation leads to the elementary harmonic sums and polylogarithm sums. We also give an explicit proof of its alternating sum.

Keywords Harmonic number · Euler sum · Logarithmic integral · Dilogarithm function · Abel's summation formula · Alternating Euler sum

Mathematics Subject Classification 40C10 · 33B30 · 40G10

1 Introduction

In the theory of the non-linear harmonic series, the quadratic series associated with the harmonic numbers H_n for all integers n greater than zero

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17\pi^4}{360} \tag{1}$$

has an interesting history. It first appeared in *The American Mathematical Monthly* proposed by H. F. Sandham [1], and its solution was due to Martin Kneser [2], which involved the series manipulation followed by the computation of the logarithmic integrals. Later this series was numerically rediscovered by an undergraduate student, Enrico Au-Yeung, in the faculty of Mathematics in Waterloo. It is famously known as the quadratic series of Au-Yeung because of its rediscovery in the literature. David Borwein and Jonathan Borwein (see [3]) used the Fourier series and the Parseval series formula to prove it.

Communicated by Rahul Roy.

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After the first appearance and rediscovery of the series, several different approaches were made in the mathematical literature. For instance, Freitas [4] calculated the series via the application of double integrals containing logarithmic integrals, Vălean et al. [5] gave a new proof of the series based on the quadratic logarithmic functions and Abel's summation formula, and a different approach proved it by calculating a special harmonic series and applying Abel's summation formula can be found in [6], and Furdui gave a new proof of the series in [7]. This series appears in different mathematical books and papers. For example, a list of papers and books regarding the appearance of the series can be found [8].

Now focusing on the present aim of this paper, we construct an alternative approach to the quadratic series of Au-Yeung (1), which solely relies on the calculation of the logarithmic integrals with the tails of dilogarithm functions. For instance, the logarithmic integrals with the tails of the dilogarithm functions are

$$\int_0^1 \frac{\log(x)\operatorname{Li}_2(x)}{1-x}\,dx \quad \text{and} \quad \int_0^1 \frac{\log(x)}{1-x}\operatorname{Li}_2\left(\frac{-x}{1-x}\right)dx.$$

In the course of working with proofs, we shall come across some more integrals that ultimately reduce to the harmonic sums and the polylogarithm sums of orders less than 5. Likewise, we provide an explicit evaluation of its alternating sum, namely

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{H_n}{n}\right)^2 &= \frac{41}{16}\zeta(4) - 2\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{7}{4}\zeta(3)\log(2) \\ &\quad + \frac{1}{2}\zeta(2)\log^2(2) - \frac{1}{12}\log^4(2). \end{aligned} \tag{2}$$

Here, $\operatorname{Li}_s(z)$ is the polylogarithm function defined by $\sum_{n=1}^{\infty} z^n/n^s$ of order $s > 1$ and $|z| \leq 1$, and $\zeta(s)$ is the Riemann zeta function, which is a special case of $\operatorname{Li}_s(z)$ at $z = 1$. The above series (2) can be found in [8, p. 75], and its compact result was evaluated via the construction of the generating function. In view of (1) and (2), we can easily conclude the results of the harmonic series containing odd-indexed and even-indexed harmonic numbers. i.e.,

$$\sum_{n=1}^{\infty}\left(\frac{H_{2n}}{n}\right)^2 \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{H_{2n-1}}{2n-1}\right)^2.$$

Next, for natural number $m > 2$, we note the classical Euler sum [9]

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^m} = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2}\zeta(m-k)\zeta(k+1). \tag{3}$$

In addition, we mention Abel's summation formula [10, p. 55], which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers such that $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1}). \tag{4}$$

These two essential results aid in the deduction of some intermediate results and Euler sums. We organize the remaining work into different sections. In Section 2, we display some lemmas, and prove them. In Sections 3 and 4, we give the proofs of our main results.

2 Lemmas, their proofs, and main results

In this section, we highlight the main aims of the paper, and we explore and prove some important results. They are as follows:

Lemma 1 *For $n > -1$, the following relation holds:*

$$\int_0^1 x^{n-1}\,Li_2(x)\,dx = \frac{\zeta(2)}{n} - \frac{H_n}{n^2},$$

where $\zeta(s)$ is the Riemann zeta function and H_n is the nth harmonic number.

Proof Integrating by parts gives us

$$\int_0^1 x^{n-1}\,\mathrm{Li}_2(x)\,dx = \frac{\zeta(2)}{n} + \frac{1}{n}\int_0^1 x^{n-1}\log(1-x)\,dx$$
$$= \frac{\zeta(2)}{n} - \sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{n-k+1}\,dx = \frac{\zeta(2)}{n} - \frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k(n+k)}.$$

In the above steps, we use the series expansion of $\log(1-x)$ for $x \in [-1, 1)$. Further, employing the series representation of the harmonic numbers $H_n = \sum_{k=1}^{\infty} n/k(k+n)$ [6, p. 5, Eqn. (3)], we obtain the desired conclusion. □

Lemma 2 *If m, p are positive integers, then the following equality holds:*

$$\sum_{k=1}^{m}\frac{H_k^{(p)}}{k^p} = \frac{1}{2}\left((H_m^{(2)})^2 + H_m^{(2p)}\right),$$

where $H_k^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ is the kth generalized harmonic number of order p.

Proof The proof of the stated lemma is straightforward. The required conclusion can be obtained by applying Abel's summation formula (4), where we choice $a_k = 1/k$ and $b_k = H_k^{(p)}$. We leave details to the reader. □

Lemma 3 *If $y \in [-1, 1)$, then*

$$\int_0^1 \frac{x\log(x)}{(1-x)(1-xy)^2}\,dx = \frac{Li_2(y)-\zeta(2)}{(1-y)^2} - \frac{\log(1-y)}{y(1-y)}.$$

Proof We note that the partial fraction decomposition of the integrand $\frac{x}{(1-x)(1-xy)^2}$ is

$$\frac{x}{(1-x)(1-xy)^2} = \frac{1}{(1-x)(1-y)^2} - \frac{y}{(1-xy)(1-y)^2} - \frac{1}{(1-y)(1-xy)}. \tag{5}$$

Now, multiplying both sides of (5) by $\log(x)$ and integrating from $x = 0$ to $x = 1$, we get

$$\int_0^1 \frac{x\log(x)}{(1-x)(1-xy)^2}\,dx = -\frac{\zeta(2)}{(1-y)^2} - \frac{1}{(1-y)^2}\int_0^1 \frac{y\log(x)}{1-xy}\,dx$$
$$- \frac{1}{1-y}\int_0^1 \frac{\log(x)}{(1-xy)^2}\,dx$$
$$= -\frac{\zeta(2)}{(1-y)^2} + \frac{\mathrm{Li}_2(y)}{(1-y)^2} - \frac{\log(1-y)}{y(1-y)}.$$

Rearranging the terms yields the proposed result. □

Remark 1 We can express Lemma 3 in other direction by using the series of the factor $\frac{1}{(1-x)(1-xy)}$.

$$\frac{\partial}{\partial y}\frac{1}{(1-x)(1-xy)} = \frac{\partial}{\partial y}\sum_{k=1}^{\infty} x^{k-1}\frac{1-y^k}{1-y}$$
$$= \sum_{k=1}^{\infty} x^{k-1}\left(\frac{1}{(1-y)^2} + \frac{y^k}{(1-y)^2}\left(k-1-\frac{k}{y}\right)\right).$$

We note that the partial derivatives of $\frac{\partial}{\partial y}\frac{1}{(1-x)(1-xy)} = \frac{x}{(1-x)(1-xy)^2}$. Next, multiplying both sides by $\log(x)$ and integrating $\int_0^1 \cdots dx$, we get

$$\int_0^1 \frac{x\log(x)}{(1-x)(1-xy)^2}\,dx = -\sum_{n=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{(1-y)^2} + \frac{y^k}{(1-y)^2}\left(k-1-\frac{k}{y}\right)\right)$$
$$\frac{\mathrm{Li}_2(y)-\zeta(2)}{(1-y)^2} - \frac{\log(1-y)}{y(1-y)} = -\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{(1-y)^2} + \frac{y^k}{(1-y)^2}\left(k-1-\frac{k}{y}\right)\right).$$

Remark 2 In a similar fashion, for $y \in [-1, 1)$, we can easily conclude that

$$\int_0^1 \frac{\log(x)}{(1-x)(1-xy)^2}\, dx = \frac{\mathrm{Li}_2(y) - \zeta(2)}{(1-y)^2} - \frac{\log(1-y)}{1-y}.$$

Now we highlight our main results as follows:

Theorem 1 *The following equality holds:*

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17\pi^4}{360},$$

where H_n is the nth harmonic number defined by $\sum_{k=1}^{n} 1/k$.

Theorem 2 *The following equality holds:*

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{H_n}{n}\right)^2 = \frac{41}{16}\zeta(4) - 2\, Li_4\left(\frac{1}{2}\right) - \frac{7}{4}\zeta(3)\log(2) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2),$$

where $Li_4(z)$ is the tetralogarithm function defined by $\sum_{n=1}^{\infty} 1/k^4$.

3 Proof of Theorem 1

Proof Since the ordinary generating function of the harmonic number is given $\sum_{n=1}^{\infty} H_n y^n = -\frac{\log(1-y)}{1-y}$ for $y \in [-1, 1)$. Dividing both sides of it by y and applying Lemma 3, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n y^{n-1} &= \int_0^1 \frac{x \log^2(x)}{(1-x)(1-xy)^2}\, dx - \frac{\mathrm{Li}_2(y) - \zeta(2)}{(1-y)^2} \\
\sum_{n=1}^{\infty} \frac{H_n}{n} z^n &= \int_0^1 \frac{\log(x)}{1-x} \left(\int_0^z \frac{dy}{(1-xy)^2}\right) dx - \int_0^z \frac{\mathrm{Li}_2(y) - \zeta(2)}{(1-y)^2}\, dy \\
&= \int_0^1 \frac{\log(x)}{1-x} \left(\frac{1}{1-xz} - 1\right) dx - \frac{z\,\mathrm{Li}_2(z) - \zeta(2)}{1-z} \\
&\quad + \frac{1}{2}\log^2(1-z) - \zeta(2).
\end{aligned}$$

Furthermore, multiplying both sides by factor $\frac{\log(1-z)}{z}$ and integrating from $z = 0$ to $z = 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 &= -\int_0^1 \int_0^1 \frac{x \log(x) \log(1-z)}{(1-x)(1-xz)}\, dx\, dz - \zeta^2(2) \\
&\quad + \int_0^1 \frac{\log(1-z)}{z(1-z)} \left(z\,\mathrm{Li}_2(z) - \frac{\pi^2}{6}\right) dz - \frac{1}{2}\int_0^1 \frac{\log^3(1-z)}{z}\, dz \\
&= I + J + K - \zeta^2(2).
\end{aligned}$$

We get the above conclusions by using the result $\int_0^1 \frac{\log(1-z)}{z}\,dz = -\zeta(2)$ and the well-known identity $\int_0^1 x^{n-1}\log(1-x)\,dx = -\frac{H_n}{n}$ for all $n > -1$. Here the integrals

$$
\begin{aligned}
I + K - \zeta^2(2) &= -\int_0^1\int_0^1 \frac{\log(x)\log(1-z)}{(1-x)(1-xz)}\,dx\,dz - \frac{1}{2}\int_0^1 \frac{\log^3(1-z)}{z}\,dz - \zeta^2(2)\\
&= -\int_0^1 \frac{\log(x)}{1-x}\left(\int_0^1 \frac{x\log(1-z)}{1-xz}\,dz\right)dx - \frac{1}{2}\int_0^1 \frac{\log^3(z)}{1-z}\,dz - \zeta^2(2)\\
&= -\int_0^1 \frac{\log(x)}{1-x}\left(\int_0^1 \frac{x\log(z)}{1-x(1-z)}\,dz\right)dx - \frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 z^k\log^3(z)\,dz\\
&\quad - \zeta^2(2) = -\int_0^1 \frac{\log(x)}{1-x}\mathrm{Li}_2\left(\frac{-x}{1-x}\right)dx + 3\sum_{k=1}^{\infty}\frac{1}{k^4} - \zeta^2(2).
\end{aligned}
$$

Since the well-known Landen identity [11, p. 5, Eqn. (1.12)] is

$$
\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{-x}{1-x}\right) = -\frac{1}{2}\log^2(1-x) \tag{6}
$$

for $x \in [-1, 1)$. We now calculate the remaining integral in the last expression by using (6).

$$
\begin{aligned}
I + K - \zeta^2(2) &= \int_0^1 \frac{\log(x)}{1-x}\left(\frac{\log^2(1-x)}{2} + \mathrm{Li}_2(x)\right)dx + 3\zeta(4) - \frac{\pi^4}{36}\\
&= \frac{1}{2}\int_0^1 \frac{\log(x)\log^2(1-x)}{(1-x)}\,dx + \int_0^1 \frac{\log(x)\,\mathrm{Li}_2(x)}{1-x}\,dx + \frac{\pi^4}{180}.
\end{aligned}
$$

Applying integration by parts, we get

$$
\frac{1}{2}\int_0^1 \frac{\log(x)\log^2(1-x)}{1-x}\,dx = -\frac{1}{6}\int_0^1 \frac{\log^3(1-x)}{x}\,dx = -\frac{1}{6}\int_0^1 \frac{\log^3(x)}{1-x}\,dx = -\zeta(4).
$$

Also, in view of Lemma 1, we observe that

$$
\begin{aligned}
\int_0^1 \frac{\log(x)\,\mathrm{Li}_2(x)}{1-x}\,dx &= \sum_{n=1}^{\infty}\int_0^1 x^{n-1}\log(x)\,\mathrm{Li}_2(x)\,dx = \sum_{n=1}^{\infty}\frac{\partial}{\partial n}\int_0^1 x^{n-1}\,\mathrm{Li}_2(x)\,dx\\
&= \sum_{n=1}^{\infty}\left(\frac{2H_n}{n^3} + \frac{H_n^{(2)}}{n^2} - \frac{\pi^2}{3n^2}\right) = \frac{\pi^4}{36} + \frac{7\pi^4}{360} - \frac{\pi^4}{18} = -\frac{\pi^4}{120}.
\end{aligned}
$$

We use two different Euler sums $\sum_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{\pi^4}{72}$ and $\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} = \frac{7\pi^4}{360}$, which are the immediate consequence of (3) and (4), respectively. Combining the obtained results, we conclude that the value of $I + K - \zeta^2(2) = -\frac{\pi^4}{72}$. We note that $\frac{d}{dz}z\,\mathrm{Li}_2(z) = \mathrm{Li}_2(z) - \log(1-z)$ and $\int \frac{\log(1-z)}{z(1-z)}\,dz = -\mathrm{Li}_2(z) - \frac{\log^2(1-z)}{2} + C$, where C is the integration constant. If $z \to 0$, then $C = 0$. Now applying integration by parts to the integral J yields

$$
\begin{aligned}
J &= \int_0^1 (\mathrm{Li}_2(z) - \log(1-z))\left(\mathrm{Li}_2(z) + \frac{\log^2(1-z)}{2}\right)dz\\
&= \int_0^1 (\mathrm{Li}_2(z))^2\,dz - \int_0^1 \frac{\log^3(1-z)}{2}\,dz + \int_0^1 \mathrm{Li}_2(z)\log(1-z)\left(\frac{\log(1-z)}{2} - 1\right)dz.
\end{aligned}
$$

We show that $J = \frac{11\pi^4}{180}$. To reach this conclusion, we have expanded the integrand in the above step. And next, we calculate each of the above integrals as follows:

$$\begin{aligned}
\int_0^1 (\mathrm{Li}_2(z))^2\, dz &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 z^{n+1-1}\, \mathrm{Li}_2(z)\, dz \overset{\text{Lemma 1}}{=} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{\zeta(2)}{n+1} - \frac{H_{n+1}}{(n+1)^2} \right) \\
&= \sum_{n=1}^{\infty} \frac{\zeta(2)}{n^2(n+1)} - \sum_{n=1}^{\infty} \frac{H_{n+1}}{n^2(n+1)^2} \\
&= \frac{\pi^2}{36}\left(\pi^2 - 6\right) - \sum_{n=1}^{\infty} H_{n+1} \left(\frac{1}{n} - \frac{1}{n+1} \right)^2 \\
&= \frac{\pi^2}{36}\left(\pi^2 - 6\right) - \sum_{n=1}^{\infty} \frac{H_{n+1}}{n^2} + 2\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)} - \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^2} \\
&= \frac{\pi^2}{36}\left(\pi^2 - 6\right) - \sum_{n=1}^{\infty} \frac{H_n + \frac{1}{n+1}}{n^2} + 4 - \left(-1 + \sum_{n=1}^{\infty} \frac{H_n}{n^2} \right) \\
&= \frac{\pi^2}{36}\left(\pi^2 - 6\right) - 2\sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} + 5 \\
&= \frac{\pi^4}{36} - \frac{\pi^2}{6} - 4\zeta(3) - \left(\frac{\pi^2}{6} - 1 \right) + 5 = \frac{\pi^4}{36} - 4\zeta(3) - \frac{\pi^2}{3} + 6.
\end{aligned}$$

In the course of the calculation, we use the particular case of the Euler sum (3) at $m = 2$, recurrence relation of the harmonic number $H_{n+1} = H_n + \frac{1}{n+1}$, and $\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)} = 2$. It is easy to see that $\int_0^1 \log^3(1-z)\, dz = \int_0^1 \log^3(z)\, dz = -6$. Further, we calculate the remaining integrals as follows:

$$\begin{aligned}
\int_0^1 \mathrm{Li}_2(z) \log(1-z)\, dz &= -\frac{\pi^2}{6} - \int_0^1 \log(z) \log^2(1-z)\, dz - \int_0^1 \mathrm{Li}_2(1-z) \log(1-z)\, dz \\
&= -\frac{\pi^2}{6} - \int_0^1 \log(1-z) \log^2(z)\, dz - \int_0^1 \mathrm{Li}_2(z) \log(z)\, dz \\
&= -\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 z^n \log^2(z)\, dz - \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 z^n \log(z)\, dz \\
&= -\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n(n+1)^3} - \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} \\
&= -\frac{\pi^2}{6} + 2\left(-\zeta(3) + 3 - \frac{\pi^2}{6} \right) + \left(\frac{\pi^2}{3} - 3 \right) \\
&= 3 - 2\zeta(3) - \frac{\pi^2}{6}.
\end{aligned}$$

We use the elementary integral result

$$\int_0^1 z^n \log^q(z)\, dz = (-1)^q \frac{q!}{(n+1)^{q+1}}$$

for $\Re(n), \Re(q) > -1$ and a dilogarithm identity [11, p. 5, Eqn. (1.11)] for $z \in (-1, 1)$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z).$$

The sums $\sum_{n\geq 1}\frac{1}{n(n+1)^3}$ and $\sum_{n\geq 1}\frac{1}{n^2(n+1)^2}$ are straightforward that can be evaluated by the partial fraction decomposition of the summands of both sums. Also employing the same trick, we can easily deduce that

$$\begin{aligned}\int_0^1 \mathrm{Li}_2(z)\log^2(1-z)\,dz &= \frac{\pi^2}{3}-\int_0^1 \log(1-z)\log^3(z)\,dz-\int_0^1 \mathrm{Li}_2(z)\log^2(z)\,dz\\ &= \frac{\pi^2}{3}-\sum_{n=1}^{\infty}\frac{6}{n(n+1)^4}-\sum_{n=1}^{\infty}\frac{2}{n^2(n+1)^3}\\ &= \frac{\pi^2}{3}-6\left(-\zeta(3)+4-\frac{\pi^2}{6}-\frac{\pi^2}{90}\right)-2\left(\zeta(3)-6+\frac{\pi^2}{2}\right)\\ &= 4\zeta(3)-12+\frac{\pi^2}{3}+\frac{\pi^4}{15}.\end{aligned}$$

In a similar fashion, the value of the sums $\sum_{n\geq 1}\frac{1}{n(n+1)^4}$ and $\sum_{n\geq 1}\frac{1}{n^2(n+1)^3}$ are easily determined. We leave the details to the reader. Therefore, the value of the integral

$$\int_0^1 \mathrm{Li}_2(z)\log(1-z)\left(\frac{\log(1-z)}{2}-1\right)dz = 4\zeta(3)-9+\frac{\pi^2}{3}+\frac{\pi^4}{30}.$$

Combining the pieces of the integral J, we conclude that $J=\frac{11\pi^4}{180}$. Finally, collecting the values I, J, K, and putting them together produces

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = I+J+K-\zeta^2(2)=\frac{17\pi^4}{360},$$

which is the required conclusion of Theorem 1. □

4 Proof of Theorem 2

Proof Rewrite Lemma 3 in the following manner:

$$\sum_{n=1}^{\infty}\frac{H_n}{n}z^n = \int_0^1 \frac{xz\log(x)}{(1-x)(1-xz)}\,dx-\frac{z\,\mathrm{Li}_2(z)-\zeta(2)}{1-z}+\frac{\log^2(1-z)}{2}-\zeta(2). \tag{7}$$

We replace z with $-z$, multiple both sides of (7) by factor $\frac{\log(1-z)}{z}$, and integrating from $z=0$ to $z=1$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{H_n}{n}\right)^2 &= -\int_0^1\int_0^1 \frac{x\log(z)\log(1-z)}{(1-x)(1+xz)}\,dx\,dz+\zeta^2(2)\\ &\quad+\int_0^1 \frac{\log^2(1+z)\log(1-z)}{2z}\,dz\\ &\quad+\int_0^1 \frac{\log(1-z)}{z(1+z)}\left(z\,\mathrm{Li}_2(-z)+\frac{\pi^2}{6}\right)dz.\end{aligned} \tag{8}$$

Now we separately calculate each of these integrals. Before we begin to evaluate, we note that $\mathrm{Li}_2(-z) = \int_0^1 \frac{z\log(x)}{1+xz}\,dx$. Using the noted result and interchange of integrals, we get

$$\begin{aligned}
\int_0^1 \frac{\log(1-z)\,\mathrm{Li}_2(-z)}{1+z}\,dz &= \int_0^1 \log(x)\left(\int_0^1 \frac{z\log(1-z)}{(1+z)(1+zx)}\,dz\right)dx\\
&= \int_0^1 \log(x)\left(\int_0^1 \frac{z\log(1-z)}{x-1}\left(\frac{x}{1+zx}-\frac{1}{1+z}\right)dz\right)dx\\
&= \int_0^1 \log(x)\left(\frac{\zeta(2)}{2(1-x)}-\frac{\log^2(2)}{2(1-x)}-\frac{\mathrm{Li}_2\left(\frac{x}{1+x}\right)}{x(1-x)}\right)dx\\
&= \frac{\zeta(2)-\log^2(2)}{2}\int_0^1 \frac{\log(x)}{1-x}\,dx - \int_0^1 \frac{\log(x)}{x(1-x)}\,\mathrm{Li}_2\left(\frac{x}{1+x}\right)dx\\
&= -\frac{\pi^2}{12}\left(\frac{\pi^2}{6}-\log^2(2)\right) - \int_0^1 \frac{\log(x)}{x(1-x)}\,\mathrm{Li}_2\left(\frac{x}{1+x}\right)dx. \qquad (9)
\end{aligned}$$

Also, we note that the double integral

$$\begin{aligned}
\int_0^1\int_0^1 \frac{x\log(x)\log(1-z)}{(1-x)(1+xz)}\,dx\,dz &\overset{z\to 1-z}{=} \int_0^1 \frac{x\log(x)}{1-x}\left(\int_0^1 \frac{\log(z)}{1+x(1-z)}\,dz\right)dx\\
&= -\int_0^1 \frac{\log(x)}{1-x}\,\mathrm{Li}_2\left(\frac{x}{1+x}\right)dx. \qquad (10)
\end{aligned}$$

Further, we have the algebraic identity $6ab^2 = (a+b)^3+(a-b)^3-2a^3$. Setting $a=\log(1-z)$ and $b=\log(z)$, we get

$$\begin{aligned}
&6\int_0^1 \frac{\log^2(1+z)\log(1-z)}{z}\,dz\\
&= \int_0^1 \left(\frac{\log^3(1-z^2)}{z}+\frac{\log^3\left(\frac{1-z}{1+z}\right)}{z}-\frac{2\log^3(1-z)}{z}\right)dz.
\end{aligned}$$

Substituting $1-z^2\to z$, $\frac{1-z}{1+z}\to z$, $1-z\to z$, in the first, second, and the third integrals, respectively produces

$$\begin{aligned}
&6\int_0^1 \frac{\log^2(1+z)\log(1-z)}{z}\,dz\\
&= \int_0^1 \log^3(z)\left(\frac{1}{2(1-z)}+\frac{1}{(1-z)(1+z)}-\frac{2}{1-z}\right)dz\\
&= \int_0^1 \log^3(z)\left(\frac{1}{1+z}-\frac{1}{2(1-z)}\right)dz\\
&= \sum_{n=0}^{\infty}\int_0^1 \log^3(z)\left((-1)^n z^n-\frac{z^n}{2}\right)dz\\
&= 6\sum_{n=0}^{\infty}\left(-\frac{(-1)^n}{(n+1)^4}+\frac{1}{2(n+1)^4}\right)
\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_0^1 \frac{\log^2(1+z)\log(1-z)}{z}\,dz = -\frac{1}{2}\cdot\frac{\pi^4}{240} = -\frac{\pi^4}{480}. \qquad (11)$$

Substituting the results (9), (10), and (11) into (8) gives us

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{H_n}{n}\right)^2 = \int_0^1 \frac{\log\left(\frac{1}{x}\right)}{x}\operatorname{Li}_2\left(\frac{x}{1+x}\right)dx - \frac{\pi^4}{480}$$

$$\overset{\text{IBP}}{=} \frac{1}{2}\int_0^1 \frac{\log^2(x)\log(1+x)}{x(1+x)}\,dx - \frac{\pi^4}{480}. \tag{12}$$

It is easy to note that $\int \frac{\log(x)}{x}\,dx = \frac{\log^2(x)}{2}$ and $\frac{d}{dx}\operatorname{Li}_2\left(\frac{x}{1+x}\right) = \frac{\log(1+x)}{x(1+x)}$. Employing integration by parts (IBP), we yield (12). By the partial fraction decomposition, the last integral boils down to

$$\int_0^1 \frac{\log^2(x)\log(1+x)}{x(1+x)}\,dx = \int_0^1 \frac{\log^2(x)\log(1+x)}{x}\,dx - \int_0^1 \frac{\log^2(x)\log(1+x)}{1+x}\,dx$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} - \int_0^1 \frac{\log^2(x)\log(1+x)}{1+x}\,dx = \frac{7\pi^4}{360} - B.$$

To calculate integral B, we make the use of the elementary algebraic identity $(a-b)^3 = a^3 - b^3 + 3ab^2 - 3a^2b$, where we set $a = \log(x)$ and $b = \log(1+x)$. Thus,

$$3B = \int_0^1 \frac{\log^3(x)}{1+x}\,dx - \int_0^1 \frac{\log^3(1+x)}{1+x}\,dx + 3\int_0^1 \frac{\log(x)\log^2(1+x)}{1+x}\,dx$$

$$-\int_0^1 \frac{\log^3\left(\frac{x}{1+x}\right)}{1+x}\,dx$$

$$= -\frac{7\pi^4}{120} - \frac{\log^4(2)}{4} - \int_0^1 \frac{\log^3(1+x)}{x}\,dx - \int_0^{\frac{1}{2}} \frac{\log^3(x)}{1-x}\,dx.$$

Since $\int_0^1 \frac{\log^3(x)}{1+x}\,dx = -\frac{7\pi^4}{120}$ and $\int_0^1 \frac{\log^3(1+x)}{1+x}\,dx = \frac{\log^4(2)}{4}$. Here, we note that the integral $\int_0^{1/2} \frac{\log^3(x)}{1+x}\,dx$ can be further written as follows:

$$\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}} x^n\log^3(x)\,dx = \frac{3}{2}\log^2(2)\zeta(2) - 6\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{21}{4}\log(2)\zeta(3) - \frac{\log^4(2)}{2}.$$

For the remaining middle integral, we substitute $\frac{1}{1+x} \to x$ to get

$$\int_0^1 \frac{\log^3(1+x)}{x}\,dx = \int_1^{\frac{1}{2}} \frac{\log^3(x)}{x(1-x)}\,dx = \int_1^{\frac{1}{2}} \frac{\log^3(x)}{x}\,dx + \int_1^{\frac{1}{2}} \frac{\log^3(x)}{1-x}\,dx$$

$$= \frac{\log^4(2)}{4} + \sum_{n=0}^{\infty}\int_1^{\frac{1}{2}} x^n\log^3(x)\,dx$$

$$= \frac{\log^4(2)}{4} + \frac{\pi^4}{15} - \frac{21}{4}\zeta(3)\log(2) - 6\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{3}{2}\zeta(2)\log^2(2) - \frac{\log^4(2)}{2}$$

$$= \frac{\pi^4}{15} - \frac{\log^4(2)}{4} - \frac{21}{4}\zeta(3)\log(2) - 6\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{3}{2}\zeta(2)\log^2(2).$$

We employ

$$\int x^n\log^3(x)\,dx = \left(\frac{\log^3(x)}{n+1} - \frac{3\log^2(x)}{(n+1)^2} + \frac{6\log(x)}{(n+1)^3} - \frac{6}{(n+1)^4}\right)x^{n+1} + K,$$

where K is the integration constant. If $x \to 0$, then $K = 0$. And to simply the results, we use the well-known identities

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2},$$
$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{\log^3(2)}{6} - \frac{\pi^2}{12}\log(2),$$

which are found in [11, p. 11, Eqn. (1.16)] and [11, p. 296, Entry A.2.6.3], respectively. Collecting all the values of integral $3B$ yields

$$\frac{7\pi^4}{360} - B = -4\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{2}\zeta(3)\log(2) + \frac{11\pi^4}{180} - \frac{\log^4(2)}{6} + \zeta(2)\log^2(2). \tag{13}$$

Dividing both sides of (13) by 2 and substituting the obtained value into (12) completes the proof of Theorem 2. □

Corollary 1 *The following relations hold:*

$$\sum_{n=1}^{\infty}\left(\frac{H_{2n}}{n}\right)^2 = 4\,Li_4\left(\frac{1}{2}\right) + \frac{27}{8}\zeta(4) + \frac{7}{2}\zeta(3)\log(2) - \zeta(2)\log^2(2) + \frac{\log^4(2)}{6},$$
$$\sum_{n=1}^{\infty}\left(\frac{H_{2n-1}}{2n-1}\right)^2 = \frac{109}{32}\zeta(4) + \frac{1}{4}\zeta(2)\log^2(2) - Li_4\left(\frac{1}{2}\right) - \frac{7}{8}\zeta(3)\log(2) - \frac{\log^4(2)}{24}.$$

Proof The proofs of these two corollaries directly follow from Theorem 1 and 2. We encourage interested reader to pursue the results. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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