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ORIGINAL RESEARCH

A note on Whitney embedding theorem

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Abstract In this note, we describe a method for constructing a given compact manifold inside a euclidean space using its handlebody decomposition. We use this method to show that every smooth compact n-manifold consisting of handles up to index $k < n$ smoothly embeds in $\mathbb{R}^{n+k}$. An immediate consequence of this shows that every closed smooth n–manifold admits a topological embedding in $\mathbb{R}^{2n}$. We also show that every rank—k vector bundle over a closed m–manifold M can be realized as a subbundle of the trivial vector bundle $M \times \mathbb{R}^{m+k}$.

Keywords Manifold · Embedding

Mathematics Subject Classification Primary: 57Q35, 57R40

1 Introduction

Embeddings of manifolds in a fixed euclidean space is one of the classical problems in geometric topology. H. Whitney [1] showed that every closed smooth n–manifold smoothly embeds in $\mathbb{R}^{2n}$. In this note, we use handlebody decomposition of a closed smooth manifold and some well-known facts about the Stiefel manifolds and embeddings of spheres in euclidean space to prove the following results.

Theorem 1.1 *Let W be a smooth compact n–manifold with handlebody decomposition consisting of handles of index $k < n$. Then W admits smooth embedding in $\mathbb{R}^{n+k}$.*

Theorem 1.2 *Every closed smooth n–manifold admits a topological embedding in $\mathbb{R}^{2n}$.*

Both the results mentioned in this note are well-known classical results. In order to prove the version of Theorem 1.2 in the category of smooth embeddings, one needs to use the Whitney trick. Here, we take a constructive and geometric approach to these results to achieve topological embedding and avoid the use of the Whitney trick.

2 Embeddings of compact manifolds

Let us begin by recalling the following definitions.

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Definition 2.1 Let W be a compact n–manifold with non-empty boundary ∂W and $\mathcal{S}^{k-1}$ be an embedded $(k-1)$–sphere in ∂W with a trivial tubular neighborhood $\mathcal{N}_{\partial W}(\mathcal{S}^{k-1})$ in ∂W. Let $H^k = D^k \times D^{n-k}$ and $\phi : \partial D^k \times D^{n-k} \to \mathcal{N}_{\partial W}(\mathcal{S}^{k-1}) \subset \partial W$ be a diffeomorphism which maps $\partial D^k \times \{0\}$ to $\mathcal{S}^{k-1}$. We say that a compact n–manifold W' is obtained by attaching a k–handle H^k to W along $\mathcal{S}^{k-1}$ with the framing ϕ if W' is diffeomorphic to the quotient:

$$(D^k \times D^{n-k}) \bigsqcup_{\phi} W$$

In this case, we call $D^k \times \{0\}$ the core of H^k, $\mathcal{S}^{k-1}$ an attaching sphere of H^k and $\mathcal{N}_{\partial W}(\mathcal{S}^{k-1})$ an attaching region of H^k.

Definition 2.2 Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. A fiberwise linear embedding of E into E' is an embedding $f : E \to E'$ such that

(1) the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{Id} & M \end{array}$$

(2) the map $f : \pi^{-1}(x) \hookrightarrow (\pi')^{-1}(x)$ is linear.

Let us fix the following notations for rest of the note.

- We denote the projection of $M \times N$ onto its first factor M by the map $\pi_1 : M \times N \to M$.
- We denote the identity map of a manifold by Id.

Now, we prove the following lemma which we require to prove our main theorems of the note.

Lemma 2.3 *Let $D^{k+1} \times \mathbb{R}^m$ and $D^{k+1} \times \mathbb{R}^n$ be the trivial vector bundles over D^{k+1} of rank m and n, respectively, where $m < n$ and $k < n - m$. Then any fiberwise linear embedding $g : S^k \times \mathbb{R}^m \hookrightarrow S^k \times \mathbb{R}^n$ can be extended to a fiberwise linear embedding $\widetilde{g} : D^{k+1} \times \mathbb{R}^m \hookrightarrow D^{k+1} \times \mathbb{R}^n$.*

Proof Let $\{e_1, e_2, \dots, e_m\}$ be the standard basis of $\mathbb{R}^m$ and for each $x \in S^k$, let $\{v_1(x), v_2(x), \dots, v_m(x)\}$ be the k–frame $\mathbb{R}^n$ such that $(x, v_i(x)) = f(x, e_i)$. Note that the fiberwise linear embedding $f : S^k \times \mathbb{R}^m \hookrightarrow S^k \times \mathbb{R}^n$ determines the map $\gamma : S^k \to V_m(\mathbb{R}^n)$ (an element of $\pi_k(V_m(\mathbb{R}^n))$ given by

$$\gamma(x) = (v_1(x), v_2(x), \dots, v_m(x)),$$

where $V_m(\mathbb{R}^n)$ is the Stiefel manifold consisting of all m–frames in $\mathbb{R}^n$. It is well known that for $k < n - m$, the k^{th} homotopy group $\pi_k(V_m(\mathbb{R}^n)$ is trivial. Therefore, there exists a homotopy $\gamma^t : S^k \to V_m(\mathbb{R}^n)$ such that $\gamma^0 : S^k \to V_m(\mathbb{R}^n)$ is a constant map given by $\gamma^0(x) = (v_1^0, v_2^0, \dots, v_m^0)$ for all $x \in S^k$ and $\gamma^1 = \gamma$. We write $\gamma^t(x) = (v_1^t(x), v_2^t(x), \dots, v_m^t(x))$, $x \in S^k$. For each $y \in D^{k+1}$, the homotopy γ^t allows us to define the linear embedding

$$g_y : \{y\} \times \mathbb{R}^m \to \{y\} \times \mathbb{R}^n,$$

which is determined by sending $(y, e_i) \to (y, v_i^{\|y\|}(\frac{y}{\|y\|}))$. Thus, the desired fiberwise linear embedding $\widetilde{g} : D^{k+1} \times \mathbb{R}^m \hookrightarrow D^{k+1} \times \mathbb{R}^n$ is given by $\widetilde{g}(y, v) = g_y(y, v)$. This completes the proof of the lemma. □

Next, we prove the following inductive lemma.

Lemma 2.4 *Let $f : W \hookrightarrow \mathbb{R}^{n+k}$ be a smooth embedding of a compact n–manifold W in $\mathbb{R}^{n+k}$, where $k < n-1$. Let W' be a smooth manifold obtained by attaching $(k+1)$–handles $H_1^{k+1}, H_2^{k+1}, \dots H_l^{k+1}$ to W along the disjointly embedded k–spheres $\mathcal{S}_1^k, \mathcal{S}_2^k, \dots, \mathcal{S}_l^k$ in ∂W with the framings $\phi_1, \phi_2, \dots \phi_l$, respectively. Then W' admits a smooth embedding in $\mathbb{R}^{n+k+1}$.*

Proof Recall that

$$W' = (H_1^{k+1} \sqcup H_2^{k+1} \sqcup \cdots \sqcup H_l^{k+1}) \bigsqcup_{(\phi_1,\phi_2,\ldots,\phi_l)} W.$$

By considering $\mathbb{R}^{n+k}$ as $R^{n+k} \times \{0\}$ in $\mathbb{R}^{n+k} \times \mathbb{R}$, we can regard the embedding f as the embedding $f : W \hookrightarrow \mathbb{R}^{n+k+1}$. In order to get the desired embedding of W' in $\mathbb{R}^{n+k+1}$, we first show that the embedding f can be extended to the cores $C_1, C_2, \ldots C_l$ of the $(k+1)$–handles H_1^{k+1}, H_2^{k+1}, $\ldots$ H_l^{k+1} attached to W, respectively.

Notice that as $k < n-1$, we have $2(k+1) < n+k+1$. Now, it follows from the well known fact that each embedded k–sphere $\mathbf{S}_i^k = f(\mathcal{S}_i^k)$ in $\mathbb{R}^{n+k} \times \{0\}$ bounds properly embedded $(k+1)$–disc $\mathbf{D}_i^{k+1}$ in $\mathbb{R}^{n+k} \times \mathbb{R}_{\geq 0}$. By Thom transversality, we can assume that the discs $\mathbf{D}_i^{k+1}$'s are mutually disjoint. Thus, we can extend the embedding f to an embedding

$$\widetilde{f} : (C_1 \sqcup C_2 \sqcup \cdots \sqcup C_l) \bigsqcup_{(\phi_1,\phi_2,\ldots,\phi_l)} W \hookrightarrow \mathbb{R}^{n+k} \times \mathbb{R}_{\geq 0}.$$

Now, we show that the embedding $\widetilde{f}$ can be extended to the $(k+1)$–handles H_1^{k+1}, H_2^{k+1}, ..., H_l^{k+1} to get the desired embedding of W' into $\mathbb{R}^{n+k+1}$. Let $\mathcal{N}(\mathbf{D}_i^{k+1})$'s be the mutually disjoint tubular neighborhoods of $\mathbf{D}_i^{k+1}$'s in $\mathbb{R}^{n+k} \times \mathbb{R}_{\geq 0}$ and let $\psi_i : \mathcal{N}(\mathbf{D}_i^{k+1}) \to (D^{k+1} \times D^n)_i$ be a trivialization of the tubular neighborhood $\mathcal{N}(\mathbf{D}_i^{k+1})$, for each i. Recall that $H_i^{k+1} = (D^{k+1} \times D^{n-k-1})_i$. For each i, let $F_i : (\partial D^{k+1} \times D^{n-k-1})_i \hookrightarrow (\partial D^{k+1} \times D^n)_i$ be the embedding given by the map $\psi_i \circ f \circ \phi_i$. Without loss of generality, we can assume that F_i is a fiberwise linear embedding, i.e. the following diagram commutes.

$$\begin{array}{ccc} (S^k \times D^{n-k-1})_i & \xrightarrow{F_i} & (S^k \times D^n)_i \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_1} \\ S^k & \xrightarrow{\quad Id \quad} & S^k \end{array}$$

Notice that $k < n - (n-k-1) = k+1$. Therefore by Lemma 2.3, it follows that, each embedding F_i can be extended to a fiberwise linear embedding $\widetilde{F}_i : H_i^{k+1} = (D^{k+1} \times D^{n-k-1})_i \hookrightarrow (D^{k+1} \times D^n)_i$ such that the following diagram commutes.

$$\begin{array}{ccc} H_i^{k+1} = (D^{k+1} \times D^{n-k-1})_i & \xrightarrow{\widetilde{F}_i} & (D^{k+1} \times D^n)_i \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_1} \\ D^{k+1} & \xrightarrow{\quad Id \quad} & D^{k+1} \end{array}$$

Let $F_i' : H_i^{k+1} \hookrightarrow \mathcal{N}(\mathbf{D}_i^{k+1})$ be the embedding given by $F_i' = \psi_i^{-1} \circ \widetilde{F}_i$, for each i. Observe that each embedding F_i' and the embedding $\widetilde{f}$ agree on their overlapping region in W'. Therefore, the embeddings $F_1', F_2', \ldots, F_l'$ together with the embedding $\widetilde{f}$ defines the desired embedding of W' in R^{n+k+1}. □

Remark 2.5 The proof of Lemma 2.4 shows the importance of studying embeddings of the k–spheres in Euclidean spaces and Stiefel manifolds in the study of embeddings of manifolds in Euclidean spaces.

Let us define the notion of a handlebody decomposition of a compact manifold.

Definition 2.6 Let W be a compact n–manifold. A Handlebody decomposition of W is a sequence $W_0, W_1, \ldots, W_n = W$ of compact 4–manifolds, where

1. the compact n–manifold W_0 is the n–ball D^n.
2. for each $1 \leq k \leq n$, W_k is obtained from W_{k-1} by attaching finitely many k–handles to W_{k-1}.

Remark 2.7 If W_k is obtained from W_{k-1} by attaching zero k–handles, then $W_k = W_{k-1}$.

Now, we state the following fundamental theorem in Morse theory which we need to prove Theorem 1.1 and Theorem 1.2.

Theorem 2.8 *Every smooth connected compact n–manifold W admits a handlebody decomposition* $W_0, W_1, \ldots, W_n = W$. *Moreover, if W is closed, we can assume that W is obtained by attaching only one n–handle to* W_{n-1}.

Now, we are ready to give the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 Let $W_0, W_1, \ldots, W_k = W, W_{k+1} = W, \ldots, W_n = W$ be a handlebody decomposition of W. As W_0 is a n–ball, W_0 embeds in $\mathbb{R}^n$ as a standard radius one n–ball in $\mathbb{R}^n$. Therefore, by Lemma 2.4, W_1 embeds into $\mathbb{R}^{n+1}$. Now, by Lemma 2.4, it follows inductively that $W_k = W$ embeds in $\mathbb{R}^{n+K}$. This completes the proof. □

Proof of Theorem 1.2 Let W be a closed smooth connected n–manifold. By Theorem 2.8, W admits a handlebody decomposition $W_0, W_1, \ldots, W_n = W$. By Theorem 1.1, there exists an embedding $f : W_{n-1} \hookrightarrow \mathbb{R}^{n+n-1} \times \{0\} \subset \mathbb{R}^{2n}$. Recall that ∂W_{n-1} is the $n-1$ sphere $\mathcal{S}^{n-1}$. Thus, $f(W_{n-1})$ together with a cone $\mathcal{C}$ over the embedded sphere $f(\mathcal{S}^{n-1})$ in $\mathbb{R}^{n+n-1} \times \mathbb{R}_{\geq 0}$ gives a topological embedding of W in $\mathbb{R}^{2n}$. □

We end this note by proving the following result regarding embeddings of vector bundles into a trivial vector bundle.

Theorem 2.9 *Let $\pi : E \to M$ be a rank k vector bundle over a closed m–manifold M and $\pi_1 : M \times \mathbb{R}^{m+k} \to M$ be the trivial vector bundle over M. Then, there is a fiberwise linear embedding $f : E \hookrightarrow M \times R^{m+k}$ of E into the trivial vector bundle $M \times R^{m+k}$.*

Proof Let Δ be a triangulation of M. We denote the i^{th} skeleton of Δ by M_i. Let $f_0 : \pi^{-1}(M_0) \hookrightarrow M_0 \times \mathbb{R}^{m+k}$ be any fiberwise linear embedding. Now, by repeated application of Lemma 2.3, f_0 can be extended to a desired embebedding f. □

Reference

1. H. Whitney, *The self-intersections of a smooth n–manifold in* $2n$*–space*, Ann. of Math, vol. 45(2), (1944), 200–246.