



On diagonal pluriharmonic metrics of G -Higgs bundles

Natsuo Miyatake

Received: 1 May 2023 / Accepted: 15 May 2023 / Published online: 20 May 2023
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Abstract Let $(E, \Phi) \rightarrow (X, \omega_X)$ be a Higgs bundle over a compact Kähler manifold. We suppose that the holomorphic vector bundle E decomposes into a direct sum of holomorphic line bundles. In this paper, we give the necessary and sufficient condition for the existence of a diagonal metric which is a solution to the Hermitian-Einstein equation. Our theorem can easily be generalized to G -Higgs bundles. We also describe the relationship between the stability condition and our condition using the torus action on the space of Higgs fields.

Keywords Higgs bundle · Harmonic bundle · Stability condition · Torus action

1 Main theorem and proof

Let X be a compact connected Riemann surface of genus $g(X) \geq 2$ with the canonical bundle $K_X \rightarrow X$. We define a holomorphic vector bundle E as the direct sum of holomorphic line bundles $L_1, \dots, L_r$ over X with $\deg(L_1) + \dots + \deg(L_r) = 0$. Let Φ_j be a holomorphic section of $(L_j^{-1} L_{j+1}) \otimes K_X$ for each $j = 1, \dots, r-1$, and Φ_r a holomorphic section of $(L_r^{-1} L_1) \otimes K_X$. We define Φ as the sum of all the Φ_j , $j = 1, \dots, r$, and it belongs to the space $H^0(\text{End } E \otimes K_X)$. The pair (E, Φ) is called a cyclic Higgs bundle [3, 6]. Suppose that $\Phi_j \neq 0$ for all $j = 1, \dots, r$. Then (E, Φ) is stable, and thus, there uniquely exists a harmonic metric h on (E, Φ) . The harmonic metric h splits as $h = (h_1, \dots, h_r)$ concerning to the decomposition since the Higgs field Φ is an eigensection of the gauge transformation $g = \text{diag}(1, \omega, \dots, \omega^{r-1})$ ($\omega = e^{2\pi\sqrt{-1}/r}$) [3]. This property of the harmonic metric makes it possible to use techniques of maximum principles when we investigate it (see [6]). The present paper aims to establish the following two results regarding Higgs bundles over compact Kähler manifolds: First, we provide a necessary and sufficient condition for the existence of a diagonal metric that solves the Hermitian-Einstein equation (see Theorem 1). Second, we establish the relationship between the condition which we give in Theorem 1 and the usual stability condition of Higgs bundles (see Proposition 1 and Proposition 2). The precise statement of Theorem 1 is as follows: Let $(E, \Phi) \rightarrow (X, \omega_X)$ be a Higgs bundle over a compact Kähler manifold. Suppose that the holomorphic vector bundle E decomposes as $E = L_1 \oplus \dots \oplus L_r$ with holomorphic line bundles $L_1, \dots, L_r \rightarrow X$. Then the Higgs field Φ decomposes as $\Phi = \Phi_0 + \sum_{i,j=1,\dots,r} \Phi_{i,j}$, where $\Phi_{i,j}$ is a holomorphic $(1, 0)$ -form with values in $L_j^{-1} L_i$ and Φ_0 is the diagonal part. For each $j = 1, \dots, r$, let γ_j be the degree of the holomorphic line bundle L_j with respect to the Kähler form ω_X . We assume that $\deg_{\omega_X}(E) = \gamma_1 + \dots + \gamma_r = 0$ for simplicity. Let V be a real vector space defined as $V := \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid x_1 + \dots + x_r = 0\}$. For each

Communicated by Indranil Biswas.

N. Miyatake (✉)

Mathematical Science Center for Co-creative Society, Tohoku University, 468-1 Aramaki Azaoaba, Aoba-ku, Sendai, Miyagi 9800845, Japan

E-mail: natsuo.miyatake.e8@tohoku.ac.jp

$i, j = 1, \dots, r$, let $v_{i,j} \in V$ be a vector defined as $v_{i,j} := u_i - u_j$, where $u_1, \dots, u_r$ is the canonical basis of $\mathbb{R}^r$. We define a vector $\gamma \in V$ as $\gamma := (\gamma_1, \dots, \gamma_r)$. Then the following holds:

Theorem 1 *The Higgs bundle (E, Φ) admits a solution to the Hermitian-Einstein equation which is diagonal concerning the decomposition $E = L_1 \oplus \dots \oplus L_r$ if and only if the following (i) and (ii) hold:*

- (i) *The off-diagonal part of $\Lambda[\Phi \wedge \Phi^{*h}]$ vanishes for a diagonal metric $h = (h_1, \dots, h_r)$, where Λ denotes the adjoint of $\omega_X \wedge$.*
- (ii) *The following holds:*

$$-\gamma \in \sum_{i,j=1,\dots,r, \Phi_{i,j} \neq 0} \mathbb{R}_{>0} v_{i,j}. \quad (1)$$

In particular, a Higgs bundle satisfying (i) and (ii) is polystable.

Remark 1 From [10, Proposition 3.4], if the above condition (i) and (ii) hold and in addition, $c_2(E) = 0$, then the solution of the Hermitian-Einstein equation is a diagonal pluriharmonic metric. The corresponding pluriharmonic map from the universal covering space $\tilde{X}$ to $\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$ naturally lifts to the map to $\mathrm{SL}(r, \mathbb{C})/T$, where T is the maximal torus of $\mathrm{SU}(r)$ which consists of all diagonal matrices of $\mathrm{SU}(r)$ (see [3, 4]).

Remark 2 Theorem 1 can easily be generalized to G -Higgs bundles. In particular, the following holds: let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let $H \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{h}$. Let $\alpha_1, \dots, \alpha_l$ be a base of the root space $\Delta \subseteq \mathfrak{h}^*$ and δ the highest root. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition. Consider a G -Higgs bundle $(P_G, \Phi) \rightarrow (X, \omega_X)$ with holomorphic G -bundle P_G which admits a reduction $P_H \subseteq P_G$ to a holomorphic H -subbundle P_H . Suppose that the Higgs field Φ is of the following form:

$$\Phi = \sum_{\alpha \in \{-\alpha_1, \dots, -\alpha_l, \delta\}} \Phi_\alpha.$$

We call (P_G, Φ) a G -cyclic Higgs bundle over (X, ω_X) . For a G -cyclic Higgs bundle, if $\Phi_\alpha \neq 0$ for all $\alpha \in \{-\alpha_1, \dots, -\alpha_l, \delta\}$, then there uniquely exists a diagonal solution to the Hermitian-Einstein equation of the G -cyclic Higgs bundle (P_G, Φ) . This can be verified by the following observation: For each $\alpha \in \{-\alpha_1, \dots, -\alpha_l, \delta\}$, let h_α be the coroot of α . Then we can observe that

$$\sum_{\alpha \in \{-\alpha_1, \dots, -\alpha_l, \delta\}} \mathbb{R}_{>0} h_\alpha = \sqrt{-1}t,$$

where t is the Lie algebra of the maximal compact torus $T \subseteq H$. Therefore, no matter which direction the vector γ points in, we see that condition (1) holds for the G -cyclic Higgs bundle. This implies the claim (see also [8, Section 2]).

Remark 3 We prove Theorem 1 by applying [8, Theorem 1]. However, Theorem 1 can also be derived by showing the stability of quiver bundles [1] coincides with condition (1) in the setting we are considering. We also note that the same condition as (1) is obtained in [2] for a slightly different PDE.

We prove Theorem 1. We can easily check that condition (i) is a necessary condition for the existence of a diagonal metric which solves the Hermitian-Einstein equation. Suppose that (i) holds. We fix a diagonal metric $h = (h_1, \dots, h_r)$ such that $\det(h) = 1$. Let $\xi := (f_1, \dots, f_r)$ ($f_1 + \dots + f_r = 0$) be a pair of $\mathbb{R}$ -valued functions. Then the Hermitian-Einstein equation for a metric $(e^{f_1} h_1, \dots, e^{f_r} h_r)$ is the following:

$$\Delta_{\omega_X} \xi + \sum_{i,j=1,\dots,r} 4|\Phi_{i,j}|_{h,\omega_X}^2 e^{(v_{i,j}, \xi)} v_{i,j} = -2\sqrt{-1} \Lambda F_h, \quad (2)$$

where we denote by F_h the curvature of the Chern connection of h and by Δ_{ω_X} the geometric Laplacian. We apply [8, Theorem 1] to equation (2). In order to apply [8, Theorem 1], we must check that for each $i, j = 1, \dots, r$, if $\Phi_{i,j}$ is not a zero section, then $\log |\Phi_{i,j}|_{h,\omega_X}^2$ is integrable. This follows from the following lemma, which is an immediate consequence of the fact that plurisubharmonic functions are locally integrable:

Lemma 1 *Let $U \subseteq \mathbb{C}^n$ be a domain and $f : U \rightarrow V$ a holomorphic section of a trivial bundle $V = U \times \mathbb{C}^r \rightarrow U$. Then for any smooth Hermitian metric h_V on V , $\log |f|_{h_V} \in L_{loc}^1(U)$.*



Proof Let $v_1, \dots, v_r : U \rightarrow V$ be a holomorphic frame of V . Then f is denoted as $f = f_1 v_1 + \dots + f_r v_r$. We denote by $\hat{h}_V : U \rightarrow \text{Herm}(r)$ the Hermitian matrix valued smooth function whose (i, j) component is $h_V(v_i, v_j)$. Then $\hat{h}_V$ is diagonalized as ${}^t \bar{B} \hat{h}_V B = \text{diag}(\lambda_1, \dots, \lambda_r)$ for a unitary matrix valued function B and positive functions $\lambda_1, \dots, \lambda_r$ over U , where we denote by $\text{diag}(\lambda_1, \dots, \lambda_r)$ the diagonal matrix whose diagonal entries are $\lambda_1, \dots, \lambda_r$. We set $(f'_1, \dots, f'_r) := (f_1, \dots, f_r)B$. Let $F \subseteq U$ be a compact subset of U . We define a positive constant C as $C := \min_{1 \leq i \leq r} \{\min_{x \in F} \lambda_1(x), \dots, \min_{x \in F} \lambda_r(x)\}$. Then we have

$$\begin{aligned} \log |f|_{h_V}^2 &= \log\{\lambda_1 |f'_1|^2 + \dots + \lambda_r |f'_r|^2\} \\ &\geq \log\{C |f'_1|^2 + \dots + C |f'_r|^2\} \\ &= \log C + \log\{|f'_1|^2 + \dots + |f'_r|^2\} \\ &= \log C + \log\{|f_1|^2 + \dots + |f_r|^2\} \end{aligned}$$

for each point of F . Since $f_1, \dots, f_r$ are holomorphic functions, $\log\{|f_1|^2 + \dots + |f_r|^2\}$ is a plurisubharmonic function (see [5]). In particular, it is in $L^1_{loc}(U)$. This implies the claim. $\square$

Then we prove Theorem 1:

Proof of Theorem 1 As already remarked, under the assumption of (i), the Hermitian-Einstein equation for a diagonal metric $h = (e^{f_1} h_1, \dots, e^{f_r} h_r)$ is equation (2). Then from [8, Theorem 1], equation (2) has a smooth solution if and only if (ii) holds. This implies the claim.

2 Relationship between condition (1) and the stability condition

We describe the relationship between condition (1) and the usual stability condition of Higgs bundles by using the torus action on the space of Higgs fields. Suppose that $\gamma_1, \dots, \gamma_r$ are all rational numbers. For each subset $I \subseteq \{1, \dots, r\}$, we define a subbundle $E_I \subseteq E$ as $E_I := \bigoplus_{i \in I} L_i$. Then the following holds:

Proposition 1 *The following are equivalent:*

(i) *The following holds:*

$$-\gamma \in \sum_{\substack{i,j=1,\dots,r, \\ \Phi_{i,j} \neq 0}} \mathbb{Q}_{\geq 0} v_{i,j}. \quad (3)$$

(ii) *For any subset $I \in \{1, \dots, r\}$, if E_I is preserved by Φ , then $\deg_{\omega_X}(E_I) \leq 0$.*

In particular, if (E, Φ) is semistable, then (E, Φ) satisfies (3) for any holomorphic splitting.

Proposition 2 *The following (i) and (ii) are equivalent:*

(i) *The following holds:*

$$-\gamma \in \sum_{\substack{i,j=1,\dots,r, \\ \Phi_{i,j} \neq 0}} \mathbb{Q}_{>0} v_{i,j}. \quad (4)$$

(ii) *There exist subsets $I_1, \dots, I_k \subseteq \{1, \dots, r\}$ such that*

- *The holomorphic vector bundle E is a direct sum of $E_{I_1}, \dots, E_{I_k}$: $E = E_{I_1} \oplus \dots \oplus E_{I_k}$ and for each $j = 1, \dots, k$, E_{I_j} is preserved by Φ .*
- *For each $j = 1, \dots, k$, $\deg_{\omega_X}(E_{I_j}) = 0$.*
- *For each $j = 1, \dots, k$, if there exists a subset $I \subsetneq I_j$ and if E_I is preserved by Φ , then $\deg_{\omega_X}(E_I) < 0$.*

In particular, if (E, Φ) is stable, then (E, Φ) satisfies (4) for any holomorphic splitting.

Remark 4 The choice of the holomorphic splitting of E is of course not unique. For example, suppose that the rank of E is 2, E decomposes as $E = L_1 \oplus L_2$, and there exists a non-trivial holomorphic bundle map $f : L_1 \rightarrow L_2$. Then $E = L'_1 \oplus L_2$ is another decomposition, where L'_1 denotes a holomorphic line bundle defined as $L'_1 := \{(v_1, v_2) \in L_1 \oplus L_2 \mid v_2 = f(v_1)\}$.



Before starting the proof of Proposition 1 and Proposition 2, we make some preparations. Let T be the maximal torus of $SU(r)$ which consists of all diagonal matrices of $SU(r)$. We denote by $\mathfrak{t}$ the Lie algebra of T . Let $H \subseteq SL(r, \mathbb{C})$ be the complexification of T with Lie algebra $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$. We define a lattice $\mathfrak{h}_{\mathbb{Z}}$ of $\sqrt{-1}\mathfrak{t}$ as $\mathfrak{h}_{\mathbb{Z}} := \{\text{diag}(n_1, \dots, n_r) \mid n_1, \dots, n_r \in \mathbb{Z}, n_1 + \dots + n_r = 0\}$. Note that $\mathfrak{h}_{\mathbb{Z}}$ coincides with the kernel of the exponential map $\text{Exp} : \mathfrak{h} \rightarrow H$. We regard the vector γ and the vectors $v_{i,j}$ ($i, j = 1, \dots, r$) as elements of $\mathfrak{h}_{\mathbb{Q}} := \mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by identifying the canonical basis $u_1, \dots, u_r$ with $\text{diag}(1, \dots, 0), \dots, \text{diag}(0, \dots, 1)$. Let $(\cdot, \cdot)$ be a bilinear form on $\mathfrak{h}$ defined as $(u, v) := \text{Tr}(uv)$ for $u, v \in \mathfrak{h}$. We denote by $\gamma^* \in \mathfrak{h}_{\mathbb{Q}}^*$ be the dual of the vector γ with respect to the $(\cdot, \cdot)$. We take a positive integer n so that $n\gamma^* \in \mathfrak{h}_{\mathbb{Z}}^*$. Then we define an element $\gamma^{\vee} \in \mathfrak{h}_{\mathbb{Z}}^*$ as $\gamma^{\vee} := -n\gamma^*$. Also, we define a character $\chi_{\gamma^{\vee}} : H \rightarrow \mathbb{C}^*$ as $\chi_{\gamma^{\vee}}(\text{Exp}(v)) := e^{\langle v, \gamma^{\vee} \rangle}$ for $v \in \mathfrak{h}$, where we denote by $\langle \cdot, \cdot \rangle$ the coupling between $\mathfrak{h}$ and $\mathfrak{h}^*$. We define an action of the algebraic torus H on $H^0(\text{End} E \otimes \bigwedge^{1,0})$ as $g \cdot (\theta, z) := (\theta_0 + \sum_{i,j=1,\dots,r} g_j^{-1} g_i \theta_{i,j}, \chi_{\gamma^{\vee}}^{-1}(g)z)$ for $(\theta, z) \in H^0(\text{End} E \otimes \bigwedge^{1,0}) \times \mathbb{C}$, where $g = \text{diag}(g_1, \dots, g_r)$ is an element of H and we denote by θ_0 and $\theta_{i,j}$ ($i, j = 1, \dots, r$) the diagonal component and the (i, j) -component of θ , respectively. For each $i, j = 1, \dots, r$, let $\alpha_{i,j}$ be the element of $\mathfrak{h}^*$ defined as $\alpha_{i,j} := \lambda_i - \lambda_j$, where $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ is the dual basis of $u_1, \dots, u_r$. For each $s \in \sqrt{-1}\mathfrak{t}$, we define a set A_s as $A_s := \{(i, j) \mid i, j \in \{1, \dots, r\}, \alpha_{i,j}(s) \geq 0\}$. Then we start the proof of Proposition 1 and Proposition 2.

Proof of Proposition 1 Condition (3) holds if and only if for each $z \neq 0$, the closure of $H \cdot (\Phi, z)$ does not intersects $\mathbb{C} \times \{0\}$ (see [8, Proposition 3]). Furthermore, this is equivalent to the following:

- It holds that $\gamma^{\vee}(s) \geq 0$ for any $s \in \sqrt{-1}\mathfrak{t}$ such that

$$\Phi \in H^0\left(\bigoplus_{(i,j) \in A_s} L_j^{-1} L_i \otimes \bigwedge^{1,0}\right). \quad (5)$$

Let $s = \text{diag}(s_1, \dots, s_r)$ be an element of $\sqrt{-1}\mathfrak{t}$ such that (5) holds. We may assume that $s_1 \geq \dots \geq s_r$. Let $I_j \subseteq \{1, \dots, r\}$ be a subset defined as $I_j := \{1, \dots, j\}$. From (5), the Higgs field Φ preserves the sequence of subbundles $0 \subsetneq E_{I_{j_1}} \subsetneq E_{I_{j_2}} \subsetneq \dots \subsetneq E_{I_{j_k}} \subsetneq E$, where $1 \leq j_1 < \dots < j_k < r$ are all the elements of $J_s := \{j \in \{1, \dots, r-1\} \mid s_j > s_{j+1}\}$. Then $\gamma^{\vee}(s)$ can be calculated as

$$\begin{aligned} \gamma^{\vee}(s) &= -n \text{Tr}(\gamma s) \\ &= -n(\gamma_1 s_1 + \dots + \gamma_r s_r) \\ &= -n(\gamma_1(s_1 - s_2) + (\gamma_1 + \gamma_2)(s_2 - s_3) + \dots \\ &\quad + (\gamma_1 + \dots + \gamma_{r-1})(s_{r-1} - s_r)) \\ &= -n \sum_{j \in J_s} (s_j - s_{j+1}) \deg_{\omega_X}(E_{I_j}). \end{aligned} \quad (6)$$

Therefore, it holds that $\gamma^{\vee}(s) \geq 0$ if and only if $\deg_{\omega_X}(E_{I_j}) \leq 0$ for all $j \in J_s$. This implies the claim. $\square$

Proof of Proposition 2 Condition (4) holds if and only if $H \cdot (\Phi, z)$ is closed for each $z \neq 0$ (see [8, Proposition 4]). Furthermore, this is equivalent to that the following holds for all $s \in \sqrt{-1}\mathfrak{t}$ such that (5) holds:

- The element s of $\sqrt{-1}\mathfrak{t}$ satisfies $\gamma^{\vee}(s) \geq 0$ and if $\gamma^{\vee}(s) = 0$, then Φ lies in $H^0(\bigoplus_{(i,j) \in A_s^0} L_j^{-1} L_i \otimes \bigwedge^{1,0})$, where A_s^0 is defined as $A_s^0 := \{(i, j) \mid i, j \in \{1, \dots, r\}, \alpha_{i,j}(s) = 0\}$.

Then from (6), we see that (i) and (ii) are equivalent. $\square$

Remark 5 For the proof of Proposition 1 and Proposition 2, the author referred the definition of the stability of Higgs bundles by using parabolic subalgebras (see, for example, [7]).

We refer the reader to [9] for the relationship between the condition (4) and Donaldson's functional restricted to diagonal metrics on Higgs bundles.

Acknowledgements I wish to express my gratitude to Ryushi Goto and Hisashi Kasuya for their valuable comments and helpful advice. I am very grateful to Yoshinori Hashimoto for his valuable discussions, and for asking me about the relationship between condition (1) and the definition of the usual stability condition.



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